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# SOME REMARKS ON THE DIFFERENCES BETWEEN ORDINATES OF CONSECUTIVE ZETA ZEROS

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**Abstract:** If $0 < \gamma_1 \leqslant \gamma_2 \leqslant \gamma_3 \leqslant \dots$ denote ordinates of complex zeros of the Riemann zeta-function $\zeta(s)$, then several results involving the maximal order of $\gamma_{n+1} - \gamma_n$ and the sum

$$\sum_{0<\gamma_n\leqslant T} (\gamma_{n+1} - \gamma_n)^k \qquad (k > 0)$$

are proved.

**Keywords:** Riemann zeta-function, consecutive zeta-zeros, large differences, Riemann hypothesis.

## 1. Introduction

Let $0 < \gamma_1 \leqslant \gamma_2 \leqslant \gamma_3 \leqslant \dots$ denote ordinates of complex zeros of the Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad (\Re s > 1).$$

For $\Re s \leqslant 1$ one defines $\zeta(s)$ by analytic continuation (see the monographs of A. Ivić [11] and E.C. Titchmarsh [20] for the properties of $\zeta(s)$). Here the Riemann Hypothesis (RH), that all complex zeros of $\zeta(s)$ satisfy $\Re s = \frac{1}{2}$, is not assumed. Thus if equality among the $\gamma_n$'s occurs for some $n$, it does not necessarily mean that the zero $\rho_n = \beta_n + i\gamma_n$ is not simple, i.e., $\zeta(\rho_n) = 0$ and $\zeta'(\rho_n) = 0$. Namely one could have $\gamma_n = \gamma_{n+1}$, $\rho_n = \beta_n + i\gamma_n, \rho_{n+1} = \beta_{n+1} + i\gamma_{n+1}$ with $\beta_n \neq \beta_{n+1}$, and both $\rho_n$ and $\rho_{n+1}$ simple. Although all numerical evidence points to the simplicity of all zeta zeros, proving this is an open and difficult question. In fact, it seems that the simplicity of zeta-zeros and the RH are independent statements in the sense that, as far as it is known, both statements could be true or false, or one true and the other one false.

Problems involving $\gamma_{n+1} - \gamma_n$, the difference between consecutive ordinates of the zeros (if the zeros are arranged according to the size of their imaginary parts)

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are of great interest. Since $\zeta(\beta_n - i\gamma_n) = 0$ if $\zeta(\beta_n + i\gamma_n) = 0$, one may consider without loss of generality that $\gamma_n > 0$ for all $n$. One of the natural problems is to investigate the sum

$$S_k(T) := \sum_{0<\gamma_n\leqslant T} (\gamma_{n+1} - \gamma_n)^k, \tag{1.1}$$

where $k$ is a fixed positive number. A. Fujii [5], [6] proved in 1975 that, for a fixed $k \in \mathbb{N}$,

$$C_1 \frac{N(T)}{(\log T)^k} \;\leqslant\; S_k(T) \;\leqslant\; C_2 \frac{N(T)}{(\log T)^k}. \tag{1.2}$$

In (1.2) we have $0 < C_1 = C_1(k) < C_2 = C_2(k), T \geqslant T_0 = T_0(k)$, and $N(T)$ denotes the number of $\gamma_n$'s not exceeding $T$, counted with multiplicities. Recall that by the classical Riemann–von Mangoldt formula (see e.g., [11] or [20]) we have

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right), \qquad S(T) = \frac{1}{\pi}\arg\zeta(\tfrac{1}{2} + iT). \tag{1.3}$$

Here $\arg\zeta(\frac{1}{2} + iT)$ is obtained by continuous variation along the segments joining the points $2, 2+iT, \frac{1}{2}+iT$, starting with the value 0. If $T$ is the ordinate of a zero lying on the critical line, then $S(T) = S(T+0)$. One has (see [20]) the bounds

$$S(T) \ll \log T, \qquad S(T) = o(\log T) \quad \text{(LH)}, \qquad S(T) \ll \frac{\log T}{\log\log T} \quad \text{(RH)}, \tag{1.4}$$

where LH denotes the (hitherto unproved) Lindelöf hypothesis that

$$\zeta(\tfrac{1}{2} + it) \;\ll_\varepsilon\; |t|^\varepsilon.$$

The LH is a consequence of the RH (see [11] or [20]), but it is not known whether the converse is true. Here $f(x) \ll_\varepsilon g(x)$ (same as $f(x) = O_\varepsilon(g(x))$) means that the implied $\ll$ (or $O$) constant depends only on $\varepsilon$. The bounds in (1.2) are explicit, but they are stated to hold only if $k \in \mathbb{N}$. The last restriction can be easily removed. Indeed, we shall show in Section 3 that (1.2) holds for any fixed $k > 1$.

Note that from (1.3) and the first bound in (1.4) we have unconditionally

$$\gamma_{n+1} - \gamma_n \;\ll\; 1. \tag{1.5}$$

From Theorem 9.12 of [20] it follows that (1.5) can be improved to

$$\gamma_{n+1} - \gamma_n \;\leqslant\; \frac{A}{\log\log\log\gamma_n} \qquad (A > 0,\ n \geqslant n_0). \tag{1.6}$$

R.R. Hall and W.K. Hayman [10] showed that any constant $A > \pi/2$ is permissible in (1.6). Also from (1.3) and (1.4), on the RH, the bound (1.6) can be improved to

$$\gamma_{n+1} - \gamma_n \;\ll\; \frac{1}{\log\log\gamma_n}. \tag{1.7}$$

The purpose of this article is to investigate $S_k(T)$, as well as some problems involving the order of $\gamma_{n+1} - \gamma_n$ and the frequency of values of $n$ for which this difference is large. Our results are primarily explicit.

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## 2. The maximal order of $\gamma_{n+1} - \gamma_n$

Although improving the upper bounds (1.6) and (1.7) seems difficult, one can derive explicit bounds, namely replace the $\ll$-constant in (1.7) by an explicit value. This is contained in

**Theorem 1.** *Under the Riemann hypothesis one has*

$$\gamma_{n+1} - \gamma_n \leqslant \Big(\frac{\pi}{2} + o(1)\Big)\frac{1}{\log\log\gamma_n} \qquad (n \to \infty). \tag{2.1}$$

**Proof.** To prove (2.1), we shall use the bound, valid under the RH,

$$|S(T)| \leqslant \left(\frac{1}{4} + o(1)\right)\frac{\log T}{\log\log T} \qquad (T \to \infty). \tag{2.2}$$

This is Theorem 2 of E. Carneiro, V. Chandee and M. Milinovich [3]. It improves the previous bound of K. Ramachandra and A. Sankaranarayanan [17], and of D.A. Goldston and S.M. Gonek [8], who had the constant $\frac{1}{2} + o(1)$ in (2.2), which yields (2.1) (see their Corollary 1) with the constant $\pi + o(1)$. Actually, in [3] Carneiro et al. have shown that

$$|S(T)| \leqslant \left(\frac{1}{4} + O\Big(\frac{\log\log\log T}{\log\log T}\Big)\right)\frac{\log T}{\log\log T}.$$

We use (1.3) with $T = \gamma_n, H > 0$, and (2.2) to obtain

$$\begin{aligned} N(T+H) - N(T) &= \frac{1}{2\pi}\int_T^{T+H} \log\left(\frac{t}{2\pi}\right) \mathrm{d}t + S(T+H) - S(T) + O\left(\frac{1}{T}\right) \\ &\geqslant \frac{H}{2\pi}\log\left(\frac{T}{2\pi}\right) - \left(\frac{1}{2} + o(1)\right)\frac{\log T}{\log\log T} > 0 \end{aligned}$$

for

$$H = \Big(\frac{\pi}{2} + o(1)\Big)\frac{1}{\log\log T} \qquad (T \to \infty).$$

Thus

$$\gamma_{n+1} \in \left[\gamma_n, \gamma_n + \frac{\frac{1}{2}\pi + o(1)}{\log\log\gamma_n}\right],$$

which implies then (2.1). Clearly the term $o(1)$ in (2.1) can be replaced by the more explicit

$$O\left(\frac{\log\log\log\gamma_n}{\log\log\gamma_n}\right). \qquad \blacksquare$$

We remark that, although the unconditional bound (1.5) is weaker than (1.6), one can obtain relatively simply an explicit bound for $\gamma_{n+1}-\gamma_n$. Namely we take advantage of the recent bound of T. Trudgian [21]

$$|S(T)| \leqslant 0.112\log T + 0.278\log\log T + 2.510 \qquad (T \geqslant \mathrm{e}). \tag{2.3}$$

If we take $T=\gamma_n$ in (1.3) and use (2.3) we obtain, for $H>0, T\geqslant T_0$ and some number $\theta$ for which $|\theta|\leqslant 1$,

$$\begin{aligned} N(T+H)-N(T) &= \frac{1}{2\pi}\int_T^{T+H}\log\left(\frac{t}{2\pi}\right)\mathrm{d}t + S(T+H)-S(T)+O\left(\frac{1}{T}\right)\\ &\geqslant \frac{H}{2\pi}\log\left(\frac{T}{2\pi}\right) + \theta 0.225\log T > 0 \end{aligned}$$

for $H=1.414$ and $T\geqslant T_0$. This gives then unconditionally

$$\gamma_{n+1}-\gamma_n \ \leqslant\ 1.414 \qquad (n\geqslant n_0), \tag{2.4}$$

and with some effort one could determine $n_0$ in (2.4) explicitly.

Determining the maximal order of $\gamma_{n+1}-\gamma_n$ is a difficult problem. Note that from (1.5) one has unconditionally

$$\sum_{0<\gamma_n\leqslant T}(\gamma_{n+1}-\gamma_n) = \sum_{0<\gamma_n\leqslant T,\gamma_{n+1}\neq\gamma_n}(\gamma_{n+1}-\gamma_n) = T+O(1). \tag{2.5}$$

Hence

$$T+O(1) = \sum_{0<\gamma_n\leqslant T}(\gamma_{n+1}-\gamma_n) \leqslant N(T)\max_{0<\gamma_n\leqslant T}(\gamma_{n+1}-\gamma_n),$$

and from (1.3) one obtains

$$\max_{0<\gamma_n\leqslant T}(\gamma_{n+1}-\gamma_n) \geqslant \frac{2\pi\bigl(1+o(1)\bigr)}{\log(T/2\pi)} \qquad (T\to\infty). \tag{2.6}$$

The lower bound in (2.6) is quite explicit, but it is weak and probably far from the true order of the quantity on the left-hand side. In his paper [16], A.M. Odlyzko states that under the GUE (Gaussian Unitary Ensemble hypothesis, see [16] and [12]) it is plausible that

$$\max_{0<\gamma_n\leqslant T}(\gamma_{n+1}-\gamma_n) \ \sim\ \frac{8}{\sqrt{2\log T}} \qquad (T\to\infty). \tag{2.7}$$

On the other hand, D. Joyner in [14] brings forth that under the so-called Dyson–Montgomery hypothesis, explained in [14], one has

$$\max_{0<\gamma_n\leqslant T}(\gamma_{n+1}-\gamma_n) \ \ll\ \frac{1}{\sqrt{\log T\log\log T}}. \tag{2.8}$$

Note that (2.7) and (2.8) cannot both be true, since they contradict one another. The very slow variation of $\sqrt{\log\log T}$ makes a numerical comparison of (2.7) and (2.8) difficult.

## 3. Some remarks on the moments of $\gamma_{n+1} - \gamma_n$

In this section we shall show that (1.2) holds for any fixed $k > 1$, not necessarily an integer. We assume that $k > 1$ is fixed and start from (2.5). Then, by Hölder's inequality,

$$T + O(1) \leqslant \left\{ \sum_{0<\gamma_n\leqslant T} (\gamma_{n+1} - \gamma_n)^k \right\}^{1/k} \Big\{N(T)\Big\}^{1-1/k}.$$

Since $T \sim 2\pi N(T)/\log T$ by (1.3), the above inequality yields immediately

$$\sum_{0<\gamma_n\leqslant T} (\gamma_{n+1} - \gamma_n)^k \geqslant \frac{\big(2\pi + o(1)\big)^k}{(\log T)^k} N(T) \qquad (T \to \infty). \tag{3.1}$$

Note that (3.1) is the lower bound inequality in (1.2), with the explicit value

$$C_1 = C_1(k) = \big(2\pi + \varepsilon\big)^k$$

for any given $\varepsilon > 0$.

To obtain the upper bound inequality, recall that the upper bound in (1.2) holds for $k \in \mathbb{N}$ and suppose that $\alpha$ satisfies $k < \alpha < k + 1$ for some $k \in \mathbb{N}$. Then write

$$\sum_{0<\gamma_n\leqslant T} (\gamma_{n+1} - \gamma_n)^\alpha = F_1(T) + F_2(T), \tag{3.2}$$

say. We have, on using the upper bound in (1.2),

$$\begin{aligned} F_1(T) &:= \sum_{0<\gamma_n\leqslant T, \gamma_{n+1}-\gamma_n\leqslant 1/\log T} (\gamma_{n+1} - \gamma_n)^\alpha \\ &\leqslant (\log T)^{k-\alpha} \sum_{0<\gamma_n\leqslant T} (\gamma_{n+1} - \gamma_n)^k \leqslant C_2(k) \frac{N(T)}{(\log T)^\alpha}. \end{aligned}$$

Similarly, using (1.2) with $k + 1$ in place of $k$, we have

$$\begin{aligned} F_2(T) &:= \sum_{0<\gamma_n\leqslant T, \gamma_{n+1}-\gamma_n > 1/\log T} (\gamma_{n+1} - \gamma_n)^\alpha \\ &\leqslant (\log T)^{k+1-\alpha} \sum_{0<\gamma_n\leqslant T} (\gamma_{n+1} - \gamma_n)^{k+1} \leqslant C_2(k+1) \frac{N(T)}{(\log T)^\alpha}. \end{aligned}$$

Inserting the bounds for $F_1(T)$ and $F_2(T)$ in (3.2) we obtain the desired upper bound for $S_\alpha(T)$.

An asymptotic formula for $S_k(T)$, when $k \neq 0, 1$, is hard to obtain. One can obtain such a formula if one assumes the RH and the Gaussian Unitary Ensemble (GUE) conjecture (see A.M. Odlyzko [16] for a detailed account). This says that, for

$$0 \leqslant \alpha < \beta < \infty, \quad \delta_n = \frac{1}{2\pi}(\gamma_{n+1} - \gamma_n)\log(\frac{\gamma_n}{2\pi}),$$

we have

$$\sum_{\gamma_n \leqslant T, \delta_n \in [\alpha,\beta]} 1 = \left(\int_\alpha^\beta p(0,u)\,\mathrm{d}u + o(1)\right)\frac{T}{2\pi}\log(\frac{T}{2\pi}) \qquad (T \to \infty).$$

Then one has, as shown by the author in [12],

$$\sum_{\gamma_n \leqslant T} (\gamma_{n+1} - \gamma_n)^k = \big(c_1(k) + o(1)\big)\left(\frac{2\pi}{\log(\frac{T}{2\pi}) - 1}\right)^{k-1} T \qquad (T \to \infty) \tag{3.3}$$

for fixed $k \geqslant 0$, thus not necessarily an integer. Here $c_1(0) = c_1(1) = 1$, and in general

$$c_1(k) := \int_0^\infty p(0,u)u^k\,\mathrm{d}u,$$

where $p(0,u)$ is the function appearing in the GUE conjecture. We have

$$\begin{aligned}
1 - \left(\frac{\sin \pi u}{\pi u}\right)^2 &= \sum_{k=0}^\infty p(k,u),\\
p(0,u) &= \frac{1}{3}\pi^3 u^2 - \frac{2}{15}\pi^4 u^4 + \frac{1}{315}\pi^6 u^6 + \cdots \qquad (u \to 0+),\\
\log p(0,u) &= -\frac{\pi^2}{8} + o(1) \qquad (u \to \infty).
\end{aligned}$$

From (1.3) one infers that the *average* distance $\gamma_{n+1} - \gamma_n$ is $2\pi/\log(\gamma_n/(2\pi))$. Thus a natural question is to investigate the quantities

$$\mu := \liminf_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{2\pi/\log(\gamma_n/(2\pi))}, \qquad \lambda := \limsup_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{2\pi/\log(\gamma_n/(2\pi))}. \tag{3.4}$$

A. Selberg [19] in 1946 indicated (without proof) that $\mu < 1$ and $\lambda > 1$ holds unconditionally, but no particular values of $\mu$ and $\lambda$ have been found yet. On the RH, several authors worked on this problem over the years and produced explicit values of $\mu$ and $\lambda$. For example, Feng and Wu [4] obtained the values $\mu \leqslant 0.514$ and $\lambda \geqslant 2.7327$. J. Bredberg [1] proved that for sufficiently large $T$ there is a subinterval of $[T, 2T]$ of length at least $2.766 \times \frac{2\pi}{\log(T/2\pi)}$ in which $\zeta(\frac{1}{2} + it)$ does not vanish. Thus, on the RH, one has $\lambda \geqslant 2.766$.

A stronger variant of (3.4) is that there exist constants $\mu < 1$ and $\lambda > 1$ such that

$$\frac{\gamma_{n+1} - \gamma_n}{2\pi/\log(\gamma_n/(2\pi))} \leqslant \mu, \qquad \frac{\gamma_{n+1} - \gamma_n}{2\pi/\log(\gamma_n/(2\pi))} \geqslant \lambda \tag{3.5}$$

for a positive proportion of $n$'s. This was stated by A. Fujii in [6], and a detailed proof of (3.5) may be found on pp. 246-249 of E.C. Titchmarsh's monograph [20].

It is interesting to investigate what is the number of $\gamma_n$'s not exceeding $T$ for which the distance $\gamma_{n+1} - \gamma_n$ is larger or smaller than the average distance. This problem, and some related questions, will be discussed in the next section.

## 4. Lower bounds for sums of large differences of $\gamma_{n+1} - \gamma_n$

We begin our discussion on the frequency of occurrences of $\gamma_{n+1} - \gamma_n$. First note that, for a given positive constant $C$,

$$\begin{aligned}\sum_{0<\gamma_n\leqslant T} (\gamma_{n+1} - \gamma_n)^2 &= \sum_{0<\gamma_n\leqslant T;\gamma_{n+1}-\gamma_n\leqslant C/\log(T/2\pi)} (\gamma_{n+1} - \gamma_n)^2 \\ &\quad + \sum_{0<\gamma_n\leqslant T;\gamma_{n+1}-\gamma_n> C/\log(T/2\pi)} (\gamma_{n+1} - \gamma_n)^2 \\ &\leqslant (C^2 + o(1))\frac{N(T)}{\log^2 T} \\ &\quad + \left(\sum_{0<\gamma_n\leqslant T;\gamma_{n+1}-\gamma_n> \frac{C}{\log(T/2\pi)}} 1\right)^{\frac12} \left(\sum_{\gamma_n\leqslant T} (\gamma_{n+1} - \gamma_n)^4\right)^{\frac12}.\end{aligned}$$

Thus it follows, on using (1.2), that

$$(C_1(2) - C^2 + o(1))\frac{N(T)}{\log^2 T} \leqslant \left(\sum_{0<\gamma_n\leqslant T;\gamma_{n+1}-\gamma_n> \frac{C}{\log(T/2\pi)}} 1\right)^{1/2} \left(\frac{C_2(4)N(T)}{\log^4 T}\right)^{1/2},$$

which yields unconditionally

$$\sum_{0<\gamma_n\leqslant T;\gamma_{n+1}-\gamma_n> C/\log(T/2\pi)} 1 \geqslant \frac{(C_1(2) - C^2 + o(1))^2}{C_2(4)} N(T) \qquad (T\to\infty), \tag{4.1}$$

and the bound (4.1) is non-trivial if $0 < C < \sqrt{C_1(2)}$.

If one assumes the RH, then the $\gamma_n$'s are exactly the zeros of Hardy's function (see the author's monograph [13] for an extensive account)

$$Z(t) := \zeta(\tfrac12 + it)\big(\chi(\tfrac12 + it)\big)^{-1/2}, \qquad \zeta(s) = \chi(s)\zeta(1-s),$$

which is real-valued and satisfies $|Z(t)| = |\zeta(\frac12 + it)|$. Hardy's function is thus an invaluable tool for the investigation of zeros of $\zeta(s)$ on the *critical line* $\Re s = \frac12$.

If one also assumes that almost all the $\gamma_n$'s are simple, then (4.1) can be used for obtaining an alternative proof of Theorem 2 in the paper of Gonek–Ivić [9]. Following Fujii's arguments one can find numerical values of the constants in (4.1), but they will certainly produce poor values of the constant in Theorem 2 in [9]. However, the proof of this result assumes both the RH and the Pair Correlation conjecture, and both of these are strong assumptions.

The quantity $\log(T/2\pi)$ appearing in (4.1) is natural, because of (1.3) we already noted that the average spacing between the $\gamma_n$'s is $2\pi/\log(\gamma_n/2\pi)$. Moreover, with increasing $C$ the sum in (4.1) decreases, so one has to have an expression such as $C_1(2) - C^2$ on the right-hand side of (4.1).

In view of (1.3) one can rewrite (3.3) as

$$\sum_{\gamma_n\leqslant T}(\gamma_{n+1}-\gamma_n)^k = \big(c_1(k)+o(1)\big)\left(\frac{2\pi}{\log(\frac{T}{2\pi})-1}\right)^k N(T) \qquad (T\to\infty). \tag{4.2}$$

With this notation (4.1) becomes then

$$\sum_{0<\gamma_n\leqslant T;\gamma_{n+1}-\gamma_n>C/\log(T/2\pi)} 1 \geqslant \frac{\Big\{(2\pi)^2c_1(2)-C^2+o(1)\Big\}^2}{(2\pi)^2c_1(4)}N(T), \tag{4.3}$$

and one has then only to calculate explicitly the values of $c_1(2)$ and $c_1(4)$ and insert them in (4.3). This will produce an explicit bound in the range

$$0 < C < 2\pi\sqrt{c_1(2)}.$$

A variant of the approach leading to (4.3) is as follows. Recall that we have (2.4), namely

$$\sum_{0<\gamma_n\leqslant T}(\gamma_{n+1}-\gamma_n) = T + O(1). \tag{4.4}$$

Write, for a given $C>0$,

$$\begin{aligned}\sum_{\gamma_n\leqslant T}(\gamma_{n+1}-\gamma_n) &= \sum_{\gamma_n\leqslant T;\gamma_{n+1}-\gamma_n\leqslant C/\log(T/2\pi)}(\gamma_{n+1}-\gamma_n)\\ &\quad+\sum_{\gamma_n\leqslant T;\gamma_{n+1}-\gamma_n>C/\log(T/2\pi)}(\gamma_{n+1}-\gamma_n)\\ &= S_1(T;C)+S_2(T;C),\end{aligned} \tag{4.5}$$

say. One has trivially

$$S_1(T;C) \leqslant \frac{C}{\log(T/2\pi)}\sum_{\gamma_n\leqslant T}1 = \frac{C}{\log(T/2\pi)}N(T). \tag{4.6}$$

On the other hand, by the Cauchy-Schwarz inequality, we obtain

$$S_2(T;C) \leqslant \left\{ \sum_{\gamma_n \leqslant T; \gamma_{n+1}-\gamma_n > \frac{C}{\log(T/2\pi)}} 1 \sum_{\gamma_n \leqslant T; \gamma_{n+1}-\gamma_n > \frac{C}{\log(T/2\pi)}} (\gamma_{n+1}-\gamma_n)^2 \right\}^{\frac{1}{2}}. \tag{4.7}$$

We have trivially

$$\sum_{\gamma_n \leqslant T; \gamma_{n+1}-\gamma_n > \frac{C}{\log(T/2\pi)}} (\gamma_{n+1}-\gamma_n)^2 \leqslant \sum_{\gamma_n \leqslant T} (\gamma_{n+1}-\gamma_n)^2,$$

and one can estimate the last sum by (1.2). However, if one assumes the Riemann hypothesis, then A. Fujii [7] showed that one has

$$\sum_{\gamma_n \leqslant T} (\gamma_{n+1}-\gamma_n)^2 \leqslant 9 \cdot \frac{2\pi T}{\log(T/2\pi)} \qquad (T \geqslant T_0). \tag{4.8}$$

Consequently from (4.4)–(4.8), on the RH, we have

$$\sum_{0<\gamma_n \leqslant T; \gamma_{n+1}-\gamma_n > C/\log(T/2\pi)} 1 \geqslant \frac{T\log(T/2\pi)}{18\pi}\left(1-\frac{C}{2\pi}\right)^2 + O(T). \tag{4.9}$$

Note that (4.9) is an explicit inequality, and it is non-trivial for $0 < C < 2\pi$, that is, for the difference between consecutive ordinates which is smaller than the average difference.

In the above two approaches we have exploited the sum in (1.1) with $k = 1$ and $k = 2$. One can work with general $k$ in (1.2), but it is unclear which $k$ will yield the best lower bound for the sum in (4.9).

We summarize the preceding discussion in

**Theorem 2.** *With the notation introduced above we have unconditionally, if* $0 < C < \sqrt{C_1(2)}$,

$$\sum_{0<\gamma_n \leqslant T; \gamma_{n+1}-\gamma_n > C/\log(T/2\pi)} 1 \geqslant \frac{(C_1(2) - C^2 + o(1))^2}{C_2(4)} N(T) \qquad (T \to \infty).$$

*Moreover, if the RH is assumed, then for for* $0 < C < 2\pi, T \geqslant T_0 > 0$ *we have*

$$\sum_{0<\gamma_n \leqslant T; \gamma_{n+1}-\gamma_n > C/\log(T/2\pi)} 1 \geqslant \frac{T\log(T/2\pi)}{18\pi}\left(1-\frac{C}{2\pi}\right)^2 + O(T).$$

## 5. Upper bounds for sums of large differences of $\gamma_{n+1} - \gamma_n$

A natural problem is to consider upper bounds the sum in Theorem 2. An explicit upper bound for this sum is easily obtained. Namely, by using (2.5), we have

$$\begin{aligned}\sum_{0<\gamma_n\leqslant T;\gamma_{n+1}-\gamma_n>C/\log(T/2\pi)} 1 &\leqslant \frac{1}{C}\log(T/2\pi)\sum_{0<\gamma_n\leqslant T}(\gamma_{n+1}-\gamma_n)\\ &= \frac{1}{C}\log(T/2\pi)\Bigl(T+O(1)\Bigr)\\ &= \frac{2\pi T}{2\pi C}\log(T/2\pi)+O(\log T)\\ &= \frac{2\pi}{C}N(T)+O\Bigl(\frac{T}{C}\Bigr).\end{aligned}$$

This gives, unconditionally and uniformly for any $C > 0$,

$$\sum_{0<\gamma_n\leqslant T;\gamma_{n+1}-\gamma_n>C/\log(T/2\pi)} 1 \leqslant \frac{2\pi}{C}N(T)+O\Bigl(\frac{T}{C}\Bigr). \tag{5.1}$$

Using the upper bound in (1.2) with general $k$ one obtains similarly

$$\sum_{0<\gamma_n\leqslant T;\gamma_{n+1}-\gamma_n>C/\log(T/2\pi)} 1 \leqslant \frac{C_2(k)}{C^k}\Bigl(1+o(1)\Bigr)N(T) \qquad (T\to\infty), \tag{5.2}$$

but it is unclear for what range of $C$ and the value of $k$ this bound is optimal. Note that (5.1) and (5.2) are superseded, for $C$ large enough, by the bound

$$\sum_{0<\gamma_n\leqslant T;\gamma_{n+1}-\gamma_n>C/\log(T/2\pi)} 1 \ll N(T)\exp(-AC) \qquad (A>0, C\geqslant C_0). \tag{5.3}$$

The bound (5.3) is Corollary 2 on p. 35 of A. Fujii [5]. By (2.3) and $C = \lambda\log(T/2\pi)$ with $\lambda > 0$ sufficiently large, the sum in (5.3) is empty. In that case

$$N(T)\exp(-AC) = N(T)\exp\Bigl(-A\lambda\log(T/2\pi)\Bigr) = N(T)(T/2\pi)^{-A\lambda} < 1$$

if $\lambda > 1/A, T \geqslant T_0 > 0$. This shows that the bound in (5.3) is quite strong.

## 6. Sums of reciprocals of $\gamma_{n+1} - \gamma_n$

The sum $S_k(T)$ in (1.1) clearly makes sense not only for $k > 0$, but for $k < 0$ as well (for $k = 0$ the sum is just $N(T)$, so it need not be considered). When $k < 0$ one has obviously to assume the condition $\gamma_{n+1} \neq \gamma_n$, or equivalently $\gamma_{n+1} > \gamma_n$, to avoid zeros in the denominator. Such a condition is also natural when $k > 0$, since

$$(\gamma_{n+1}-\gamma_n)^k \equiv 0 \qquad (k>0, \gamma_{n+1}=\gamma_n).$$

There seem to be no results concerning $S_k(T)$ in the literature when $k < 0$. Even the sum $S_{-1}(T)$ seems elusive.

We shall consider here the somewhat less difficult sum

$$H(T) := \sum_{0<t_n\leqslant T, t_{n+1}\neq t_n} (t_{n+1}-t_n)^{-1},$$

where $0 < t_1 \leqslant t_2 \leqslant t_3 \leqslant \ldots$ are the ordinates of zeta zeros on the critical line $\Re s = \frac{1}{2}$, or equivalently, the zeros of Hardy's function $Z(t)$. Further let

$$R(T) := \sum_{0<t_n\leqslant T, t_{n+1}\neq t_n} 1.$$

If $\rho_n = \frac{1}{2} + it_n$ is a simple zero of $\zeta(s)$, then we cannot have $t_{n+1} = t_n$. Thus $R(T)$ counts all simple zeros on the critical line, and the number of those for which $0 < t_n \leqslant T$ is $\gg T \log T$. In fact, H.M. Bui, B. Conrey and M.P. Young [2] showed that more than 40% of the zeros counted by $N(T)$ are simple and on the critical line. More recently N. Robles, A. Roy and A. Zaharescu [18] proved that at least 41.0725% of the zeros of $\zeta(s)$ are on the critical line and at least 40.5824% of the zeros of $\zeta(s)$ are both on the critical line and simple.

Thus for some $D$ satisfying $D > 2/5$ we have

$$R(T) \geqslant \frac{DT}{2\pi}\log\frac{T}{2\pi} \qquad (T \geqslant T_0 > 0). \tag{6.1}$$

On the other hand, by using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} R(T) &= \sum_{0<t_n\leqslant T, t_{n+1}\neq t_n} \frac{1}{\sqrt{t_{n+1}-t_n}} \cdot \sqrt{t_{n+1}-t_n} \\ &\leqslant \left\{ H(T) \sum_{0<t_n\leqslant T, t_{n+1}\neq t_n} (t_{n+1}-t_n) \right\}^{1/2}. \end{aligned} \tag{6.2}$$

Since $t_{n+1} - t_n \ll t_n^{1/6}$ (see Chapter 9 of [11]), it follows that

$$\sum_{0<t_n\leqslant T, t_{n+1}\neq t_n} (t_{n+1}-t_n) = T + o(T) \qquad (T \to \infty). \tag{6.3}$$

From (6.1)–(6.3) we obtain that

$$\frac{DT}{2\pi}\log\frac{T}{2\pi} \leqslant \sqrt{H(T)(T+o(T))},$$

which gives

**Theorem 3.** *We have*

$$H(T) = \sum_{0<t_n\leqslant T, t_{n+1}\neq t_n} (t_{n+1}-t_n)^{-1} \geqslant \frac{T}{(5\pi)^2}\left(\log\frac{T}{2\pi}\right)^2 \qquad (T \geqslant T_1 > 0). \tag{6.4}$$

An upper bound for $H(T)$ seems difficult to obtain.

We have

$$H(T) \leqslant \max_{0<t_n\leqslant T, t_{n+1}\neq t_n} (t_{n+1}-t_n)^{-1} N(T)$$
$$\leqslant \max_{0<t_n\leqslant T, t_{n+1}\neq t_n} (t_{n+1}-t_n)^{-1} \left(\frac{T}{2\pi}\log\frac{T}{2\pi} + O(T)\right),$$

and from (6.4) it follows that we obtain

$$\max_{0<t_n\leqslant T, t_{n+1}\neq t_n} (t_{n+1}-t_n)^{-1} \geqslant \frac{2}{25\pi}\log\frac{T}{2\pi} \qquad (T \geqslant T_1 > 0). \tag{6.5}$$

or equivalently

$$\min_{0<t_n\leqslant T, t_{n+1}\neq t_n} (t_{n+1}-t_n) \leqslant \frac{25\pi}{2\log\frac{T}{2\pi}} \qquad (T \geqslant T_1 > 0). \tag{6.6}$$

If one considers the analogous problem with the sequence $\{t_n\}$ replaced by the sequence $\{\gamma_n\}$, then only the analogue of (6.1) is not obvious, namely

$$\sum_{0<\gamma_n\leqslant T, \gamma_{n+1}\neq\gamma_n} 1 \gg T\log T. \tag{6.7}$$

However, the sum in (6.7) certainly counts simple zeros (with $\gamma_n \leqslant T$) on the critical line, and as already mentioned, there are $\gg T\log T$ of these. Thus (6.7) holds, and the rest of the preceding argument easily carries through. Alternatively, since the sum in (6.7) certainly also counts *distinct* zeros of $\zeta(s)$, and there are at least 70% of distinct zeta-zeros (see H. Ki and Y. Lee [15]), we can obtain an even better bound for the sum in (6.7). Therefore we can obtain the analogues of (6.4)–(6.6) for the sequence $\{\gamma_n\}$, with different explicit constants, of course.

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