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WEIGHTED REAL EGYPTIAN NUMBERS

Melvyn B. Nathanson

Abstract: Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a sequence of nonempty finite sets of positive real numbers, and let $\mathcal{B} = (B_1, \ldots, B_n)$ be a sequence of infinite discrete sets of positive real numbers. A *weighted real Egyptian number with numerators $\mathcal{A}$ and denominators $\mathcal{B}$* is a real number c that can be represented in the form

$$c = \sum_{i=1}^{n} \frac{a_i}{b_i}$$

with $a_i \in A_i$ and $b_i \in B_i$ for $i \in \{1, \ldots, n\}$. In this paper, classical results of Sierpiński for Egyptian fractions are extended to the set of weighted real Egyptian numbers.

Keywords: Egyptian fractions, representation functions, nowhere dense sets.

1. Weighted Egyptian numbers

Let $\mathbf{N} = \{1, 2, 3, \ldots\}$ denote the set of positive integers.

An *Egyptian fraction of length n* is a rational number that can be represented as the sum of n pairwise distinct unit fractions, that is, a rational number of the form

$$\sum_{i=1}^{n} \frac{1}{b_i}$$

for some n-tuple $(b_1, \ldots, b_n)$ of pairwise distinct positive integers. Deleting the requirement that the denominators be pairwise distinct, we define an *Egyptian number of length n* as a rational number that is the sum of n unit fractions, that is, a rational number of the form

$$\sum_{i=1}^{n} \frac{1}{b_i}$$

for some n-tuple $(b_1, \ldots, b_n)$ of positive integers. Because $1/b = 1/2b + 1/2b$, an Egyptian number of length at most n is also an Egyptian number of length n.

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For numbers in the open interval $(0,1)$, repeated use of the elementary identities

$$\frac{2}{2k} = \frac{1}{k} = \frac{1}{k+1} + \frac{1}{k(k+1)}$$

and

$$\frac{2}{2k+1} = \frac{1}{k+1} + \frac{1}{(k+1)(2k+1)}$$

allows us to write an Egyptian number of length n as an Egyptian fraction of length n, and also to write an Egyptian fraction of length n as an Egyptian fraction of length n' for every $n' \geqslant n$.

Richard K. Guy's book *Unsolved Problems in Number Theory* [1, pp. 252–262] contains an ample bibliography and many open questions about Egyptian fractions.

There is a natural extension of Egyptian numbers from the the set of positive rational numbers to the set of positive real numbers. Let A be a finite set of positive real numbers, and let B be an infinite discrete set of positive real numbers. (The set B is *discrete* if $B \cap X$ is finite for every bounded set X.) We consider "unit fractions" of the form $1/b$ with $b \in B$, and finite sums of these unit fractions with weights $a \in A$. This gives real numbers of the form $\sum_{i=1}^{n} a_i/b_i$.

More generally, let $\mathcal{A} = (A_1, \ldots, A_n)$ be a sequence of nonempty finite sets of positive real numbers, and let $\mathcal{B} = (B_1, \ldots, B_n)$ be a sequence of infinite discrete sets of positive real numbers. A *weighted real Egyptian number with numerators $\mathcal{A}$ and denominators $\mathcal{B}$* is a real number c that can be represented in the form

$$c = \sum_{i=1}^{n} \frac{a_i}{b_i}$$

for some

$$(a_1, \ldots, a_n, b_1, \ldots, b_n) \in A_1 \times \cdots \times A_n \times B_1 \times \cdots \times B_n.$$

Let

$$\mathcal{E}(\mathcal{A}, \mathcal{B}) = \left\{ \sum_{i=1}^{n} \frac{a_i}{b_i} : a_i \in A_i \text{ and } b_i \in B_i \text{ for } i \in \{1, \ldots, n\} \right\}$$

be the set of all weighted real Egyptian numbers with numerators $\mathcal{A}$ and denominators $\mathcal{B}$. The set $\mathcal{E}(\mathcal{A}, \mathcal{B})$ is a set of positive real numbers.

For all $c \in \mathbf{R}$, we define the *representation function*

$$r_{\mathcal{A},\mathcal{B}}(c) = \text{card}\left((a_1, \ldots, a_n, b_1, \ldots, b_n) \in A_1 \times \cdots \times A_n \times B_1 \times \cdots \times B_n : \sum_{i=1}^{n} \frac{a_i}{b_i} = c \right).$$

The purpose of this note is to show that the topological results about Egyptian numbers in Sierpiński's classic paper [2], "Sur les decompositions de nombres rationnels en fractions primaires" extend to weighted Egyptian numbers.

Note that an Egyptian number of length n is a weighted real Egyptian number with numerators $\mathcal{A} = (\{1\}, \ldots, \{1\})$ and denominators $\mathcal{B} = (\mathbf{N}, \ldots, \mathbf{N})$. Conversely, for all $a, b \in \mathbf{N}$, we have

$$\frac{a}{b} = \underbrace{\frac{1}{b} + \cdots + \frac{1}{b}}_{a \text{ summands}}.$$

Thus, every weighted real Egyptian number with numerators $\mathcal{A} = (A_1, \ldots, A_n)$ such that A_i is a finite set of positive integers for $i \in \{1, \ldots, n\}$, and with denominators $\mathcal{B} = (\mathbf{N}, \ldots, \mathbf{N})$, is an Egyptian number of length at most $\sum_{i=1}^n \max(A_i)$.

Theorem 1. *Let $A_1, \ldots, A_n$ be nonempty finite sets of positive real numbers, and let $B_1, \ldots, B_n$ be infinite discrete sets of positive real numbers. Let*

$$((a_{m,1}, \ldots, a_{m,n}, b_{m,1}, \ldots, b_{m,n}))_{m \in \mathbf{N}} \tag{1}$$

be an infinite sequence of pairwise distinct $2n$-tuples in $A_1 \times \cdots \times A_n \times B_1 \times \cdots \times B_n$, that is,

$$(a_{m,1}, \ldots, a_{m,n}, b_{m,1}, \ldots, b_{m,n}) = (a_{m',1}, \ldots, a_{m',n}, b_{m',1}, \ldots, b_{m',n})$$

if and only if $m = m'$. For $m \in \mathbf{N}$, let

$$c_m = \sum_{i=1}^n \frac{a_{m,i}}{b_{m,i}} \in \mathcal{E}(\mathcal{A}, \mathcal{B}).$$

The sequence $(c_m)_{m \in \mathbf{N}}$ contains a strictly decreasing subsequence.

Equivalently, there exists a strictly increasing sequence $(m_j)_{j=1}^{\infty}$ of positive integers such that

$$c_{m_j} > c_{m_{j+1}} > 0$$

for all $j \in \mathbf{N}$.

Proof. For $i \in \{1, 2, \ldots, n\}$, let

$$B_{0,i} = \{b_{m,i} : m = 1, 2, 3, \ldots\}$$

where $b_{m,i}$ is the $(n+i)$-th coordinate of the mth $2n$-tuple in the sequence (1). We have $B_{0,i} \subseteq B_i$ and

$$(a_{m,1}, \ldots, a_{m,n}, b_{m,1}, \ldots, b_{m,n}) \in A_1 \times \cdots \times A_n \times B_{0,1} \times \cdots \times B_{0,n}$$

for all $m \in \mathbf{N}$. If the set $B_{0,i}$ is finite for all $i = 1, \ldots, n$, then the set $A_1 \times \cdots \times A_n \times B_{0,1} \times \cdots \times B_{0,n}$ is finite. This implies that the sequence (1) is finite, which is absurd. Therefore, $B_{0,i}$ is infinite for some $i \in \{1, \ldots, n\}$. Without loss of generality, we can assume that $i = 1$ and $B_{0,1}$ is infinite.

Because $B_{0,1}$ is contained in the discrete set B_1, there is a strictly increasing sequence of positive integers $(m_{j,1})_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} b_{m_{j,1},1} = \infty.$$

Let $k \in \{1,\ldots,n\}$, and let $(m_{j,k})_{j=1}^{\infty}$ be a strictly increasing sequence of positive integers such that

$$\lim_{j\to\infty} b_{m_{j,k},i} = \infty$$

for $i \in \{1,\ldots,k\}$. If $k \leqslant n-1$, then, for $i \in \{k+1,k+2,\ldots,n\}$, we consider the set

$$B_{k,i} = \{b_{m_{j,k},i} : j \in \mathbf{N}\}.$$

Suppose that the set $B_{k,i}$ is infinite for some $i \in \{k+1,k+2,\ldots,n\}$. Without loss of generality, we can assume that $i = k+1$. Because $B_{k,k+1}$ is an infinite subset of the discrete set B_{k+1}, the sequence $(m_{j,k})_{i=1}^{\infty}$ contains a strictly increasing subsequence $(m_{j,k+1})_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} b_{m_{j,k+1},k+1} = \infty.$$

It follows that

$$\lim_{j\to\infty} b_{m_{j,k+1},i} = \infty$$

for all $i \in \{1,2,\ldots,k,k+1\}$. Continuing inductively, we obtain an integer $s \in \{1,2,\ldots,n\}$ and a strictly increasing sequence of positive integers $(m_{j,s})_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} b_{m_{j,s},i} = \infty \tag{2}$$

for all $i \in \{1,2,\ldots,s\}$, and the sets

$$B_{s,i} = \{b_{m_{j,s},i} : j \in \mathbf{N}\}$$

are finite for all $i \in \{s+1,\ldots,n\}$.

The sets $A_1,\ldots,A_n$ and $B_{s,s+1},\ldots,B_{s,n}$ are finite. Therefore, the set of $(2n-s)$-tuples

$$A_1 \times \cdots \times A_n \times B_{s,s+1} \times \cdots \times B_{s,n}$$

is finite. By the pigeonhole principle, there exists a $(2n-s)$-tuple

$$\left(a_1^*,\ldots,a_n^*,b_{s+1}^*,\ldots,b_n^*\right) \in A_1 \times \cdots \times A_n \times B_{s,s+1} \times \cdots \times B_{s,n}$$

and a strictly increasing subsequence $(m_{j,s+1})_{t=1}^{\infty}$ of the sequence $(m_{j,s})_{j=1}^{\infty}$ such that

$$\left(a_{m_{j,s+1},1},\ldots,a_{m_{j,s+1},n},b_{m_{j,s+1},s+1},\ldots,b_{m_{j,s+1},n}\right) = \left(a_1^*,\ldots,a_n^*,b_{s+1}^*,\ldots,b_n^*\right)$$

for all $j \in \mathbf{N}$. It follows that, for all $j \in \mathbf{N}$,

$$c_{m_{j,s+1}} = \sum_{i=1}^{s} \frac{a_i^*}{b_{m_{j,s+1},i}} + \sum_{i=s+1}^{n} \frac{a_i^*}{b_i^*} = \sum_{i=1}^{s} \frac{a_i^*}{b_{m_{j,s+1},i}} + c_0^*$$

where

$$c_0^* = \sum_{i=s+1}^{n} \frac{a_i^*}{b_i^*} \geqslant 0.$$

Note that $c_0^* > 0$ if $s < n$ and $c_0^* = 0$ if $s = n$.

The limit condition (2) implies that there exists a strictly increasing sequence of positive integers $(m_{j,s+2})_{j=1}^{\infty}$ such that

$$b_{m_{j,s+2},i} < b_{m_{j+1,s+2},i}$$

for all $i \in \{1, \ldots, s\}$ and for all $j \in \mathbf{N}$. Let

$$m_j = m_{j,s+2}$$

for $j \in \mathbf{N}$. We have

$$b_{m_j,i} < b_{m_{j+1},i}$$

for all $i \in \{1, \ldots, s\}$, and so

$$c_{m_j} = \sum_{i=1}^{s} \frac{a_i^*}{b_{m_j,i}} + c_0^* > \sum_{i=1}^{s} \frac{a_i^*}{b_{m_{j+1},i}} + c_0^* = c_{m_{j+1}} > 0$$

for all $j \in \mathbf{N}$. This completes the proof. ■

Corollary 1. *If $\mathcal{A} = (A_1, \ldots, A_n)$ is a sequence of nonempty finite sets of positive real numbers and $\mathcal{B} = (B_1, \ldots, B_n)$ is a sequence of infinite discrete sets of positive real numbers, then*

$$r_{\mathcal{A},\mathcal{B}}(c) < \infty$$

for all $c \in \mathbf{R}$.

Proof. Because $\mathcal{E}(\mathcal{A}, \mathcal{B})$ is a set of positive real numbers, we have $r_{\mathcal{A},\mathcal{B}}(c) = 0$ for all $c \leqslant 0$.

If $r_{\mathcal{A},\mathcal{B}}(c) = \infty$ for some $c > 0$, then there exists an infinite sequence of pairwise distinct $2n$-tuples of the form (1) such that $c_m = c$ for all $m \in \mathbf{N}$, and the constant sequence $(c_m)_{m \in \mathbf{N}}$ contains no strictly decreasing subsequence. This is impossible by Theorem 1. ■

Corollary 2. *For every $c \in \mathbf{R}$ there exists $\delta = \delta(c) > 0$ such that $(c - \delta, c) \cap \mathcal{E}(\mathcal{A}, \mathcal{B}) = \emptyset$.*

Proof. Let $c \in \mathbf{R}$. If, for every positive integer m, there exists

$$c_m \in \left(c - \frac{1}{m}, c\right) \cap \mathcal{E}(\mathcal{A}, \mathcal{B}),$$

then the sequence $(c_m)_{m \in \mathbf{N}}$ contains a strictly increasing subsequence, and this subsequence contains no strictly decreasing subsequence. This is impossible by Theorem 1. Therefore, there exists $m \in \mathbf{N}$ such that $\delta = 1/m > 0$ satisfies the condition $(c - \delta, c) \cap \mathcal{E}(\mathcal{A}, \mathcal{B}) = \emptyset$. ■

Corollary 3. *The set $\mathcal{E}(\mathcal{A}, \mathcal{B})$ is nowhere dense.*

Proof. Let $\overline{\mathcal{E}(\mathcal{A}, \mathcal{B})}$ denote the closure of $\mathcal{E}(\mathcal{A}, \mathcal{B})$, and let U be a nonempty open set in $\mathbf{R}$. If $U \cap \overline{\mathcal{E}(\mathcal{A}, \mathcal{B})} \neq \emptyset$, then there exists $c \in U \cap \mathcal{E}(\mathcal{A}, \mathcal{B})$. By Corollary 2, there exists $\delta > 0$ such that $(c - \delta, c) \cap \mathcal{E}(\mathcal{A}, \mathcal{B}) = \emptyset$, and so $U \not\subseteq \overline{\mathcal{E}(\mathcal{A}, \mathcal{B})}$. It follows that the set $\mathcal{E}(\mathcal{A}, \mathcal{B})$ of weighted real Egyptian numbers is nowhere dense. ■

2. Signed weighted Egyptian numbers

Notation. Let $j_1, \ldots, j_s \in \mathbf{N}$. We write

$$(j_1, \ldots, j_s) \preceq (1, \ldots, n\}$$

if $1 \leqslant j_1 < j_2 < \cdots < j_s \leqslant n$. For $s \in \{1, \ldots, n-1\}$ and

$$J = (j_1, \ldots, j_s) \preceq (1, \ldots, n\}$$

let

$$L = (1, \ldots, n) \setminus J = (\ell_1, \ldots, \ell_{n-s})$$

be the strictly increasing $(n-s)$-tuple obtained by deleting the integers $j_1, \ldots, j_s$ from $(1, \ldots, n)$. To the n-tuple of sets $\mathcal{A} = (A_1, A_2, \ldots, A_n)$, we associate the s-tuple of sets

$$\mathcal{A}_J = (A_{j_1}, A_{j_2}, \ldots, A_{j_s}).$$

and the $(n-s)$-tuple of sets

$$\mathcal{A}_L = \left(A_{\ell_1}, A_{\ell_2}, \ldots, A_{\ell_{n-s}}\right).$$

For example, the 2-tuple

$$J = (3, 5) \preceq (1, 2, 3, 4, 5, 6)$$

and the 4-tuple

$$L = (1, 2, 3, 4, 5, 6) \setminus (3, 5) = (1, 2, 4, 6)$$

determine the set sequences $\mathcal{A}_J = (A_3, A_5)$ and $\mathcal{A}_L = (A_1, A_2, A_4, A_6)$.

Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a sequence of nonempty finite sets of positive real numbers, and let $\mathcal{B} = (B_1, \ldots, B_n)$ be a sequence of infinite discrete sets of positive real numbers. A *signed weighted real Egyptian number with numerators* $\mathcal{A}$ *and denominators* $\mathcal{B}$ is a real number c that can be represented in the form

$$c = \sum_{i=1}^{n} \frac{\varepsilon_i a_i}{b_i} \tag{3}$$

for some $3n$-tuple

$$\begin{aligned}(a_1, \ldots, &a_n, b_1, \ldots, b_n, \varepsilon_1, \ldots, \varepsilon_n) \\ &\in A_1 \times \cdots \times A_n \times B_1 \times \cdots \times B_n \times \{1, -1\}^n.\end{aligned} \tag{4}$$

Let

$$\mathcal{E}^{\pm}(\mathcal{A}, \mathcal{B}) = \left\{ \sum_{i=1}^{n} \frac{\varepsilon_i a_i}{b_i} : a_i \in A_i,\ b_i \in B_i,\ \text{and } \varepsilon_i \in \{1, -1\} \text{ for } i \in \{1, \ldots, n\} \right\}$$

be the set of all signed weighted Egyptian numbers with numerators $\mathcal{A}$ and denominators $\mathcal{B}$. For all $c \in \mathbf{R}$, the *representation function* $r^{\pm}_{\mathcal{A},\mathcal{B}}(c)$ counts the number of $3n$-tuples of the form (4) that satisfy equation (3). We have $r^{\pm}_{\mathcal{A},\mathcal{B}}(c) \geqslant 1$ if and only if $c \in \mathcal{E}^{\pm}(\mathcal{A}, \mathcal{B})$.

The proofs in this section are simple modifications of proofs in [2].

Theorem 2. *Let* $\mathcal{A} = (A_1, \ldots, A_n)$ *be a sequence of nonempty finite sets of positive real numbers, and let* $\mathcal{B} = (B_1, \ldots, B_n)$ *be a sequence of infinite discrete sets of positive real numbers. If* $n = 1$*, then*

$$r^{\pm}_{\mathcal{A},\mathcal{B}}(c) < \infty$$

for all $c \in \mathbf{R}$*. If* $n = 2$*, then*

$$r^{\pm}_{\mathcal{A},\mathcal{B}}(c) < \infty$$

for all $c \in \mathbf{R} \setminus \{0\}$*, but it is possible that* $r^{\pm}_{\mathcal{A},\mathcal{B}}(0) = \infty$*.*

Let $n \geqslant 3$*. Let* $s \in \{2, 3, \ldots, n-1)$*,* $J = (j_1, \ldots, j_s) \preceq (1, \ldots, n)$*, and* $L = (1, \ldots, n) \setminus J$*. If* $r^{\pm}_{\mathcal{A}_J,\mathcal{B}_J}(0) = \infty$*, then* $r^{\pm}_{\mathcal{A},\mathcal{B}}(c) = \infty$ *for all* $c \in \mathcal{E}^{\pm}(\mathcal{A}_L, \mathcal{B}_L)$*.*

Proof. If $n = 1$, $\mathcal{A} = (A_1)$, and $\mathcal{B} = (B_1)$, then

$$\mathcal{E}^{\pm}(\mathcal{A}, \mathcal{B}) = \left\{ \frac{\varepsilon_1 a_1}{b_1} : a_1 \in A_1,\ b_1 \in B_1,\ \text{and } \varepsilon_1 \in \{1, -1\} \right\}$$

is a set of nonzero numbers, and so $r^{\pm}_{\mathcal{A},\mathcal{B}}(0) = 0$.

Let $c \in \mathbf{R} \setminus \{0\}$. If $r^{\pm}_{\mathcal{A},\mathcal{B}}(c) \geqslant 1$, then $c = \varepsilon_1 a_1 / b_1$ for some $a_1 \in A_1$, $b_1 \in B_1, \varepsilon_1 \in \{1, -1\}$. If $c > 0$, then $\varepsilon_1 = 1$. If $c < 0$, then $\varepsilon_1 = -1$. For each $a_1 \in A_1$ there is at most one $b_1 \in B_1$ such that $c = \varepsilon_1 a_1 / b_1$, and so $r^{\pm}_{\mathcal{A},\mathcal{B}}(c) \leqslant |A_1| < \infty$.

Let $n = 2$. Suppose that $\mathcal{A} = (A_1, A_2)$ and $\mathcal{B} = (B_1, B_2)$. Let $A = A_1 \cap A_2$ and $B = B_1 \cap B_2$. If A is nonempty and B is infinite, then for all $a \in A$ and $b \in B$ we have

$$(a, a, b, b, 1, -1) \in A_1 \times A_2 \times B_1 \times B_2 \times \{1, -1\}^2$$

and

$$0 = \frac{a}{b} + \frac{(-a)}{b}$$

and so $r^{\pm}_{\mathcal{A},\mathcal{B}}(0) = \infty$.

Let $c \in \mathbf{R} \setminus \{0\}$. Let $a^* = \max(A_1 \cup A_2)$. If

$$c = \frac{\varepsilon_1 a_1}{b_1} + \frac{\varepsilon_2 a_2}{b_2} \tag{5}$$

is a representation of c in $\mathcal{E}^{\pm}(\mathcal{A},\mathcal{B})$, then

$$|c| \leqslant \frac{a_1}{b_1} + \frac{a_2}{b_2} \leqslant a^* \left(\frac{1}{b_1} + \frac{1}{b_2}\right) \leqslant \frac{2a^*}{\min(b_1, b_2)}$$

and so

$$0 < \min(b_1, b_2) \leqslant \frac{na^*}{|c|}$$

Because the sets B_1 and B_2 are discrete, the sets

$$\tilde{B}_i = \left\{ b_i \in B_i : b_i \leqslant \frac{na^*}{|c|} \right\}$$

are finite for $i = 1$ and 2, and so the set of fractions

$$\mathcal{F} = \bigcup_{i=1}^{2} \left\{ \frac{\varepsilon_i a_i}{b_i} : a_i \in A_i, b_i \in \tilde{B}_i, \varepsilon_i \in \{1, -1\} \right\}$$

is also finite. Every representation of c of the form (5) must include at least one fraction in the set $\mathcal{F}$, and this fraction uniquely determines the other fraction in the representation (5). Therefore, $r^{\pm}_{\mathcal{A},\mathcal{B}}(c) < \infty$ for $c \neq 0$.

The statement for $n \geqslant 3$ follows immediately from the observation that if $J \preceq (1, \ldots, n)$ and $L = (1, \ldots, n) \setminus J$, then

$$\mathcal{E}^{\pm}(\mathcal{A}_J, \mathcal{B}_J) + \mathcal{E}^{\pm}(\mathcal{A}_L, \mathcal{B}_L) = \mathcal{E}^{\pm}(\mathcal{A}, \mathcal{B}).$$

This completes the proof. ■

Theorem 3. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a sequence of nonempty finite sets of positive real numbers, and let $\mathcal{B} = (B_1, \ldots, B_n)$ be a sequence of infinite discrete sets of positive real numbers. Let*

$$\mathcal{J}(\mathcal{A}, \mathcal{B}) = \bigcup_{s=1}^{n-2} \bigcup_{\substack{J_s = (j_1, \ldots, j_s) \\ \preceq (1, \ldots, n\}}} \mathcal{E}^{\pm}(\mathcal{A}_{J_s}, \mathcal{B}_{J_s}).$$

For all $c \in \mathbf{R} \setminus \mathcal{J}(\mathcal{A}, \mathcal{B})$,

$$r^{\pm}_{\mathcal{A},\mathcal{B}}(c) < \infty.$$

Proof. The sets $A_1, \ldots, A_n$ are nonempty and finite. Let

$$a^* = \max \left(\bigcup_{i=1}^{n} A_i \right).$$

Let $c \in \mathbf{R} \setminus \{0\}$. If

$$c = \sum_{i=1}^{n} \frac{\varepsilon_i a_i}{b_i}$$

is a representation of c in $\mathcal{E}^{\pm}(\mathcal{A}, \mathcal{B})$, then

$$|c| \leqslant \sum_{i=1}^{n} \frac{a_i}{b_i} \leqslant a^* \sum_{i=1}^{n} \frac{1}{b_i} \leqslant \frac{na^*}{\min\{b_1, \ldots, b_n\}}$$

and so

$$\min\{b_1, \ldots, b_n\} \leqslant \frac{na^*}{|c|}.$$

Because the sets $B_1, \ldots, B_n$ are discrete, the sets

$$\tilde{B}_i = \left\{ b_i \in B_i : b_i \leqslant \frac{na^*}{|c|} \right\}$$

are finite for $i = 1, \ldots, n$, and so the set of fractions

$$\mathcal{F} = \bigcup_{i=1}^{n} \left\{ \frac{\varepsilon_i a_i}{b_i} : a_i \in A_i, b_i \in \tilde{B}_i, \varepsilon_i \in \{1, -1\} \right\}$$

is also finite. Every representation of c in $\mathcal{E}^{\pm}(\mathcal{A}, \mathcal{B})$ must include at least one fraction in the set $\mathcal{F}$. By the pigeonhole principle, if $r^{\pm}_{\mathcal{A},\mathcal{B}}(c) = \infty$, then there must exist $j_1 \in \{1, \ldots, n\}$ such that the fraction $\varepsilon_{j_1} a_{j_1}/b_{j_1} \in \mathcal{F}$ occurs in infinitely many representations. Let j_1 be the smallest integer in $\{1, \ldots, n\}$ with this property, and let $J_1 = (j_1)$. Let L_1 be the $(n-1)$-tuple obtained by deleting j_1 from $(1, \ldots, n)$, that is,

$$L_1 = (\ell_1, \ldots, \ell_{n-1}) = (1, \ldots, n) \setminus J_1.$$

We obtain

$$c_1 = c - \frac{\varepsilon_{j_1} a_{j_1}}{b_{j_1}} \in \mathcal{E}^{\pm}(\mathcal{A}_{L_1}, \mathcal{B}_{L_1})$$

and

$$r^{\pm}_{\mathcal{A}_{L_1}, \mathcal{B}_{L_1}}(c_1) = r^{\pm}_{\mathcal{A}_{L_1}, \mathcal{B}_{L_1}}\left(c - \frac{\varepsilon_{j_1} a_{j_1}}{b_{j_1}} \right) = \infty.$$

If $c_1 = 0$, then

$$c = \frac{\varepsilon_{j_1} a_{j_1}}{b_{j_1}} \in \mathcal{E}^{\pm}(\mathcal{A}_{J_1}, \mathcal{B}_{J_1}) \subseteq \mathcal{J}(\mathcal{A}, \mathcal{B}).$$

If $c_1 \neq 0$, then we repeat this procedure. Because $r^{\pm}_{\mathcal{A}_{L_1}, \mathcal{B}_{L_1}}(c_1) = \infty$, we obtain $j_2 \in \{1, \ldots, n\}$ with $j_2 > j_1$ and a fraction $\varepsilon_{j_2} a_{j_2}/b_{j_2} \in \mathcal{F}$ that occurs in infinitely many representations of c_1. Let j_2 be the smallest integer in $\{j_1+1,\ldots,n\}$ with this property. Let L_2 be the $(n-2)$-tuple obtained by deleting j_1 and j_2 from $(1,\ldots,n)$, that is, $J_2=(j_1,j_2)$ and $L_2=(1,\ldots,n)\setminus J_2$. Let

$$c_2 = c_1 - \frac{\varepsilon_{j_2}a_{j_2}}{b_{j_2}} = c - \left(\frac{\varepsilon_{j_1}a_{j_1}}{b_{j_1}} + \frac{\varepsilon_{j_2}a_{j_2}}{b_{j_2}}\right) \in \mathcal{E}^{\pm}(\mathcal{A}_{L_2},\mathcal{B}_{L_2}).$$

We have proved that

$$r^{\pm}_{\mathcal{A}_{L_2},\mathcal{B}_{L_2}}(c_2) = r^{\pm}_{\mathcal{A}_{L_2},\mathcal{B}_{L_2}}\left(c - \left(\frac{\varepsilon_{j_1}a_{j_1}}{b_{j_1}} + \frac{\varepsilon_{j_2}a_{j_2}}{b_{j_2}}\right)\right) = \infty.$$

If $c_2 = 0$, then

$$c = \frac{\varepsilon_{j_1}a_{j_1}}{b_{j_1}} + \frac{\varepsilon_{j_2}a_{j_2}}{b_{j_2}} \in \mathcal{E}^{\pm}(\mathcal{A}_{J_2},\mathcal{B}_{J_2}) \subseteq \mathcal{J}(\mathcal{A},\mathcal{B}).$$

If $c_2 \neq 0$, then we repeat this procedure.

After s iterations, we obtain the s-tuple

$$J_s = (j_1,\ldots,j_s) \preceq (1,\ldots,n),$$

the $(n-s)$-tuple

$$L_s = (1,\ldots,n)\setminus(j_1,\ldots,j_s),$$

and fractions $\varepsilon_{j_i}a_{j_i}/b_{j_i}$ for $i=1,\ldots,s$ such that the weighted Egyptian number

$$c_s = c - \sum_{i=1}^{s}\frac{\varepsilon_{j_i}a_{j_i}}{b_{j_i}} \in \mathcal{E}^{\pm}(\mathcal{A}_{L_s},\mathcal{B}_{L_s})$$

satisfies

$$r^{\pm}_{\mathcal{A}_{L_s},\mathcal{B}_{L_s}}(c_s) = \infty.$$

By Theorem 2, this is impossible if $n-s=2$ and $c_s \neq 0$. Therefore, if $r^{\pm}_{\mathcal{A},\mathcal{B}}(c)=\infty$, then $c_s = 0$ for some $s \in \{1,\ldots,n-2\}$ and so

$$c \in \mathcal{E}^{\pm}(\mathcal{A}_{J_s},\mathcal{B}_{J_s}) \subseteq \mathcal{J}(\mathcal{A},\mathcal{B}).$$

This completes the proof. ∎

Lemma 1. *Let A be a nonempty finite set of positive real numbers and let B be an infinite discrete set of positive real numbers. If X is a nowhere dense set of real numbers, then*

$$Y = \left\{x + \frac{\varepsilon a}{b} : x \in X, a \in A, b \in B, \text{ and } \varepsilon \in \{1,-1\}\right\}$$

is also a nowhere dense set of real numbers.

Proof. Let $a^* = \max(A)$. Because X is nowhere dense, for every open interval (u', v') there is a nonempty subinterval (u, v) contained in in (u', v') such that $X \cap (u, v) = \emptyset$. Let $0 < \delta < (v - u)/2$ and let

$$y \in (u + \delta, v - \delta)$$

for some

$$y = x + \frac{\varepsilon a}{b} \in Y.$$

If $\varepsilon = 1$ and $y = x + a/b$, then

$$x < y < v - \delta < v$$

and so $x \leqslant u$. Therefore,

$$x \leqslant u < u + \delta < y = x + \frac{a}{b}$$

and so $\delta < a/b$. If $\varepsilon = -1$ and $y = x - a/b$, then

$$u < u + \delta < y < x$$

and so $x \geqslant v$. Therefore,

$$x - \frac{a}{b} = y < v - \delta < v \leqslant x$$

and $\delta < a/b$. In both cases, $b < a/\delta \leqslant a^*/\delta$. Because A is finite and B is discrete, the set

$$K = \left\{ \frac{\varepsilon a}{b} : a \in A, b \in B, \varepsilon \in \{1, -1\}, \text{ and } b \leqslant \frac{a^*}{\delta} \right\}$$

is finite. We have

$$Z = \{x + \kappa : x \in X \text{ and } \kappa \in K \} = X + K \subseteq Y$$

and

$$Y \cap (u + \delta, v - \delta) = Z \cap (u + \delta, v - \delta).$$

The set Z is the union of a finite number of translates of the nowhere dense set X. Because a translate of a nowhere dense set is nowhere dense, and because a finite union of nowhere dense sets is nowhere dense, it follows that Z is nowhere dense. Therefore, the interval $(u + \delta, v - \delta)$ contains a nonempty open subinterval that is disjoint from Y. This completes the proof. ■

Theorem 4. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a sequence of nonempty finite sets of positive real numbers, and let $\mathcal{B} = (B_1, \ldots, B_n)$ be a sequence of infinite discrete sets of positive real numbers. The set $\mathcal{E}^{\pm}(\mathcal{A}, \mathcal{B})$ is nowhere dense.*

Proof. The proof is by induction on n. If $n = 1$, then $\mathcal{A} = (A_1)$, $\mathcal{B} = (B_1)$, the set $\mathcal{E}^{\pm}(\mathcal{A}, \mathcal{B})$ is discrete, and a discrete set is nowhere dense. The inductive step follows immediately from Lemma 1. ■

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Address: Melvyn B. Nathanson: Department of Mathematics, Lehman College (CUNY), Bronx, NY 10468, USA.

E-mail: melvyn.nathanson@lehman.cuny.edu

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