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POWERS IN $\prod_{k=1}^{n}\left(ak^{2^l\cdot 3^m}+b\right)$

Zhongfeng Zhang

Abstract: Let $f(x)=ax^{2^l\cdot 3^m}+b\in\mathbb{Z}[x]$ be a polynomial with $l\geqslant 1, l+m\geqslant 2, ab\neq 0$ and such that $f(k)\neq 0$ for any $k\geqslant 1$. We prove, under ABC conjecture, that the product $\prod_{k=1}^{n} f(k)$ is not a $2^l\cdot 3^m$-th power for n large enough.

Keywords: powers, the greatest prime factor, ABC conjecture.

1. Introduction

In [1], J. Cilleruelo proved that the product $\prod_{k=1}^{n}(k^2+1)$ is not a square when $n>3$. Using similar arguments, Erhan Gürel, Ali Ulaş Özgür Kişisel [2] proved that $\prod_{k=1}^{n}(k^3+1)$ is not a square for any positive integer n. For any irreducible quadratic polynomial $f(x)\in\mathbb{Z}[x]$, Zhang and Yuan[4] proved that the product $\prod_{k=1}^{n} f(k)$ is not a square when $n>C(f)$. Their proof also tells us how to calculate the constant $C(f)$. For higher degree polynomials, it is not easy to obtain a similar result.

For the special family of polynomials $f(x)=ax^{2^l\cdot 3^m}+b\in\mathbb{Z}[x]$, we obtain a result of this type under the ABC conjecture.

Theorem 1.1. *Let l,m be non-negative integers, $l\geqslant 1, l+m\geqslant 2$, and let $f(x)=ax^{2^l\cdot 3^m}+b\in\mathbb{Z}[x]$ be a polynomial such that $ab\neq 0$ and $f(k)\neq 0$ for $k\geqslant 1$. Then under ABC conjecture, the product $T_n=\prod_{k=1}^{n} f(k)$ is not a $2^l\cdot 3^m$ power for sufficiently large n.*

2. Proof of Theorem 1.1

First, we introduce the ABC conjecture.

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ABC Conjecture. *Let $\epsilon > 0$, then there is a constant C_ϵ, depending only on ϵ, such that for all triples $A, B, C \in \mathbb{Z}$, with $A + B + C = 0$ and $\gcd(A, B, C) = 1$, the following inequality holds:*

$$\max\{|A|, |B|, |C|\} < C_\epsilon \prod_{p|ABC} p^{1+\epsilon}.$$

The following lemma is obtained by Nagell[3] .

Lemma 2.1. *Let $f(x)$ be any polynomial with integer coefficients which is not the product of linear factors with integral coefficients. Denote by P_n, the greatest prime factor of $\prod_{k=1}^{n} f(k)$. Then*

$$P_n > C_1 n \log n,$$

where the positive constant C_1 depends on $f(x)$.

Proof of Theorem 1.1. We give two propositions, and then Theorem 1.1 follows. ■

Proposition 2.2. *Let $l \geqslant 2$ be an integer, $f(x) = ax^{2^l} + b \in \mathbb{Z}[x]$, $ab \neq 0$, $f(k) \neq 0$, $k \geqslant 1$. Then, under ABC conjecture, there is a positive constant C_f, depending only on $f(x)$, such that the product $\prod_{k=1}^{n} f(k)$ is not a 2^l-th power when $n > C_f$.*

Proof. Let $T_n = \prod_{k=1}^{n} f(k)$ be a 2^l-th power, $n > \max\{|a|, |b|\}$, and p any prime which divides T_n. First, we prove that there exists a constant $C_1 = C_1(f)$, such that $p < C_1 n$. We distinguish three cases which cover all the situations and assume $p > n$ in the following discussion.

Case 1: $p^3 | f(k)$ for some $1 \leqslant k \leqslant n$.

Let $ak^{2^l} + b = p^3 e$, then $d = \gcd\left(ak^{2^l}, b, p^3 e\right) = \gcd\left(ak^{2^l}, b, e\right)$ because $p > n > |b|$. We have now to consider the equality

$$\frac{ak^{2^l}}{d} + \frac{b}{d} = p^3 \frac{e}{d}.$$

There is a constant $C_2 = C_2(f)$, such that

$$|a|k \frac{|b|}{d} p \frac{|e|}{d} = \frac{|ab|kp}{d^2} \frac{|ak^{2^l} + b|}{p^3} = \frac{|ab|}{d^2} \frac{|ak^{2^l+1} + bk|}{p^2} < C_2 k^{2^l - 1}$$

since $p > n \geqslant k$. Take $\epsilon = 2^{-(l+1)}$ in the ABC conjecture, we have

$$|a|k^{2^l} < C_\epsilon (C_2 k^{2^l - 1})^{1 + 2^{-(l+1)}},$$

which yields $k < C_3 = C_3(f)$, and then we obtain $p < C_4 = C_4(f)$. Therefore, we have $p < C_5 n = C_5(f) n$ in Case 1.

Case 2: $p^2 | f(r_j)$ for some $1 \leqslant r_1 < r_2 < ... < r_{2^{l-1}} \leqslant n$.

In this case, one has $p^2|a\left(r_j^{2^l}-r_i^{2^l}\right)=a\left(r_j^{2^{l-1}}-r_i^{2^{l-1}}\right)\left(r_j^{2^{l-1}}+r_i^{2^{l-1}}\right)$ for any $1\leqslant i<j\leqslant 2^{l-1}$. Since

$$r_j^{2^{l-1}}+r_i^{2^{l-1}}=r_j^{2^{l-1}}-r_i^{2^{l-1}}+2r_i^{2^{l-1}},$$

and $p>n$, we get $\gcd\left(p,r_j^{2^{l-1}}-r_i^{2^{l-1}},r_j^{2^{l-1}}+r_i^{2^{l-1}}\right)=1$.

Define $S_l=2^{l-2}+1$ for $l\geqslant 2$, then $S_{l+1}=2S_l-1$. We will prove, by induction on l, that if $p^2|a\left(t_j^{2^l}-t_1^{2^l}\right),2\leqslant j\leqslant S_l$ for some $1\leqslant t_1<t_2<...<t_{S_l}\leqslant n$, then $p<2n$.

When $l=2$, we have $p<2n$ from $p^2|a(t_2^4-t_1^4)=a(t_2^2-t_1^2)(t_2^2+t_1^2)$ and $p>n>|a|$.

If the statements holds for $l-1$, we prove that it is also true for l.

Since $p^2|f(r_j)$, we have $p^2|f(r_j)-f(r_1)$, that is

$$p^2|a\left(t_j^{2^l}-t_1^{2^l}\right)=a\left(t_j^{2^{l-1}}-t_1^{2^{l-1}}\right)\left(t_j^{2^{l-1}}+t_1^{2^{l-1}}\right),\qquad 2\leqslant j\leqslant S_l=2S_{l-1}-1.$$

Together with $\gcd\left(p,r_j^{2^{l-1}}-r_i^{2^{l-1}},r_j^{2^{l-1}}+r_i^{2^{l-1}}\right)=1$, $p>n>|a|$ and pigeonhole principle, we have

(i) $p^2|t_{j_i}^{2^{l-1}}-t_1^{2^{l-1}}$ for some $2\leqslant j_1<...<j_{S_{l-1}-1}\leqslant S_l$,

or

(ii) $p^2|t_{j_i}^{2^{l-1}}+t_1^{2^{l-1}}$ for some $2\leqslant j_1<...<j_{S_{l-1}}\leqslant S_l$.

Case (i) is just the situation of $l-1$, by induction, we obtained $p<2n$. Case (ii) leads to $p^2|t_{j_i}^{2^{l-1}}-t_{j_1}^{2^{l-1}}$, $2\leqslant i\leqslant S_{l-1}$, and is also the situation of $l-1$, thus we get $p<2n$ by induction. Since $S_l=2^{l-2}+1\leqslant 2^{l-1}$, we have $p<2n$ in Case 2.

Case 3: $p|f(r_j)$ for some $1\leqslant r_1<r_2<...<r_{2^{l-1}+1}\leqslant n$.

In this case, one has

$$p|a\left(r_j^{2^l}-r_i^{2^l}\right)=a\left(r_j^{2^{l-1}}-r_i^{2^{l-1}}\right)\left(r_j^{2^{l-1}}+r_i^{2^{l-1}}\right),\qquad 1\leqslant i<j\leqslant 2^{l-1}+1.$$

Similar to Case 2, we can get $p<2n$ by induction. Actually, since $2^{l-1}+1=S_{l+1}$, we replace p^2 by p in the induction of Case 2, and the same argument leads to $p<2n$ if it is true for $l=2$. Thus we only need to check the case $l=2$.

When $l=2$, we have $p|a\left(r_j^4-r_1^4\right)=a\left(r_j^2-r_1^2\right)\left(r_j^2+r_1^2\right)$, $2\leqslant j\leqslant 3$. If $p|r_2^2-r_1^2$ or $r_3^2-r_1^2$, then $p<2n$. Otherwise $p|r_j^2+r_1^2$, $2\leqslant j\leqslant 3$ since $p>n>|a|$, that is, $p|\left(r_3^2+r_1^2\right)-\left(r_2^2+r_1^2\right)=r_3^2-r_2^2$, which yields $p<2n$.

From the discussion of Cases 1, 2, 3 we know that there exists a constant $C_1=C_1(f)$, such that $p<C_1n$. It is obvious that $f(x)$ can not decompose into linear factors with integral coefficients, so by Lemma 2.1 we get $n<C_f$. ■

Proposition 2.3. *Let $l\geqslant 1$, $m\geqslant 1$ be integers, $f(x)=ax^{2^l\cdot 3^m}+b\in\mathbb{Z}[x]$, $ab\neq 0$, $f(k)\neq 0$, $k\geqslant 1$. Then, under ABC conjecture, there is a positive constant C_f, depending only on $f(x)$, such that the product $\prod_{k=1}^{n}f(k)$ is not a $2^l\cdot 3^m$-th power when $n>C_f$.*

Proof. Let $T_n = \prod_{k=1}^{n} f(k)$ be an $2^l \cdot 3^m$-th power, $n > \max\{|a|,|b|\}$, p is any prime which divides T_n. Similar to Proposition 2.2, we prove that there exists a constant $C_1 = C_1(f)$, such that $p < C_1 n$. Since $2^l \cdot 3^{m-1} + 2^{l+1} \cdot 3^{m-1} = 2^l \cdot 3^m$, we distinguish three cases which cover all the situations and assume $p > \max\{n, 3\}$ in the following discussion.

Case 1: $p^3 | f(k)$ for some $1 \leqslant k \leqslant n$.

Similar to the proof of Proposition 2, Case 1, we have $p < C_2 n = C_2(f)n$ under *ABC* conjecture.

Case 2: $p^2 | f(r_j)$ for some $1 \leqslant r_1 < r_2 < ... < r_{2^l \cdot 3^{m-1}+1} \leqslant n$.

Case 2.1: We will prove $p < 2n$ by induction in the following situation: There exist $3^{m-1}+1$ different integers $1 \leqslant t_i \leqslant n$, which for any $2 \leqslant i \leqslant 3^{m-1}+1$, one has $p^2 | a(t_i^{3^m} - t_1^{3^m})$.

Define $Q_m = 3^{m-1}+1$ for $m \geqslant 1$, then $Q_{m+1} = 3Q_m - 2$. We proceed to prove the statement by induction on m.

When $m = 1$, since $Q_1 = 3^{1-1}+1 = 2$, we have

$$p^2 | a\left(t_2^3 - t_1^3\right) = a\left(t_2 - t_1\right)\left(t_2^2 + t_2 t_1 + t_1^2\right),$$

together with $p \nmid a(t_2 - t_1)$ implied by $p > n > |a|$ leads to $p < 2n$.

Assume we get $p < 2n$ for $m-1$, we will prove $p < 2n$ for m. Since

$$\begin{aligned} p^2 | a & \left(t_i^{3^m} - t_1^{3^m}\right) \\ &= a\left(t_i^{3^{m-1}} - t_1^{3^{m-1}}\right)\left(t_i^{2\cdot 3^{m-1}} + t_i^{3^{m-1}} t_1^{3^{m-1}} + t_1^{2\cdot 3^{m-1}}\right), \qquad 2 \leqslant i \leqslant Q_m, \end{aligned}$$

and

$$\begin{aligned} & t_i^{2\cdot 3^{m-1}} + t_i^{3^{m-1}} t_1^{3^{m-1}} + t_1^{2\cdot 3^{m-1}} \\ & \qquad = \left(t_i^{3^{m-1}} - t_1^{3^{m-1}}\right)^2 + 3 t_i^{3^{m-1}} t_1^{3^{m-1}}, \qquad p > \max\{n, 3\} \end{aligned}$$

implies

$$\gcd\left(p, t_i^{3^{m-1}} - t_1^{3^{m-1}},\ t_i^{2\cdot 3^{m-1}} + t_i^{3^{m-1}} t_1^{3^{m-1}} + t_1^{2\cdot 3^{m-1}}\right) = 1,$$

one has three cases which contain all the possibilities.

(i) $p^2 | t_{i_s}^{3^{m-1}} - t_1^{3^{m-1}}$ for some $2 \leqslant i_1 < i_2 < ... < i_{Q_{m-1}-1} \leqslant Q_m$, then we get $p < 2n$ by induction.

(ii) $p^2 \nmid t_i^{3^{m-1}} - t_1^{3^{m-1}}$ for some $2 \leqslant i \leqslant Q_m$. Without loss of generality, we assume $p^2 \nmid t_2^{3^{m-1}} - t_1^{3^{m-1}}$. If $p^2 | t_{j_s}^{3^{m-1}} - t_2^{3^{m-1}}$ for some $3 \leqslant j_1 < j_2 < ... < j_{Q_{m-1}-1} \leqslant Q_m$, then we also get $p < 2n$ by induction.

(iii) Assume $p^2 \nmid t_2^{3^{m-1}} - t_1^{3^{m-1}}$, and recall that $Q_m = 3Q_{m-1} - 2$, then since $3Q_{m-1} - 2 - (Q_{m-1} - 2 + Q_{m-1} - 2 + 2) = Q_{m-1}$, the left case is that

$$p^2 \nmid t_{j_s}^{3^{m-1}} - t_1^{3^{m-1}}, \qquad p^2 \nmid t_{j_s}^{3^{m-1}} - t_2^{3^{m-1}}$$

for some $3\leqslant j_1<j_2<...<j_{Q_{m-1}}\leqslant Q_m$. Therefore one has

$$p^2|\left(t_{j_s}^{2\cdot 3^{m-1}}+t_{j_s}^{3^{m-1}}t_1^{3^{m-1}}+t_1^{2\cdot 3^{m-1}}\right)-\left(t_2^{2\cdot 3^{m-1}}+t_2^{3^{m-1}}t_1^{3^{m-1}}+t_1^{2\cdot 3^{m-1}}\right)$$
$$=\left(t_{j_s}^{3^{m-1}}-t_2^{3^{m-1}}\right)\left(t_{j_s}^{3^{m-1}}+t_2^{3^{m-1}}+t_1^{3^{m-1}}\right)$$

and then

$$p^2|\left(t_{j_s}^{3^{m-1}}+t_2^{3^{m-1}}+t_1^{3^{m-1}}\right)-\left(t_{j_1}^{3^{m-1}}+t_2^{3^{m-1}}+t_1^{3^{m-1}}\right)$$
$$=t_{j_s}^{3^{m-1}}-t_{j_1}^{3^{m-1}},\qquad 2\leqslant s\leqslant Q_{m-1},$$

which yields $p<2n$ by induction.

Case 2.2: Now we use Case 2.1, together with induction on l to show $p<2n$ in Case 2.

When $l=1$, one has

$$p^2|a\left(r_i^{2\cdot 3^m}-r_1^{2\cdot 3^m}\right)=a\left(r_i^{3^m}-r_1^{3^m}\right)\left(r_i^{3^m}+r_1^{3^m}\right),$$

combined with

$$\gcd\left(p,r_i^{3^m}-r_1^{3^m},r_i^{3^m}+r_1^{3^m}\right)=1,\qquad 2\leqslant i\leqslant 2\cdot 3^{m-1}+1$$

implied by $p>n$, we obtain

(i) $p^2|r_{i_s}^{3^m}-r_1^{3^m}$ for some $2\leqslant i_1<i_2<...<i_{3^{m-1}}\leqslant 2\cdot 3^{m-1}+1$, or
(ii) $p^2|r_{j_s}^{3^m}+r_1^{3^m}$ for some $2\leqslant j_1<j_2<...<j_{3^{m-1}+1}\leqslant 2\cdot 3^{m-1}+1$.

Because (ii) implies (i), by Case 2.1, each case leads to $p<2n$.

Assume we get $p<2n$ for $l-1$, we will prove $p<2n$ for l. Since

$$p^2|a\left(r_i^{2^l\cdot 3^m}-r_1^{2^l\cdot 3^m}\right)$$
$$=a\left(r_i^{2^{l-1}\cdot 3^m}-r_1^{2^{l-1}\cdot 3^m}\right)\left(r_i^{2^{l-1}\cdot 3^m}+r_1^{2^{l-1}\cdot 3^m}\right),\qquad 2\leqslant i\leqslant 2^l\cdot 3^{m-1}+1$$

and

$$\gcd\left(p,r_i^{2^{l-1}\cdot 3^m}-r_1^{2^{l-1}\cdot 3^m},r_i^{2^{l-1}\cdot 3^m}+r_1^{2^{l-1}\cdot 3^m}\right)=1$$

implied by $p>n$, one has

(i) $p^2|r_{i_s}^{2^{l-1}\cdot 3^m}-r_1^{2^{l-1}\cdot 3^m}$ for some $2\leqslant i_1<i_2<...<i_{2^{l-1}\cdot 3^{m-1}}\leqslant 2^l\cdot 3^{m-1}+1$,

or

(ii) $p^2|r_{j_r}^{2^{l-1}\cdot 3^m}+r_1^{2^{l-1}\cdot 3^m}$ for some $2\leqslant j_1<j_2<...<j_{2^{l-1}\cdot 3^{m-1}+1}\leqslant 2^l\times 3^{m-1}+1$.

From the fact that (ii) implies (i), each case leads to $p < 2n$ by induction.

Case 3: $p|f(r_j)$ for some $1 \leqslant r_1 < r_2 < \ldots < r_{2^{l+1}\cdot 3^{m-1}} \leqslant n$.

Case 3.1: We now prove $p < 3n$ by induction in the following two situations.

Case 3.1.1: There exists $2 \cdot 3^{m-1} + 1$ different integers $1 \leqslant t_i \leqslant n$, which for any $2 \leqslant i \leqslant 2 \cdot 3^{m-1} + 1$, one has $p|t_i^{3^m} - t_1^{3^m}$.

Case 3.1.2: There exists $2 \cdot 3^{m-1} + 1$ different integers $1 \leqslant t_i \leqslant n$, which for any $2 \leqslant i \leqslant 2 \cdot 3^{m-1} + 1$, one has $p|t_i^{3^m} + t_1^{3^m}$.

Define $L_m = 2 \cdot 3^{m-1} + 1$ for $m \geqslant 1$, then $L_{m+1} = 3L_m - 2$.

In the situation Case 3.1.1, when $m = 1$, then

$$p|t_2^3 - t_1^3 = (t_2 - t_1)\left(t_2^2 + t_2t_1 + t_1^2\right), \qquad p|t_3^3 - t_1^3 = (t_3 - t_1)\left(t_3^2 + t_3t_1 + t_1^2\right).$$

Since $p > n$, we have $p \nmid t_2 - t_1,\ p \nmid t_3 - t_1$, then

$$p|\left(t_3^2 + t_3t_1 + t_1^2\right) - \left(t_2^2 + t_2t_1 + t_1^2\right) = (t_3 + t_2 + t_1)(t_3 - t_2),$$

which yields $p < 3n$. Since $L_{m+1} = 3L_m - 2$, induction on m, the same arguments as the proof of Case 2.1, we get $p < 3n$ in this situation.

We continue to prove $p < 3n$ in situation Case 3.1.2 by induction on m.

When $m = 1$, similar to the situation Case 3.1.1, we can get $p < 3n$.

For $m \geqslant 2$, from

$$p|t_i^{3^m} + t_1^{3^m} = \left(t_i^{3^{m-1}} + t_1^{3^{m-1}}\right)\left(t_i^{2\cdot 3^{m-1}} - t_i^{3^{m-1}}t_1^{3^{m-1}} + t_1^{2\cdot 3^{m-1}}\right), \qquad 2 \leqslant i \leqslant L_m,$$

one has two cases.

(i) $p|t_{i_s}^{3^{m-1}} + t_1^{3^{m-1}}$ for some $2 \leqslant i_1 < i_2 < \ldots < i_{L_{m-1}-1} \leqslant L_m$, by induction we have $p < 3n$.

(ii) $p|t_{j_s}^{2\cdot 3^{m-1}} - t_{j_s}^{3^{m-1}}t_1^{3^{m-1}} + t_1^{2\cdot 3^{m-1}}$ for some $2 \leqslant j_1 < j_2 < \ldots < j_{2L_{m-1}-1} \leqslant L_m$, that is

$$\begin{aligned} &p\,|\left(t_{j_s}^{2\cdot 3^{m-1}} - t_{j_s}^{3^{m-1}}t_1^{3^{m-1}} + t_1^{2\cdot 3^{m-1}}\right) - \left(t_{j_1}^{2\cdot 3^{m-1}} - t_{j_1}^{3^{m-1}}t_1^{3^{m-1}} + t_1^{2\cdot 3^{m-1}}\right) \\ &= \left(t_{j_s}^{3^{m-1}} - t_{j_1}^{3^{m-1}}\right)\left(t_{j_s}^{3^{m-1}} + t_{j_1}^{3^{m-1}} - t_1^{3^{m-1}}\right), \qquad 2 \leqslant s \leqslant 2L_{m-1} - 1. \end{aligned}$$

If the number of s satisfy $p|t_{j_s}^{3^{m-1}} - t_{j_1}^{3^{m-1}}$ is at least $L_{m-1} - 1$, then $p < 3n$ by the conclusion of Case 3.1.1. Otherwise the number of s satisfy $p|t_{j_s}^{3^{m-1}} + t_{j_1}^{3^{m-1}} - t_1^{3^{m-1}}$ is not less than L_{m-1}, subtract by pairs and using the conclusion of Case 3.1.1, we also get $p < 3n$.

Case 3.2: Similar to Case 2.2, together with Case 3.1 and induction on l, we get $p < 3n$ in Case 3.

From the discussion of Cases 1, 2, 3, we obtained $p < C_1 n$ for some positive constant $C_1 = C_1(f)$. It is easy to see that $f(x)$ can not decompose into linear factors with integral coefficients, so by Lemma 2.1 we obtain $n < C_f$. ■

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Address: Zhongfeng Zhang: School of Mathematics and Information Science, Zhaoqing University, Zhaoqing 526061, P. R. China.

E-mail: zh12zh31f@yahoo.com.cn

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