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THE MINIMAL NUMBER OF MONOCHROMATIC SCHUR TUPLES IN A CYCLIC GROUP

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Abstract: We discuss a question of Datskovsky [1] about the minimal number of solutions to Schur-type equation $x_1 + \ldots + x_{2n-1} = x_{2n}$ in a cyclic group $\mathbb{Z}_N$. We provide lower and upper bounds for this quantity.

Keywords: Schur k-tuples, Ramsey theory on integers.

1. Introduction

We call a Schur triple any solution (x, y, z) of the equation

$$x + y = z. \tag{1.1}$$

In 1916 Schur [6] proved that for any 2-coloring $\{1, \ldots, N\} = R \cup B$ there exists a monochromatic solution to (1.1) provided that N is large enough. This leads to a question about the minimal number of monochromatic Schur triples for 2-colorings of $\{1, \ldots, N\}$, which was first asked by Graham, Rödl and Ruciński in [2]. The answer was found to be $\frac{1}{11}N^2 + O(N)$ by Robertson and Zeilberger [4] and independently by Schoen [5]. Another proof of this fact was given later by Datskovsky [1], who also studied monochromatic Schur triples in a cyclic group $\mathbb{Z}_N$. He showed that their number for any 2-coloring $\mathbb{Z}_N = R \cup B$ depends only on the sizes of the color sets and their minimum number is equal to

$$\frac{1}{N}\left(\left\lceil \frac{N-1}{2} \right\rceil^3 + \left\lfloor \frac{N+1}{2} \right\rfloor^3\right).$$

He also remarked that we can obtain a similar result for the following generalization of Schur equation

$$x_1 + \ldots + x_{k-1} = x_k, \tag{1.2}$$

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when k is an odd number. Then, any 2-coloring $\mathbb{Z}_N = R \cup B$ contains

$$\frac{1}{N}(|R|^k + |B|^k) \tag{1.3}$$

monochromatic Schur k-tuples. Datskovsky asked what can be said about the number of these k-tuples if k is even.

Let $k = 2n$ be a positive integer and let $\mathbb{Z}_N = R \cup B$ be a partition. We will denote the number of monochromatic solutions to (1.2) modulo N by $S_{2n}(R, B)$ and its minimum by μ_{2n}, i.e.

$$S_{2n}(R,B) = |(x_1, x_2, \ldots, x_{2n}) \in R^{2n} \cup B^{2n} : x_1 + x_2 + \ldots + x_{2n-1} \equiv x_{2n} \pmod{N}|$$

and

$$\mu_{2n} = \min_{\mathbb{Z}_N = R \cup B} S_{2n}(R, B).$$

The aim of this paper is to give some estimates for μ_{2n}. We first show that this number can be in fact smaller than $\left(\frac{N}{2}\right)^{2n-1}$, which is what we can expect from a random 2-coloring or estimates (1.3) for odd k.

Theorem 1.1. *Let $k = 2n$, where $n \geqslant 2$. Then the minimal number of monochromatic solutions to* (1.2) *can be bounded from above as follows*

$$\mu_{2n} \leqslant \left(\frac{N}{2}\right)^{2n-1} \left(1 - 2\left(\frac{2}{\pi}\right)^{2n} - \frac{3}{(2n-1)}\left(\frac{2}{3\pi}\right)^{2n}\right) + o(N^{2n-1}).$$

We prove this theorem in Section 2 by giving some examples of appropriate colorings which depend on the parity of both N and n. In Section 3 we show that this result is in fact close to optimal if N is prime.

Theorem 1.2. *Let N be a prime number and $k = 2n$, where $n \geqslant 2$. Then*

$$\mu_{2n} \geqslant \left(\frac{N}{2}\right)^{2n-1} \left(1 - \left(\frac{2}{\pi}\right)^{2n-2}\right) + o(N^{2n-1}).$$

Remark. It could be also interesting to study the minimal number of monochromatic solutions to (1.2) for 2-colorings of $\{1, \ldots, N\}$ if $k > 3$, for which so far there are no estimates known.

2. Examples of colorings with small number of solutions

For a given set $A \subseteq \mathbb{Z}_N$ we will denote by $A(x)$ the characteristic function of this set. Then the Fourier coefficients of this function are defined as follows

$$\hat{A}(r) = \sum_{x=0}^{N-1} A(x) e\left(\frac{rx}{N}\right),$$

where $r \in \mathbb{Z}_N$ and $e(x) = e^{2\pi i x}$. We can use Fourier coefficients to find the number of the monochromatic solutions of (1.2) for a given partition $\mathbb{Z}_N = R \cup B$.

$$S_{2n}(R,B) = \frac{1}{N}\sum_{r\in\mathbb{Z}_N}(\hat{R}(r)^{2n-1}\hat{R}(-r)+\hat{B}(r)^{2n-1}\hat{B}(-r))$$
$$= \frac{1}{N}(\hat{R}(0)^{2n}+\hat{B}(0)^{2n}) + \frac{1}{N}\sum_{r\neq 0}(\hat{R}(r)^{2n-2}|\hat{R}(r)|^2+\hat{B}(r)^{2n-2}|\hat{B}(r)|^2).$$

However, $\hat{R}(0) = |R|, \hat{B}(0) = |B|$ and, clearly, for $r \neq 0$ $\hat{R}(r) = -\hat{B}(r)$. Therefore,

$$S_{2n}(R,B) = \frac{1}{N}(|R|^{2n}+|B|^{2n}) + \frac{2}{N}\sum_{r\neq 0}\hat{R}(r)^{2n-2}|\hat{R}(r)|^2. \tag{2.1}$$

2.1. N even

We will assume for now that both N and n are even numbers. We will consider the following 2-coloring of $\mathbb{Z}_N$

- red color set $R = \{0, 1, \ldots, \frac{N}{2}\}$,
- blue color set $B = \mathbb{Z}_N \setminus R$.

We will show that this coloring has small number of solutions $S_{2n}(R,B)$.

We begin with calculating the value of Fourier coefficients $\hat{R}(r)$ for $r \neq 0$.

$$\hat{R}(r) = \sum_{x=0}^{N-1} R(x) e\left(\frac{rx}{N}\right) = \sum_{x=0}^{\frac{N}{2}} e\left(\frac{rx}{N}\right) = \frac{e\left(\frac{r}{N}(\frac{N}{2}+1)\right)-1}{e\left(\frac{r}{N}\right)-1}$$
$$= \frac{e^{\pi i r}e^{\frac{2\pi i r}{N}}-1}{e^{\frac{2\pi i r}{N}}-1}.$$

For even $r \neq 0$ this number is equal to 1, while for odd r we have

$$\hat{R}(r) = -\frac{\cos\left(\frac{2\pi r}{N}\right)+i\sin\left(\frac{2\pi r}{N}\right)+1}{\cos\left(\frac{2\pi r}{N}\right)+i\sin\left(\frac{2\pi r}{N}\right)-1}$$
$$= -\frac{\left(\cos\left(\frac{2\pi r}{N}\right)+i\sin\left(\frac{2\pi r}{N}\right)+1\right)\left(\cos\left(\frac{2\pi r}{N}\right)-1-i\sin\left(\frac{2\pi r}{N}\right)\right)}{\left(\cos\left(\frac{2\pi r}{N}\right)+i\sin\left(\frac{2\pi r}{N}\right)-1\right)\left(\cos\left(\frac{2\pi r}{N}\right)-1-i\sin\left(\frac{2\pi r}{N}\right)\right)}$$
$$= -\frac{\cos^2\left(\frac{2\pi r}{N}\right)-1+\sin^2\left(\frac{2\pi r}{N}\right)-2i\sin\left(\frac{2\pi r}{N}\right)}{\cos^2\left(\frac{2\pi r}{N}\right)-2\cos\left(\frac{2\pi r}{N}\right)+1+\sin^2\left(\frac{2\pi r}{N}\right)} = \frac{2i\sin\left(\frac{2\pi r}{N}\right)}{2-2\cos\left(\frac{2\pi r}{N}\right)}$$
$$= i\frac{2\sin\left(\frac{\pi r}{N}\right)\cos\left(\frac{\pi r}{N}\right)}{2\sin^2\left(\frac{\pi r}{N}\right)} = i\cot\left(\frac{\pi r}{N}\right).$$

We would like to find an upper bound for the sum

$$\sum_{r\neq 0}\hat{R}(r)^{2n-2}|\hat{R}(r)|^2, \tag{2.2}$$

which we can split into two parts depending on the parity of r. We have

$$\sum_{r\text{ even},r\neq 0}\hat{R}(r)^{2n-2}|\hat{R}(r)|^2=\frac{N}{2}-1$$

and

$$\begin{aligned}\sum_{r\text{ odd}}\hat{R}(r)^{2n-2}|\hat{R}(r)|^2&=\sum_{r\text{ odd}}\left(i\cot\left(\frac{\pi r}{N}\right)\right)^{2n-2}\cot^2\left(\frac{\pi r}{N}\right)\\&=(-1)^{n-1}\sum_{r\text{ odd}}\cot^{2n}\left(\frac{\pi r}{N}\right).\end{aligned}\tag{2.3}$$

Since n is even, the above sum is negative. So we need to estimate the value of cotangent function from below. Let us first note that it is sufficient to consider only $r\leqslant\frac{N}{2}$ as

$$\cot\left(\frac{\pi r}{N}\right)=-\cot\left(\frac{\pi(N-r)}{N}\right).\tag{2.4}$$

Therefore,

$$\sum_{r\text{ odd}}\cot^{2n}\left(\frac{\pi r}{N}\right)=2\sum_{r\text{ odd},r\leqslant\frac{N}{2}}\cot^{2n}\left(\frac{\pi r}{N}\right).$$

In fact, we will mostly be interested in coefficients $r\leqslant\frac{N}{4}$ since for $r>\frac{N}{4}$ we have $0\leqslant|\cot(\frac{\pi r}{N})|\leqslant 1$, so it makes small contribution to the sum (2.3). For $r\leqslant\frac{N}{4}$ we have

$$\left|\cot\left(\frac{\pi r}{N}\right)\right|\geqslant\frac{N}{\pi r}-\frac{4-2\sqrt{2}}{\pi}$$

because in this range $|\sin(\frac{\pi r}{N})|\leqslant\frac{\pi r}{N}$ and $|\cos(\frac{\pi r}{N})|\geqslant 1-\frac{(4-2\sqrt{2})r}{N}$, which is due to concativity of this function. It follows that

$$\begin{aligned}\sum_{r\text{ odd},r\leqslant\frac{N}{2}}\cot^{2n}\left(\frac{\pi r}{N}\right)&\geqslant\sum_{r\text{ odd},r\leqslant\frac{N}{4}}\left(\frac{N}{\pi r}-\frac{4-2\sqrt{2}}{\pi}\right)^{2n}+\sum_{r\text{ odd},\frac{N}{4}<r\leqslant\frac{N}{2}}\cot^{2n}\left(\frac{\pi r}{N}\right)\\&\geqslant\sum_{r\text{ odd},r\leqslant\frac{N}{4}}\left(\frac{N}{\pi r}\right)^{2n}+O(N^{2n-1})\\&=\left(\frac{N}{\pi}\right)^{2n}\sum_{r\text{ odd},r\leqslant\frac{N}{4}}\frac{1}{r^{2n}}+O(N^{2n-1}).\end{aligned}\tag{2.5}$$

We can estimate the value of the sum over coefficients r using integral in the following way

$$\sum_{r \text{ odd}, r\leqslant \frac{N}{4}} \frac{1}{r^{2n}} \geqslant 1 + \sum_{k=1}^{\lfloor \frac{N}{8} \rfloor - 1} \frac{1}{(2k+1)^{2n}} \geqslant 1 + \int_1^{\lfloor \frac{N}{8} \rfloor} \frac{1}{(2k+1)^{2n}} dk$$
$$= 1 + \frac{1}{2}\frac{1}{(2n-1)}\frac{1}{3^{2n-1}} - \frac{1}{2}\frac{1}{(2n-1)}\frac{1}{(2\lfloor \frac{N}{8} \rfloor + 1)^{2n-1}}. \tag{2.6}$$

Finally,

$$\mu_{2n} \leqslant S_{2n}(R,B) = \frac{1}{N}\left(|R|^{2n} + |B|^{2n}\right)$$
$$+ \frac{2}{N}\left(\sum_{r\neq 0, r \text{ even}} \hat{R}(r)^{2n-2}|\hat{R}(r)|^2 + \sum_{r \text{ odd}} \hat{R}(r)^{2n-2}|\hat{R}(r)|^2\right)$$
$$= \frac{1}{N}\left(\left(\frac{N}{2}+1\right)^{2n} + \left(\frac{N}{2}-1\right)^{2n}\right)$$
$$+ \frac{2}{N}\left(\frac{N}{2} - 1 + 2(-1)^{n-1}\sum_{r \text{ odd}, r\leqslant \frac{N}{2}} \cot^{2n}\left(\frac{\pi r}{N}\right)\right)$$
$$\leqslant \frac{2}{N}\left(\frac{N}{2}\right)^{2n} - \frac{4}{N}\left(\frac{N}{\pi}\right)^{2n}\left(1 + \frac{1}{2}\frac{1}{(2n-1)}\frac{1}{3^{2n-1}}\right.$$
$$\left. - \frac{1}{2}\frac{1}{(2n-1)}\frac{1}{(2\lfloor \frac{N}{8} \rfloor + 1)^{2n-1}}\right) + O(N^{2n-2})$$
$$\leqslant \left(\frac{N}{2}\right)^{2n-1}\left(1 - 2\left(\frac{2}{\pi}\right)^{2n} - \frac{3}{(2n-1)}\left(\frac{2}{3\pi}\right)^{2n}\right) + O(N^{2n-2}).$$

We will now consider the case of odd n. It requires a slight modification of the previous coloring. We define

- red color set $R_1 = \{y, y+1, \ldots, y + \frac{N}{2}\}$,
- blue color set $B_1 = \mathbb{Z}_N \setminus R_1$,

where $y = \lfloor \frac{N}{4n-4} \rfloor$. Then, for any $r \neq 0$ we have

$$\hat{R_1}(r) = \sum_{x=0}^{\frac{N}{2}} e\left(\frac{r(y+x)}{N}\right) = e^{\frac{2\pi i r y}{N}} \sum_{x=0}^{\frac{N}{2}} e\left(\frac{rx}{N}\right) = e^{\frac{2\pi i r y}{N}} \hat{R}(r).$$

This allows us to follow our previous approach with some minor changes described below. Once again we will estimate the sum

$$\frac{1}{N}\left(|R_1|^{2n}+\sum_{r\neq 0}\hat{R}_1(r)^{2n-2}|\hat{R}_1(r)|^2\right)$$
$$=\frac{1}{N}\left(|R|^{2n}+\sum_{r\neq 0}e^{\frac{(4n-4)\pi iry}{N}}\hat{R}(r)^{2n-2}|\hat{R}(r)|^2\right). \quad (2.7)$$

We know that this sum expresses the number of red solutions of the equation (1.2) and so the imaginary parts must cancel out. In consequence, we can only study the real part of (2.7). We will also split it into two parts for even and odd r, respectively. We have

$$\operatorname{Re}\left(\sum_{r\text{ even},r\neq 0}e^{\frac{(4n-4)\pi iry}{N}}\hat{R}(r)^{2n-2}|\hat{R}(r)|^2\right)\leqslant\sum_{r\text{ even},r\neq 0}\left|e^{\frac{(4n-4)\pi iry}{N}}\right|=\frac{N}{2}-1$$

and

$$\operatorname{Re}\left(\sum_{r\text{ odd}}e^{\frac{(4n-4)\pi iry}{N}}\hat{R}(r)^{2n-2}|\hat{R}(r)|^2\right)$$
$$=\operatorname{Re}\left(\sum_{r\text{ odd}}e^{\frac{(4n-4)\pi iry}{N}}\left(i\cot\left(\frac{\pi r}{N}\right)\right)^{2n-2}\cot^2\left(\frac{\pi r}{N}\right)\right)$$
$$=\sum_{r\text{ odd}}\cos\left(\frac{(4n-4)\pi ry}{N}\right)\cot^{2n}\left(\frac{\pi r}{N}\right). \quad (2.8)$$

Let us note that

$$\cos\left(\frac{(4n-4)\pi(N-r)y}{N}\right)=\cos\left((4n-4)\pi y-\frac{(4n-4)\pi ry}{N}\right)$$
$$=\cos\left(-\frac{(4n-4)\pi ry}{N}\right)=\cos\left(\frac{(4n-4)\pi ry}{N}\right), \quad (2.9)$$

which combined with (2.4) allows us to only consider the sum over coefficients up to $\frac{N}{2}$. We will split it further into two parts.

We assume first that $\frac{N}{\log N}<r\leqslant\frac{N}{2}$. Using the inequality

$$\left|\cot\left(\frac{\pi r}{N}\right)\right|=\left|\frac{\cos\left(\frac{\pi r}{N}\right)}{\sin\left(\frac{\pi r}{N}\right)}\right|\leqslant\frac{1}{\frac{2}{\pi}\frac{\pi r}{N}}=\frac{N}{2r},$$

we get

$$\left|\sum_{r \text{ odd}, \frac{N}{\log N}<r\leqslant \frac{N}{2}} e^{\frac{(4n-4)\pi i r y}{N}} \cot^{2n}\left(\frac{\pi r}{N}\right)\right| \leqslant \left(\frac{N}{\pi}\right)^{2n} \sum_{r \text{ odd}, \frac{N}{\log N}<r\leqslant \frac{N}{2}} \frac{1}{r^{2n}}$$

$$\leqslant \left(\frac{N}{\pi}\right)^{2n} \int_{\frac{N}{\log N}-2}^{\frac{N}{2}} \frac{1}{r^{2n}} dr$$

$$= O(N \log^{2n-1} N).$$

Let now $r \leqslant \frac{N}{\log N}$ and $c = \frac{N}{(4n-4)} - y$. Then

$$\begin{aligned}\cos\left(\frac{(4n-4)\pi r y}{N}\right) &= \cos\left(\frac{(4n-4)\pi r \frac{N}{(4n-4)}}{N} - \frac{(4n-4)c\pi r}{N}\right)\\ &= \cos(\pi r)\cos\left(\frac{(4n-4)c\pi r}{N}\right) + \sin(\pi r)\sin\left(\frac{(4n-4)c\pi r}{N}\right)\\ &= -\cos\left(\frac{(4n-4)c\pi r}{N}\right). \qquad (2.10)\end{aligned}$$

Observe that the function $-\cos\left(\frac{(4n-4)c\pi r}{N}\right)$ is increasing in r in the given range provided N is large enough with respect to n. Hence, by Taylor expansion of cosine function we have

$$-\cos\left(\frac{(4n-4)c\pi r}{N}\right) \leqslant -\cos\left(\frac{(4n-4)c\pi \frac{N}{\log N}}{N}\right) \leqslant -1 + \frac{1}{2}\left(\frac{(4n-4)c\pi}{\log N}\right)^2. \qquad (2.11)$$

Therefore,

$$\sum_{r \text{ odd}, r\leqslant \frac{N}{\log N}} \cos\left(\frac{(4n-4)\pi r y}{N}\right) \cot^{2n}\left(\frac{\pi r}{N}\right)$$

$$\leqslant \left(-1 + \frac{1}{2}\left(\frac{(4n-4)c\pi}{\log N}\right)^2\right) \sum_{r \text{ odd}, r\leqslant \frac{N}{\log N}} \cot^{2n}\left(\frac{\pi r}{N}\right).$$

Let us remark that the choice of $\frac{N}{\log N}$ as a boundary value for r here is arbitrary. In fact, we could use any function that tends to infinity when $N \to \infty$ and is $o(N)$.

Finally, in view of the above estimates

$$\mu_{2n} \leqslant S_{2n}(R_1, B_1) \leqslant \left(\frac{N}{2}\right)^{2n-1} \left(1 - 2\left(\frac{2}{\pi}\right)^{2n} - \frac{3}{(2n-1)}\left(\frac{2}{3\pi}\right)^{2n}\right) + o(N^{2n-1}).$$

2.2. N odd

We will assume now that N is an odd number, while n is even. We will consider the following 2-coloring of $\mathbb{Z}_N$

- red color set $R = \{0, 1, \ldots, \frac{N-1}{2}\}$,
- blue color set $B = \mathbb{Z}_N \setminus R$.

We will show that this coloring has small number of solutions $S_{2n}(R, B)$. We will follow the approach from the previous subsection with some small changes due to the fact that Fourier coefficients $\hat{R}(r)$ for $r \neq 0$ here will have a slightly more complicated form.

$$\hat{R}(r) = \sum_{x=0}^{N-1} R(x) e\left(\frac{rx}{N}\right) = \sum_{x=0}^{\frac{N-1}{2}} e\left(\frac{rx}{N}\right) = \frac{e\left(\frac{r}{N}\frac{N+1}{2}\right) - 1}{e\left(\frac{r}{N}\right) - 1}$$
$$= \frac{e^{\pi i r} e^{\frac{\pi i r}{N}} - 1}{e^{\frac{2\pi i r}{N}} - 1} = \frac{(-1)^r e^{\frac{\pi i r}{N}} - 1}{\left(e^{\frac{\pi i r}{N}} - 1\right)\left(e^{\frac{\pi i r}{N}} + 1\right)}.$$

So, we have for even $r \neq 0$

$$\hat{R}(r) = \frac{1}{e^{\frac{\pi i r}{N}} + 1} = \frac{1}{2} - \frac{i}{2} \frac{\sin(\frac{\pi r}{N})}{1 + \cos(\frac{\pi r}{N})},$$

while for odd r

$$\hat{R}(r) = -\frac{1}{e^{\frac{\pi i r}{N}} - 1} = \frac{1}{2} + \frac{i}{2} \frac{\sin(\frac{\pi r}{N})}{1 - \cos(\frac{\pi r}{N})}.$$

Once again we want to estimate the sum (2.2), which we will split into two sums over even and odd r, respectively. Furthermore, we will only consider real parts of these sums since the imaginary parts must cancel out. Then for even $r \neq 0$ we have

$$\mathrm{Re}\left(\sum_{r \text{ even}, r\neq 0} \left(\frac{1}{2} - \frac{i}{2}\frac{\sin(\frac{\pi r}{N})}{1 + \cos(\frac{\pi r}{N})}\right)^{2n-2} \left|\frac{1}{2} - \frac{i}{2}\frac{\sin(\frac{\pi r}{N})}{1 + \cos(\frac{\pi r}{N})}\right|^2\right)$$
$$= \sum_{r \text{ even}, r\neq 0} \sum_{j=0}^{n-1} \binom{2n-2}{2j} (-1)^j \left(\frac{\sin(\frac{\pi r}{N})}{1 + \cos(\frac{\pi r}{N})}\right)^{2j} \frac{1}{2^{2n-2}} \frac{1}{2\left(1 + \cos(\frac{\pi r}{N})\right)} \tag{2.12}$$
$$= \sum_{r \text{ even}, r\neq 0} (-1)^{n-1} \left(\frac{\sin(\frac{\pi r}{N})}{1 + \cos(\frac{\pi r}{N})}\right)^{2n-2} \frac{1}{2^{2n-2}} \frac{1}{2\left(1 + \cos(\frac{\pi r}{N})\right)}$$
$$+ O\left(\frac{\left(\sin(\frac{\pi r}{N})\right)^{2n-4}}{\left(1 + \cos(\frac{\pi r}{N})\right)^{2n-3}}\right).$$

Similarly, for odd r we have

$$\operatorname{Re}\left(\sum_{r\text{ odd}}\left(\frac{1}{2}+\frac{i}{2}\frac{\sin(\frac{\pi r}{N})}{1-\cos(\frac{\pi r}{N})}\right)^{2n-2}\left|\frac{1}{2}+\frac{i}{2}\frac{\sin(\frac{\pi r}{N})}{1-\cos(\frac{\pi r}{N})}\right|^{2}\right) \tag{2.13}$$
$$=\sum_{r\text{ odd}}(-1)^{n-1}\left(\frac{\sin(\frac{\pi r}{N})}{1-\cos(\frac{\pi r}{N})}\right)^{2n-2}\frac{1}{2^{2n-2}}\frac{1}{2(1-\cos(\frac{\pi r}{N}))}$$
$$+O\left(\frac{\left(\sin(\frac{\pi r}{N})\right)^{2n-4}}{\left(1-\cos(\frac{\pi r}{N})\right)^{2n-3}}\right).$$

Let us first note that both $\frac{\left(\sin(\frac{\pi r}{N})\right)^{2n-4}}{\left(1+\cos(\frac{\pi r}{N})\right)^{2n-3}}$ and $\frac{\left(\sin(\frac{\pi r}{N})\right)^{2n-4}}{\left(1-\cos(\frac{\pi r}{N})\right)^{2n-3}}$ can have order at most N^{2n-2}. Furthermore, because n is even, the main terms of the sums (2.12) and (2.13) are negative. Observe also that

$$\sum_{r\text{ even},r\neq 0}(-1)^{n-1}\left(\frac{\sin(\frac{\pi r}{N})}{1+\cos(\frac{\pi r}{N})}\right)^{2n-2}\frac{1}{2^{2n-2}}\frac{1}{2(1+\cos(\frac{\pi r}{N}))}$$
$$=\sum_{r\text{ odd}}(-1)^{n-1}\left(\frac{\sin(\frac{\pi(N-r)}{N})}{1+\cos(\frac{\pi(N-r)}{N})}\right)^{2n-2}\frac{1}{2^{2n-2}}\frac{1}{2(1+\cos(\frac{\pi(N-r)}{N}))} \tag{2.14}$$
$$=\sum_{r\text{ odd}}(-1)^{n-1}\left(\frac{\sin(\frac{\pi r}{N})}{1-\cos(\frac{\pi r}{N})}\right)^{2n-2}\frac{1}{2^{2n-2}}\frac{1}{2(1-\cos(\frac{\pi r}{N}))},$$

which implies that these main terms are equal and so we can focus on odd coefficients. In fact, we will mostly be interested in odd coefficients $r\leqslant\frac{N}{4}$ since for $r>\frac{N}{4}$ the monotonicity of the cosine function gives

$$0\leqslant\left|\frac{\sin(\frac{\pi r}{N})}{1-\cos(\frac{\pi r}{N})}\right|\leqslant\frac{1}{1-0.5\sqrt{2}}.$$

For $r\leqslant\frac{N}{4}$ by Taylor expansions of sine and cosine functions we have

$$\sin\left(\frac{\pi r}{N}\right)\geqslant\frac{\pi r}{N}-\frac{1}{6}\left(\frac{\pi r}{N}\right)^{3}$$

and

$$1-\cos\left(\frac{\pi r}{N}\right)\leqslant\frac{1}{2}\left(\frac{\pi r}{N}\right)^{2}.$$

As a result,

$$\sum_{r\text{ odd}}\left(\frac{\sin(\frac{\pi r}{N})}{1-\cos(\frac{\pi r}{N})}\right)^{2n-2}\frac{1}{2^{2n-2}}\frac{1}{2(1-\cos(\frac{\pi r}{N}))}$$
$$\geqslant \frac{1}{2^{2n-1}}\sum_{r\text{ odd},r\leqslant\frac{N}{4}}\frac{\left(\frac{\pi r}{N}-\frac{1}{6}\left(\frac{\pi r}{N}\right)^3\right)^{2n-2}}{\left(\frac{1}{2}(\frac{\pi r}{N})^2\right)^{2n-1}}=\sum_{r\text{ odd},r\leqslant\frac{N}{4}}\left(\frac{N}{\pi r}-\frac{1}{6}\frac{\pi r}{N}\right)^{2n-2}\left(\frac{N}{\pi r}\right)^2$$
$$=\left(\frac{N}{\pi}\right)^{2n}\sum_{r\text{ odd},r\leqslant\frac{N}{4}}\frac{1}{r^{2n}}+O(N^{2n-2}).$$

To conclude, we can use (2.6) to obtain

$$\mu_{2n}\leqslant\frac{1}{N}\left(|R|^{2n}+|B|^{2n}\right)$$
$$+\frac{2}{N}\left(\sum_{r\neq0,r\text{ even}}\hat{R}(r)^{2n-2}|\hat{R}(r)|^2+\sum_{r\text{ odd}}\hat{R}(r)^{2n-2}|\hat{R}(r)|^2\right)$$
$$\leqslant\frac{1}{N}\left(\left(\frac{N+1}{2}\right)^{2n}+\left(\frac{N-1}{2}\right)^{2n}\right)$$
$$-2\frac{2}{N}\left(\sum_{r\text{ odd}}\left(\frac{\sin(\frac{\pi r}{N})}{1-\cos(\frac{\pi r}{N})}\right)^{2n-2}\frac{1}{2^{2n-2}}\frac{1}{2(1-\cos(\frac{\pi r}{N}))}+O\left(N^{2n-2}\right)\right)$$
$$\leqslant\frac{2}{N}\left(\frac{N}{2}\right)^{2n}$$
$$-\frac{4}{N}\left(\frac{N}{2}\right)^{2n}\left(\frac{2}{\pi}\right)^{2n}\left(1+\frac{1}{2}\frac{1}{(2n-1)}\frac{1}{3^{2n-1}}-\frac{1}{2}\frac{1}{(2n-1)}\frac{1}{(2\lfloor\frac{N}{8}\rfloor+1)^{2n-1}}\right)$$
$$+O(N^{2n-2})$$
$$\leqslant\left(\frac{N}{2}\right)^{2n-1}\left(1-2\left(\frac{2}{\pi}\right)^{2n}-\frac{3}{(2n-1)}\left(\frac{2}{3\pi}\right)^{2n}\right)+O(N^{2n-2}).$$

We will now discuss the case of odd n. So, as in the previous subsection, we define

- red color set $R_1=\{y,y+1,\dots,y+\frac{N-1}{2}\}$,
- blue color set $B_1=\mathbb{Z}_N\setminus R_1$,

where $y=\lfloor\frac{N}{4n-4}\rfloor$. Then, for any $r\neq0$ we have

$$\hat{R}_1(r)=\sum_{x=0}^{\frac{N-1}{2}}e\left(\frac{r(y+x)}{N}\right)=e^{\frac{2\pi iry}{N}}\sum_{x=0}^{\frac{N-1}{2}}e\left(\frac{rx}{N}\right)=e^{\frac{2\pi iry}{N}}\hat{R}(r).$$

We now have to estimate the real part of the sum (2.7). Let us first consider the sum over even r.

$$\begin{aligned}
\mathrm{Re}&\left(\sum_{r \text{ even}, r\neq 0}\left(e^{\frac{2\pi i r y}{N}}\right)^{2n-2}\left(\frac{1}{2}-\frac{i}{2}\frac{\sin(\frac{\pi r}{N})}{1+\cos(\frac{\pi r}{N})}\right)^{2n-2}\left|\frac{1}{2}-\frac{i}{2}\frac{\sin(\frac{\pi r}{N})}{1+\cos(\frac{\pi r}{N})}\right|^{2}\right)\\
&=\sum_{r \text{ even}, r\neq 0}\left[\sum_{j=0}^{n-1}\cos\left(\frac{(4n-4)\pi r y}{N}\right)\binom{2n-2}{2j}(-1)^j\left(\frac{\sin(\frac{\pi r}{N})}{1+\cos(\frac{\pi r}{N})}\right)^{2j}\right.\\
&\quad\times\frac{1}{2^{2n-2}}\frac{1}{2(1+\cos(\frac{\pi r}{N}))}+\sum_{j=0}^{n-2}\sin\left(\frac{(4n-4)\pi r y}{N}\right)\binom{2n-2}{2j+1}(-1)^{j+1}\\
&\quad\left.\times\left(-\frac{\sin(\frac{\pi r}{N})}{1+\cos(\frac{\pi r}{N})}\right)^{2j+1}\frac{1}{2^{2n-2}}\frac{1}{2(1+\cos(\frac{\pi r}{N}))}\right]\\
&=\sum_{r \text{ even}, r\neq 0}\cos\left(\frac{(4n-4)\pi r y}{N}\right)\left(\frac{\sin(\frac{\pi r}{N})}{1+\cos(\frac{\pi r}{N})}\right)^{2n-2}\frac{1}{2^{2n-2}}\frac{1}{2(1+\cos(\frac{\pi r}{N}))}\\
&\quad+O\left(\sin\left(\frac{(4n-4)\pi r y}{N}\right)\frac{\left(\sin(\frac{\pi r}{N})\right)^{2n-3}}{\left(1+\cos(\frac{\pi r}{N})\right)^{2n-2}}\right).
\end{aligned}$$

Similarly, for odd r we have

$$\begin{aligned}
\mathrm{Re}&\left(\sum_{r \text{ odd}}\left(e^{\frac{2\pi i r y}{N}}\right)^{2n-2}\left(\frac{1}{2}+\frac{i}{2}\frac{\sin(\frac{\pi r}{N})}{1-\cos(\frac{\pi r}{N})}\right)^{2n-2}\left|\frac{1}{2}+\frac{i}{2}\frac{\sin(\frac{\pi r}{N})}{1-\cos(\frac{\pi r}{N})}\right|^{2}\right)\\
&=\sum_{r \text{ odd}}\cos\left(\frac{(4n-4)\pi r y}{N}\right)\left(\frac{\sin(\frac{\pi r}{N})}{1-\cos(\frac{\pi r}{N})}\right)^{2n-2}\frac{1}{2^{2n-2}}\frac{1}{2(1-\cos(\frac{\pi r}{N}))}\\
&\quad+O\left(\sin\left(\frac{(4n-4)\pi r y}{N}\right)\frac{\left(\sin(\frac{\pi r}{N})\right)^{2n-3}}{\left(1-\cos(\frac{\pi r}{N})\right)^{2n-2}}\right).
\end{aligned}$$

Let us note that both

$$\sin\left(\frac{(4n-4)\pi r y}{N}\right)\frac{\left(\sin(\frac{\pi r}{N})\right)^{2n-3}}{\left(1+\cos(\frac{\pi r}{N})\right)^{2n-2}}$$

and

$$\sin\left(\frac{(4n-4)\pi r y}{N}\right)\frac{\left(\sin(\frac{\pi r}{N})\right)^{2n-3}}{\left(1-\cos(\frac{\pi r}{N})\right)^{2n-2}}$$

can have order at most N^{2n-1}. By (2.9) and (2.14) the main terms of the sums over even r and odd r are actually equal, which allows us to focus on the latter sum. We will split it into two parts.

We assume first that $r > \frac{N}{\log N}$. Then

$$\left| \sum_{r \text{ odd}, r > \frac{N}{\log N}} \cos\left(\frac{(4n-4)\pi r y}{N}\right) \left(\frac{\sin(\frac{\pi r}{N})}{1-\cos(\frac{\pi r}{N})}\right)^{2n-2} \frac{1}{2^{2n-2}} \frac{1}{2(1-\cos(\frac{\pi r}{N}))} \right| \leqslant \left(\frac{N}{\pi}\right)^{2n} \int_{\frac{N}{\log N}-2}^{N} \frac{1}{r^{2n}} dr = O(N \log^{2n-1} N).$$

In the case $r \leqslant \frac{N}{\log N}$ by (2.10) and (2.11) we get

$$\sum_{r \text{ odd}, r \leqslant \frac{N}{\log N}} \cos\left(\frac{(4n-4)\pi r y}{N}\right) \left(\frac{\sin(\frac{\pi r}{N})}{1-\cos(\frac{\pi r}{N})}\right)^{2n-2} \frac{1}{2^{2n-2}} \frac{1}{2(1-\cos(\frac{\pi r}{N}))} \leqslant -\left(\frac{N}{\pi}\right)^{2n} \left(1 + \frac{1}{2}\frac{1}{(2n-1)}\frac{1}{3^{2n-1}}\right) + o(N^{2n}).$$

Therefore,

$$\mu_{2n} \leqslant S_{2n}(R_1, B_1) \leqslant \left(\frac{N}{2}\right)^{2n-1} \left(1 - 2\left(\frac{2}{\pi}\right)^{2n} - \frac{3}{(2n-1)}\left(\frac{2}{3\pi}\right)^{2n}\right) + o(N^{2n-1}).$$

3. Lower estimate for prime N

In this section we will prove Theorem 1.2. We assume that N is a prime number. Let $\mathbb{Z}_N = R \cup B$ be any coloring. Recall that we can find the number of solutions to (1.2) using (2.1). Therefore, we need to estimate the sum (2.2) from below. We can assume that $|R| \leqslant |B|$.

Our approach is based on a special case of Lemma 26 from [3], which can be formulated in the following way

Lemma 3.1. *Let N be a prime number and let $0 < k < N$ be an integer. For any k-element set $R \subseteq \mathbb{Z}_N$*

$$\max_{r \neq 0} |\hat{R}(r)| \leqslant \left| \frac{\sin(\frac{\pi k}{N})}{\sin(\frac{\pi}{N})} \right|. \tag{3.1}$$

Using this lemma and Parseval's identity, we deduce that

$$\begin{aligned} S_{2n}(R, B) &\geqslant \frac{1}{N}\left(|R|^{2n} + |B|^{2n}\right) - \frac{2}{N}\left(\max_{r\neq 0} |\hat{R}(r)|\right)^{2n-2} \sum_{r \neq 0} |\hat{R}(r)|^2 \\ &\geqslant \frac{1}{N}\left(|R|^{2n} + |B|^{2n}\right) - \frac{2}{N}\left|\frac{\sin(\frac{\pi |R|}{N})}{\sin(\frac{\pi}{N})}\right|^{2n-2} \left(N|R| - |R|^2\right) \\ &= \frac{1}{N}\left(|R|^{2n} + |B|^{2n}\right) - 2\left|\frac{\sin(\frac{\pi |R|}{N})}{\sin(\frac{\pi}{N})}\right|^{2n-2} |R|\left(1 - \frac{|R|}{N}\right). \end{aligned}$$

Therefore,

$$\mu_{2n} = \min_{\mathbb{Z}_N = R\cup B} S_{2n}(R,B) \geqslant \min_{\mathbb{Z}_N = R\cup B, |R|\leqslant|B|} \frac{1}{N}\left(|R|^{2n} + (N-|R|)^{2n}\right) - 2 \max_{\mathbb{Z}_N = R\cup B, |R|\leqslant|B|} \left|\frac{\sin(\frac{\pi|R|}{N})}{\sin(\frac{\pi}{N})}\right|^{2n-2} |R|\left(1-\frac{|R|}{N}\right).$$

Clearly, in the range $|R| \in [0, \frac{N}{2}]$ the function $\frac{1}{N}(|R|^{2n}+(N-|R|)^{2n})$ is decreasing, while $\left|\frac{\sin(\frac{\pi|R|}{N})}{\sin(\frac{\pi}{N})}\right|^{2n-2} |R|\left(1-\frac{|R|}{N}\right)$ is increasing. Consequently, we will use the above estimate with $|R| = \frac{N-1}{2}$. We have

$$\left|\frac{\sin\left(\frac{\pi|R|}{N}\right)}{\sin(\frac{\pi}{N})}\right| = \left|\frac{\sin\left(\frac{\pi\frac{N-1}{2}}{N}\right)}{\sin(\frac{\pi}{N})}\right| \leqslant \frac{N}{\pi}(1+o(1)) = \frac{2}{\pi}|R|(1+o(1)).$$

Therefore,

$$\begin{aligned}\mu_{2n} &\geqslant \frac{1}{N}\left(\left(\frac{N-1}{2}\right)^{2n} + \left(\frac{N+1}{2}\right)^{2n}\right) \\ &\quad - 2\left(\frac{N}{\pi}(1+o(1))\right)^{2n-2}\frac{N-1}{2}\left(1-\frac{N-1}{2N}\right) \\ &= \left(\frac{N}{2}\right)^{2n-1}\left(1-\left(\frac{2}{\pi}\right)^{2n-2}\right) + o(N^{2n-1}).\end{aligned}$$

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