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SPACES OF ANALYTIC FUNCTIONS ON ESSENTIALLY PLURIPOLAR COMPACTA

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Abstract: Let$A(K)$ be the locally convex space of all analytic germs on a compact subset K of a Stein manifold Ω, $\dim \Omega = n$, endowed with the standard inductive topogy, let 0^n denote the origin of $\mathbb{C}^n$, The main result is the characterisation of the isomorphism $A(K) \simeq A(\{0^n\})$ in terms of pluripotential theory. It is based on the general result of Aytuna-Krone-Terzioğlu on the characterisation of power series spaces of infinite type in terms of interpolational invariants (DN) and (Ω).

Keywords: complete pluripolarity, spaces of analytic functions, interpolation invariants.

1. Preliminaries

Let X be a Fréchet space, $\{U_p,\ p \in \mathbb{N}\}$ a base of absolutely convex neighborhoods of the origin in X and $\left\{|x|_p\right\}$ the corresponding system of seminorms in X; the family

$$U_p^\circ := \{x' \in X^* : |x'(x)| \leqslant 1\}, \qquad p \in \mathbb{N}, \tag{1}$$

is a basis of the bornology of X^*, that is every bounded set M in X^* is contained in some U_p°; we consider also the corresponding system of non-bounded norms (shortly, *conorms*) on X^*:

$$|x'|_p^* := \sup\{|x'(x)| : x \in U_p\}, \qquad x' \in X^*.$$

Following interpolation invariants turn to be an important tool in theory of locally convex spaces, especially, in the structure theory of power series spaces ([6]).

Definition 1. A Fréchet space X has property (DN) if there is p so that for every q there is r and a constant C such that

$$|x|_q^2 \leqslant C\,|x|_p\,|x|_r\,, \qquad x \in X.$$

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Definition 2. A Fréchet space X has property (Ω) if for every p there is q such that for every r there is $0 < \delta < 1$ and $C > 0$ such that

$$|x'|_q^* \leqslant C \left(|x'|_p^*\right)^{1-\delta} \left(|x'|_r^*\right)^{\delta}, \qquad x' \in X.$$

Given a non-decreasing sequence of positive numbers $\alpha = (\alpha_k)$ leading to ∞, we consider the Fréchet space

$$\Lambda_\infty(\alpha) = \{x = (x_k)\} : |x|_p := \sum |x_k| \exp(p a_k) < \infty, \qquad p \in \mathbb{N},$$

endowed with the system of norms $\left\{|x|_p\right\}$; it is called the *power series space of infinite type with the exponent sequence* α.

Definition 3. For a Fréchet space X satisfying both properties (DN) and (Ω) take p as in Definition 1 and q as in Definition 2 for a chosen p. The sequence

$$\alpha_k := -\ln d_k(U_q, U_p), \tag{2}$$

where d_k means the k-th Kolmogorov diameter (see e.g., [5]), is called an associated exponent sequence of X.

The following important result is due to Aytuna-Krone-Terzioğlu [1].

Theorem 4. *Let X be a nuclear Frechet space satisfying properties (DN) and (Ω) and its associated exponent sequence $\alpha = (\alpha_k)$ is such that*

$$\limsup_{k\to\infty} \frac{\alpha_{2k}}{\alpha_k} < \infty. \tag{3}$$

Then $X \simeq \Lambda_\infty(\alpha)$.

If D is an open set on a Stein manifold Ω, then $A(D)$ denotes the Fréchet space of analytic functions on D with topology of locally uniform convergence on D. If K is a compact subset of Ω, then $A(K)$ denotes the space of all analytic germs on K with the inductive topology: $A(K) = \operatorname{limind}_{p\to\infty} A(D_p)$, where D_p is any sequence of open sets such that $D_{p+1} \subset D_p$ and $\cap D_p = K$.

As an application of the theorem 4, Aytuna-Krone-Terzioğlu obtained in [1] the following result, solving a long-standing problem on isomorphisms of spaces of analytic functions.

Theorem 5. *Let Ω be a Stein manifold. The following statements are equivalent:*

(i) *$A(\Omega) \simeq A(\mathbb{C}^n)$,*
(ii) *$A(\Omega) \in (DN)$,*
(iii) *Ω satisfies the Liouville principle for plurisubharmonic functions, i.e. a bounded plurisubharmonic function on Ω must be an identical constant.*

It is worth to notice that the equivalence (ii) $\iff$ (iii) was stated in [11] and proved in [12] (not published).

Our aim here is, applying Theorem 4, to characterize the isomorphism

$$A(K) \simeq A(\{0^n\}), \tag{4}$$

where K is a compact set on a Stein manifold Ω, $\dim \Omega = n$, and 0^n is the origin of $\mathbb{C}^n$.

It was proved in [9] that the isomorphism (4), for $K \subset \overline{\mathbb{C}}$, $K \neq \overline{\mathbb{C}}$, is equivalent to the polarity of K; on the other hand, it was proved there that the isomorphism $A(D) \simeq A(\mathbb{C})$ for $D \subset \overline{\mathbb{C}}$, $D \neq \overline{\mathbb{C}}$, is equivalent to the polarity of the compact set $\overline{\mathbb{C}} \setminus D$, what is the same that D has the Liouville principle for subharmonic functions. Due to the Grothendieck-Köthe-Silva duality, these two cases derive one from the other. On the contrary, because of the lack of a proper duality in multidimensional case, a general characterization of (4) cannot be derived from Theorem 5 and requires special consideration. Our main result on the characterization of the isomorphism (4) will be proved in Section 4 after some preparations: we consider in Section 2 a pluripotential counterpart of (4) - the essential pluripolarity, in whose terms the relation $A(K)^* \in (DN)$ is characterized in Section 3.

2. Complete and essential pluripolarity

Given a compact set K on a Stein manifold Ω and its open neighborhood D, consider the extremal functions

$$\omega^\circ(D, K; z) := \sup\{u(z) : u \in P(D, K)\},$$
$$\omega(D, K; z) := \limsup_{\zeta \to z} \omega^\circ(D, K; \zeta),$$

where $P(D, K)$ is the set of all $u \in Psh(D)$ such that $u|_K \leqslant 0$ and $u < 1$ in D.

An open set D on a Stein manifold Ω is called *strictly pluriregular,* if there is an open set $G \Supset \overline{D}$ and a continuous function $u \in Psh(G)$ such that $D = \{z \in G : u(z) < 0\}$. A compact set $K \subset \Omega$ is called *pluriregular* if $\omega(D, K; z) \equiv 0$ on K for any neighborhood $D \supset K$.

A set E on a Stein manifold Ω is called *complete pluripolar on* Ω if there exists a function $u \in Psh(\Omega)$ such that $E = \{z \in \Omega : u(z) = -\infty\}$. In the one-dimensional situation ($\Omega = \mathbb{C}$), the notions of polarity and complete polarity coincide for G_δ-sets. In several variables, this is no longer true even for compact sets: for instance, a closed disk in a one-dimensional plane $\Gamma \subset \mathbb{C}^2$ is pluripolar in $\mathbb{C}^2$ but not complete pluripolar in $\mathbb{C}^2$.

In connection with studying isomorphisms of spaces of analytic functions, a somewhat more complicated notion is needed, which we consider only for compact sets.

Definition 6. A compact set K on a Stein manifold Ω is called *essentially pluripolar* if there exists its open Runge neighborhood $D \subset \Omega$ such that $\hat{K}_{\tilde{D}}$ is complete pluripolar on the holomorphic envelope $\tilde{D}$ (which can be realized as a Riemann surface over Ω).

Let E be logarithmically polar set in $\mathbb{R}^{2n} = \mathbb{C}^n$, that is there exists a Borel non-negative measure μ supported by $\overline{E}$ such that the *logarithmic potential*

$$U^{\mu}(z) := \int \ln|\zeta - z| \; d\mu_{\zeta} \tag{5}$$

is equal to $-\infty$ on E and only on E. Then E is completely pluripolar, because the function $U^{\mu}(z)$ is plurisubharmonic on $\mathbb{C}^n$ and such that E consists of all its poles.

If a compact set $K \subset \Omega$ is complete pluripolar then, obviously, $K = \widehat{K}_{\Omega}$, so K is essentially pluripolar.

Given a pluripolar set $E \subset \Omega$, one can consider its *pluripolar hull on an open neighborhood* $D \subset \Omega$:

$$E_D^- = \{z \in D : u(z) = -\infty, \; \forall u \in \Pi_D(E)\},$$

where $\Pi_D(E)$ is the collection of all functions $u \in Psh(D)$ bounded on D and equal $-\infty$ on E.

Lemma 7 ([4]). *Let K be a compact subset of a strictly pluriregular domain D on a Stein manifold Ω. Then*

$$K_D^- = \{z \in D : \omega^0(D, K; z) < 1\}. \tag{6}$$

Proposition 8. *Let K be a Runge compact set on a Stein manifold Ω (i.e., $A(\Omega)$ is dense in the space $A(K)$). Then the following statements are equivalent:*

(i) $K_{\Omega}^- = \hat{K}_{\Omega}$;
(ii) *there exists a function $\Psi \in Psh(\Omega) \cap C(\Omega \setminus \hat{K}_{\Omega})$ such that $\hat{K}_{\Omega} = \{z \in \Omega : \Psi(z) = -\infty\}$.*

Proof. We need only to prove (i) ⇒ (ii). Take a strongly pseudoconvex open neighborhood $G \supset \hat{K}_{\Omega}$. By Lemma 7, we have $\omega^0(G, \hat{K}_{\Omega}; z) \equiv 1$ on $\overline{G} \setminus \hat{K}_{\Omega}$. Take a sequence of pluriregular compact sets $K_{\nu} \Downarrow \widehat{K}_G$. Then $\omega(G, K_{\nu}; z)$ is a non-decreasing sequence of continuous functions converging to 1 on $\overline{G} \setminus \hat{K}_G$. By Dini's theorem, for every $m \in \mathbb{N}$ we have

$$\alpha_{\nu}^{(m)} := \sup\{1 - \omega(G, K_{\nu}; z) : z \in \overline{G} \setminus \, K_m^-\} \to 0$$

as $\nu \to \infty$. Choose a sequence of positive numbers γ_{ν} so that

$$\sum_{\nu=1}^{\infty} \gamma_{\nu} = +\infty, \qquad \sum_{\nu=1}^{\infty} \gamma_{\nu} \; \alpha_{\nu}^{(m)} < \infty, \quad m \in \mathbb{N}.$$

Then the function $\varphi(z) = \sum_{\nu=1}^{\infty} \gamma_\nu(\omega(G, K_\nu; z) - 1)$ is continuous on $G \setminus \hat{K}_G$, plurisubharmonic on G and $\hat{K}_G = \{z \in G : \varphi(z) = -\infty\}$. Since Ω is a Runge neighborhood of K, we have $\hat{K}_\Omega = \hat{K}_G$. A function $\Psi \in Psh(\Omega) \cap C(\Omega \setminus \hat{K}_G)$, coinciding with φ on a neighborhood of the set K, can be constructed now with the help of Theorem 5.1.6 [2]. ■

Here we present a certain class of complete pluripolar compact sets.

Definition 9 (cf. [7]). A function f defined on a compact set $E \subset \mathbb{C}^n$ is called *quasi-entire (in a sense of S. N. Bernstein)* if there exists a sequence of polynomials $P_k(z)$ of degree s_k such that

$$\lim_{k\to\infty} \frac{\ln |f(z) - P_k(z)|_E}{s_k} = -\infty; \tag{7}$$

it is called *quasi-entire strictly on E* if for every $z_0 \in \mathbb{C}^n \setminus E$ there exists a sequence of polynomials $P_k(z)$ of degree s_k satisfying (7) and such that the sequence $\{P_k(z_0)\}$ does not converge, that is either has at least two limit points in $\mathbb{C}$ or $\limsup_{k\to\infty} |P_k(z_0)| = +\infty$.

Example 10. A lacunary power series

$$\sum_{m=1}^{\infty} \xi_m z^{s(m)}, \qquad \frac{s(m+1)}{s(m)} \to \infty,$$

satisfying the conditions

$$\lim_{m\to\infty} \frac{\ln |\xi_m|}{s(m)} = 0, \qquad \lim_{m\to\infty} \frac{\ln |\xi_{m+1}|}{s(m)} = -\infty,$$

defines a C^∞-function $f(z)$ quasi-entire strictly on the closed disk $E = \{|z| \leqslant 1\}$.

Proposition 11. *Let f be a function quasi-entire strictly on a pluriregular compact set $E \subset \mathbb{C}^n$. Then its graph $K = \{(z, f(z)) : z \in E\}$ is complete pluripolar in $\mathbb{C}^{n+1}$.*

Proof. Given a point $a = (z_0, w_0) \in \mathbb{C}^{n+1} \setminus K$, we look for a function $u \in Psh(\mathbb{C}^{n+1})$ such that $u(z, w) = -\infty$ on K, $u(z_0, w_0) \neq -\infty$. First suppose that $z_0 \notin E$. Take a sequence of polynomials $\{P_k\}$ existed for z_0 by Definition 9 and choose a subsequence (k_ν) so that

$$|w_0 - P_{k_\nu}(z_0) \geqslant \delta \tag{8}$$

for some $\delta > 0$ and all ν. Since $P_k(z)$ converges uniformly on E we have $|P_k|_E \leqslant M$ with some constant M. Hence, by the Bernstein-Siciak Lemma

$$\ln |P_k(z)| \leqslant \ln M + s_k\, g_E(z), \qquad z \in \mathbb{C}^n, \tag{9}$$

where g_E is the pluricomplex Green function. Then a desired function can be constructed as a series (cf. [8])

$$u(z,w)=\sum_{\nu=1}^{\infty}\gamma_v\left[\frac{\ln|w-P_{k_\nu}(z)|}{s_{k_\nu}}-C_\nu\right], \tag{10}$$

where the sequences of positive numbers (C_ν) and (γ_ν) are chosen so that

$$C_\nu\to\infty, \qquad \sum_{\nu=1}^{\infty}\gamma_\nu C_\nu<\infty, \qquad \sum_{\nu=1}^{\infty}\frac{\gamma_\nu\ln|f-P_{k_\nu}|_E}{s_{k_\nu}}=-\infty. \tag{11}$$

Indeed, the series converges at each point (z,w), in view of (7), (11); its sum is plurisubharmonic, since on each ball $\mathbb{B}_r$ all but finite number summands are non-positive, due to the estimate (9); by the construction $u\equiv-\infty$ on K and, taking into account (8), $u(z_0,w_0)\neq-\infty$.

Consider now the case $z_0\in E$. Since $(z_0,w_0)\notin K$, we have that $|w_0-f(z_0)|\geqslant 2\delta$ for some $\delta>0$. Take a sequence of polynomials $\{P_k\}$ satisfying (7). Then there exists $m\in\mathbb{N}$ such that $|f(z_0)-P_k(z_0)|<\delta$ for $k>m$. Therefore the condition (8) holds for the subsequence $k_\nu=m+\nu$. Now the function u is constructed as in (7), (11). ∎

Example 12. Let

$$K=\left\{(z,f(z))\in\mathbb{C}^2:|z|\leqslant 1\right\}$$

where f is the function from Example 10. Then K is a complete (hence, essentially) pluripolar compact set in $\mathbb{C}^2$, which

a) is not logarithmically polar,
b) has proper compact subsets that are not complete pluripolar.

The statement b) is contained in the following proposition.

Proposition 13. *Let L be a proper compact subset of the compact set K described above. In order that the set L be essentially pluripolar in $\mathbb{C}^2$, it is necessary and sufficient that its projection $M=\{z\in\mathbb{C}:(z,w)\in L\}$ to the plane $\mathbb{C}_z$ be polar.*

Proof. If the projection M is polar in $\mathbb{C}_z$, then $M=\{z\in\mathbb{C}:U^\mu(z)=-\infty\}$ for some measure μ supported by M. Carrying the measure μ over L by means of the map $g(\zeta)=(\zeta,f(\zeta))$, we get a measure ν in $\mathbb{C}^2$ supported by L and such that $U^\nu(z,w)=-\infty$, $(z,w)\in L$, that is, L is a logarithmically polar compact set in $\mathbb{C}^2$. Thus L is complete (hence essentially) pluripolar in $\mathbb{C}^2$.

Now assume L to be essentially pluripolar in $\mathbb{C}^2$. Then it has a Runge neighborhood D that can be chosen as

$$D=\{(z,w):z\in\Delta,\ |w-P(z)|<\varepsilon\},$$

where $P(z)$ is a polynomial sufficiently close (uniformly on $E=\{|z|\leqslant 1\}$) to the function $f(z)$, and Δ is an open neighborhood of the compact set M in $\mathbb{C}$ with the boundary consisting of finitely many smooth Jordan curves. By Proposition 8, there exists a function $\psi \in Psh(D) \cap C(D \setminus \widehat{L}_D)$ such that

$$\hat{L}_D = \{(z,w) \in D : \psi(z,w) = -\infty\}.$$

Then the function $\varphi(z) = \psi(z, f(z))$, $z \in G = \Delta \cap E^\circ$, is subharmonic in G and such that:

a) $\varphi(z) \equiv -\infty$ on the set $M \cap G$;
b) $\lim_{\zeta \to z} \varphi(\zeta) = -\infty$ if $z \in M \cap \partial G$;
c) $\varphi(z) \not\equiv -\infty$ on any connected component of the set G. By results from Potential Theory, this implies polarity of M in $\mathbb{C}$. ∎

In the definition of the essential pluripolarity of a compact K, one cannot drop out the condition on D to be a Runge neighborhood for K. We illustrate this by the next example.

Example 14. Consider a compact set $K = \{(z, f(z)) : |z| = 1\}$, where f is the function from Example 10. If we take its open pseudoconvex Runge neighborhood $D = \{1/2 < |z| < 2\} \times \mathbb{C}$, then $K = \widehat{K}_D$. So, by Proposition 13, the set K is not essentially pluripolar. But if D is a sufficiently large polydisk (which is not a Runge neighborhood for K), then $\hat{K}_D$ is complete pluripolar in D.

Problem 15. May it happen that K is essentially pluripolar, but $K \neq \widehat{K}_{\widetilde{D}}$?

3. Characterization of $A(K)^* \in (DN)$

In what follows $\hookrightarrow$ means a dense linear continuous embedding of locally convex spaces. Let $H_1 \hookrightarrow H_0$ be a couple of Hilbert spaces with compact embedding, then there is a doubly orthogonal basis $\{e_k\} \subset H_1$ such that

$$\|e_k\|_{H_0} = 1, \qquad \|e_k\|_{H_1} =: \mu_k(H_0, H_1) = \mu_k \nearrow \infty.$$

The Hilbert scale, generated by the couple H_0, H_1, is the one-parameter family $H_\alpha = (H_0)^{1-\alpha} (H_1)^\alpha$, defined by the scalar products

$$(x,y)_{H_\alpha} := \sum_{k \in \mathbb{N}} \mu_k^{2\alpha} (x, e_k)_{H_0} \overline{(y, e_k)_{H_0}}, \qquad \alpha \in \mathbb{R}.$$

The equality for Kolmogorov diameters takes place (see, e.g., [5], Corollary 3):

$$d_k\left(\|\mathbb{B}\|_{H_1}, \|\mathbb{B}\|_{H_0}\right) = \mu_{k+1}^{-1}.$$

Proposition 16 ([10]). *Let K be a pluriregular compact set on a Stein manifold Ω and $D \Subset \Omega$ its Runge neighborhood, which is strictly pluriregular and has no components disjoint with K. Let H_0, H_1 be Hilbert spaces such that*

$$A(\overline{D}) \hookrightarrow H_1 \hookrightarrow A(D) \hookrightarrow A(K) \hookrightarrow H_0 \hookrightarrow AC(K),$$

where $AC(K)$ is a completion of of $A(K)$ by the norm of $C(K)$. Then the Hilbert scale H_α, generated by the couple $H_0,\ H_1$, complies with the following embeddings:

$$A(K_\alpha) \hookrightarrow H_\alpha \hookrightarrow A(D_\alpha), \qquad 0 < \alpha < 1,$$

where $D_\alpha = \{z \in D : \omega(D, K; z) < \alpha\}$ and $K_\alpha = \{z \in D : \omega(D, K; z) \leqslant \alpha\}$. Therewith

$$\mu_k \asymp k^{1/n}, \qquad k \to \infty \tag{12}$$

Remark 17. The weak equivalence (12) goes back to Kolmogorov [3] (in terms of asymptotic behavior of ε-entropy); for a direct proof see [10]. Notice that, novadays, the exact asymptotics $\mu_k \sim \tau k^{1/n}$ is known ([13]), but we use here the weaker result (12).

Lemma 18. *Let K be a compact set on a Stein manifold Ω. The space $A(K)^*$ satisfies the property (DN) if and only if K is essentially pluripolar.*

Proof. Suppose that K is essentially pluripolar. Then there is its Runge neighborhood $G \Subset \Omega$ such that $K_{\widetilde{G}}^- = \widehat{K}_{\widetilde{G}}$, where $\widetilde{G}$ is the envelope of holomorphy of G. There is a strictly pluriregular open set $D : \widehat{K}_{\widetilde{G}} \subset D \Subset \widetilde{G}$ so that $K_D^- = \widehat{K}_D$. One can write $A(K) = A\left(\widehat{K}_D\right)$, identifying analytic germs on $\widehat{K}_D$ with their counterparts on K. Thus we assume hereafter, without loss of generality, that $K = K_D^- = \widehat{K}_D$. Then, by Lemma 7,

$$\omega^\circ(D, K; z) \equiv 1, \qquad z \in D \setminus K. \tag{13}$$

Take a sequence of pluriregular compact sets $K_q \Downarrow K$. Then, by [10, 8],

$$\omega(D, K_r; z) \uparrow \omega^\circ(D, K; z), \qquad z \in D.$$

The functions $\omega(D, K_r; z)$ are continuous on D and extendable continuously onto $\overline{D}$. Therefore, due to (13) and Dini's Theorem, the sequence $\omega(D, K_r; z)$ converges uniformly on each $D \setminus \operatorname{int} K_q,\ q \in \mathbb{N}$ to the identical unity. Hence

$$\forall q \exists r \mid \omega(D, K_r; z) > \frac{1}{2}, \qquad z \in D \setminus \operatorname{int} K_q. \tag{14}$$

Take Hilbert spaces H_0 and H_q, complying with the following linear continuous embeddings:

$$A\left(\overline{D}\right) \hookrightarrow H_0 \hookrightarrow A(D); \qquad A(K_q) \hookrightarrow H_q \hookrightarrow AC(K_q), \quad q \in \mathbb{N}. \tag{15}$$

Then

$$A(K) = \operatorname*{limind}_q H_q, \qquad A(K)^* = \operatorname*{limproj}_q H_q^*.$$

Let $K_r^\alpha := \{z \in D : \omega(D, K_r; z) \leqslant \alpha\}$ and $H_r^\alpha = (H_r)^{1-\alpha}(H_0)^\alpha$ be the Hilbert scale spanned on the spaces H_r and H_0. Applying Proposition 16 for $\alpha = \frac{1}{2}$, we get the embeddings:

$$H_r^{1/2} \hookleftarrow A\left(K_r^{1/2}\right), \qquad r \in \mathbb{N}. \tag{16}$$

The relation (14) can be written in the form

$$\forall q \exists r = r(q) \mid K_r^{1/2} \subset \operatorname{int} K_q,$$

therefore, taking into account (15) and (16), we have:

$$H_r^{1/2} \hookleftarrow A\left(K_r^{1/2}\right) \hookleftarrow A(\operatorname{int} K_q) \hookleftarrow H_q, \qquad r = r(q). \tag{17}$$

Let $G_r^\alpha = (H_r^\alpha)^* = (H_0^*)^\alpha (H_r^*)^{1-\alpha}$ be the dual Hilbert scale. Then (17) implies that there is a constant C such that

$$\|x'\|_{H_q^*} \leqslant C \|x'\|_{G_r^{1/2}}, \qquad x' \in A(K)^*.$$

Applying the multiplicative property for the Hilbert scale G_r^α, we get finally

$$\forall q \exists r \exists C \mid \|x'\|_{H_q^*} \leqslant C \left(\|x'\|_{H_0^*}\right)^{1/2} \cdot \left(\|x'\|_{H_r^*}\right)^{1/2},$$

which means that $A(K)^* \in (DN)$.

Suppose now that $A(K)^* \in (DN)$. First we show that K has a Runge neighborhood. Assuming the contrary, there is a basis $\{G_p\}_{p \in \mathbb{N}}$ of open neighborhoods of K such that $G_{p+1} \Subset G_p$ and such that the set $A(G_p)$ is not dense in the space $A(K)$ for each $p \in \mathbb{N}$. Take a sequence of Hilbert spaces H_p so that

$$A\left(\overline{G}_{p+1}\right) \hookrightarrow H_p \hookrightarrow A(G_p), \qquad p \in \mathbb{N}. \tag{18}$$

Since the closure of H_p in $A(K)$ is a proper subspace of $A(K)$ for every $p \in \mathbb{N}$, there is, by the Hahn-Banach Theorem, a non-trivial functional $x_p' \in A(K)^*$ vanishing identically on H_p, hence $\left\|x_p'\right\|_{H_p^*} = 0$. The space $A(K)^* = \operatorname{limproj}_q H_q^*$ is a Hausdorff space, hence there exists $q = q(p)$ such that $\left\|x_p'\right\|_{H_q^*} > 0$. Thus

$$\forall p \exists q \forall r \forall C \mid \left(\left\|x_p'\right\|_{H_q^*}\right)^2 > 0 = C \left\|x_p'\right\|_{H_p^*} \cdot \left\|x_p'\right\|_{H_r^*},$$

which contradicts to the assumption $A(K)^* \in (DN)$. So it is proved that K has a Runge neighborhood $G \Subset \Omega$.

We prove now, that $A(K)^* \in (DN)$ implies that K is essentially pluripolar. Let G be a Runge neighborhood of K and $\widetilde{G}$ its envelope of holomorphy. Choose a sequence of strictly pluriregular open sets $\{D_q\}$, holomorphically convex with respect to $\widetilde{G}$ and such that $D_{p+1} \Subset D_p$, $\cap D_p = \widehat{K}_{\widetilde{G}}$. Let H_q be Hilbert spaces which comply with linear continuous dense embeddings

$$A\left(\overline{D_q}\right) \hookrightarrow H_q \hookrightarrow AC\left(\overline{D_q}\right), \qquad q \in \mathbb{N} \tag{19}$$

It follows from $A(K)^* \in (DN)$ that there is p so that for every q and $0 < \delta < 1$ there is $r = r(q, \delta)$ and a constant C such that

$$\|x'\|_{H_q^*} \leqslant C \left(\|x'\|_{H_p^*}\right)^{1-\delta} \cdot \left(\|x'\|_{H_r^*}\right)^{\delta}, \qquad x' \in A(K)^*. \tag{20}$$

Since $A(K)^* \simeq A\left(\widehat{K}_{D_p}\right)^*$ and (DN) is an invariant, we can assume that $\widehat{K}_{D_p} = K$. By Proposition 8, it suffices to prove that K is pluripolar and $K^-_{D_p} = K$. Suppose the contrary, that either (a) *K is pluripolar but $K^-_{D_p} \smallsetminus K \neq \varnothing$,* or (b) *$K$ is not pluripolar.* In the case (a), due to Lemma 7, there exists $\delta : 0 < \delta < \frac{1}{4}$ such that the set

$$L_\delta := \{z \in D_p : \omega^\circ (D_p, K; z) < 1 - 4\delta\}$$

has non-empty intersection with $D_p \smallsetminus K$; on the other hand, $L_\delta \smallsetminus K \neq \varnothing$ is evident in the case (b).

Thus, it is sufficient to show that the condition $L_\delta \smallsetminus K \neq \varnothing$ with some $\delta > 0$ leads to a contradiction. Fix q so that $L_\delta \smallsetminus K_q \neq \varnothing$ then choose a pseudoconvex open set $V : \overline{D_q} \subset V \subset D_p$ so that

$$L_\delta \smallsetminus V \neq \varnothing.$$

By pseudoconvexity of V there exists a function $f \in A(V)$, such that

$$\operatorname{spec} f = V. \tag{21}$$

Let $\{e_k\}_{k\in\mathbb{N}}$ be a doubly orthogonal basis generated by the pair $(H_p,\ H_r)$ and such that

$$\|e_k\|_{H_r} = 1, \qquad \|e_k\|_{H_p} =: \mu_k \nearrow \infty.$$

Accordingly, $\|e'_k\|_{H_r^*} = 1,\ \|e'_k\|_{H_p^*} = \mu_k^{-1}$ for the biorthogonal system $\{e'_k\}$. Therefore, by (20),

$$\|e'_k\|_{H_r^*} \leqslant C\ \mu_k^{\delta-1}. \tag{22}$$

It follows from $\omega\left(D_p, \overline{D_r}; z\right) \leqslant \omega^\circ (D_p, K; z)$ that

$$L_\delta \subset \Phi_\delta := \{z \in D_p : \omega(D_p, K; z) < 1 - 3\delta\}$$

Hence, by Two Constant Theorem,

$$|e_k|_{\Phi_\delta} \leqslant C_1\ \mu_k^{1-2\delta}. \tag{23}$$

As $f \in A(V) \subset A\left(\overline{D_q}\right) \subset H_q$, we consider the expansion of f in the space H_q:

$$f = \sum_{k\in\mathbb{N}} e'_k(f)\ e_k. \tag{24}$$

Due to (22), (23), we have the estimate

$$|e'_k(f)|\,|e_k|_{\Phi_\delta} \leqslant C_1 \|e'_k\|_{H_q^*} \|f\|_{H_q} \mu_k^{1-2\delta} \leqslant CC_1 \|f\|_{H_q} \mu_k^{-\delta}.$$

Since, by Proposition 16, $\ln \mu_k \asymp k^{1/n}$ as $k \to \infty$, this estimate implies that the series (24) converges uniformly on Φ_δ to a function $g \in A(\Phi_\delta)$. Taking into account that, by the construction, L_δ has a non-empty intersection with V, we obtain the contradiction to (21). ∎

4. Isomorphism $A(K) \simeq A(\{0^n\})$

Lemma 19. *Let K be a Runge compact set on a Stein manifold Ω. Then the Fréchet space $A(K)^*$ has the property (Ω).*

Proof. Since $A(K) \simeq A\left(\widehat{K}_\Omega\right)$ and the property (Ω) is invariant, we may assume that $\widehat{K}_\Omega = K$. Take a sequence of strictly pluriregular open Runge neighborhoods $\{D_p\}$ of K, so that

$$D_{p+1} \Subset D_p, \qquad \bigcap D_p = K,$$

and every D_p has no components disjoint with K. Let $X = A(K)^*$. Since $A(K)$ is reflexive, we can identify $X^* = A(K)^{**}$ with $A(K)$, by the canonical embedding. Then the norms

$$|x|_p := \sup\{|x(z)| : z \in D_p\}$$

are conorms generating the topology in X^*. Given an arbitrary p take $q = p + 1$. For any r define the number

$$\delta = \delta(p, r) := \sup\left\{\omega\left(D_p, \overline{D_r}, z\right) : z \in D_q\right\}.$$

Then $0 < \delta < 1$ and, by the Two Constants Theorem, we have

$$|x|_q \leqslant \left(|x|_p\right)^{1-\delta} (|x|_r)^\delta, \qquad x \in A(K) = X^*,$$

that means $A(K)^* \in (\Omega)$. ■

Theorem 20. *Let K be a compact set on a Stein manifold Ω, $\dim \Omega = n$. Then the following statements are equivalent:*

(i) *$A(K)^* \simeq A(\mathbb{C}^n)$;*
(ii) *$A(K) \simeq A(\{0^n\})$;*
(iii) *$A(K)^* \in (DN)$;*
(iv) *K is essentially pluripolar on Ω.*

Proof. The equivalence (iii) $\Longleftrightarrow$ (iv) is proved in Lemma 18. The relations (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are evident. It remains to prove (iii) $\Longrightarrow$ (i). By Lemma 18, K is a Runge compactum. Hence, by Lemma 19, $A(K)^* \in (\Omega)$. It follows from (12) that the associated exponent sequence (α_k) of $A(K)^*$ is determined by the weak equivalence: $\alpha_k \asymp k^{1/n}$. All conditions of the Theorem 4, including (3), are fulfilled. Therefore $A(K)^* \simeq \Lambda_\infty\left(\left(k^{1/n}\right)\right) \simeq A(\mathbb{C}^n)$. ■

Corollary 21. *Let K be as in Proposition 11, then $A(K) \simeq A\left(\left\{0^{n+1}\right\}\right)$. In particular, $A(K) \simeq A\left(\left\{0^2\right\}\right)$ for K from Example 12.*

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