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ON THE EQUALITY BETWEEN TWO DIAMETRAL DIMENSIONS

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Abstract: The paper gives sufficient conditions to have the equality between two diametral dimensions of metrizable locally convex spaces and examples of Köthe echelon spaces satisfying them. It also provides examples for which the equality does not hold.

Keywords: diametral dimension, Schwartz spaces, Köthe sequence spaces.

1. Introduction

The *diametral dimension* of a locally convex space E is the set

$$\Delta(E) := \big\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall V \in \mathcal{V}(E),\ \exists U \in \mathcal{V}(E) \text{ such that } U \subset V \text{ and } (\xi_n \delta_n(U,V))_{n\in\mathbb{N}_0} \in c_0\big\}\,,$$

where $\mathcal{V}(E)$ is a basis of 0-neighbourhoods in E and $\delta_n(U,V)$ is the n^{th} *Kolomogorov's diameter of* U *with respect to* V:

$$\delta_n(U,V) := \inf\{\delta > 0 : \exists F \in \mathcal{L}_n(E) \text{ such that } U \subset \delta V + F\},$$

where $\mathcal{L}_n(E)$ is the set of all linear subspaces of E with a dimension less or equal to n. For more precision, we refer to [3, 4, 7, 9, 14, 16, 2].

The diametral dimension is a topological invariant which is a very useful tool for the characterization of nuclear and Schwartz spaces (see for example [7, 4] and references therein).

In [13], Terzioglu gives the definition of another "diametral dimension" as follows:

$$\Delta_b(E) := \{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall V \in \mathcal{V}(E),\ \forall B \text{ bounded subset of } E, (\xi_n \delta_n(B,V))_{n\in\mathbb{N}_0} \in c_0\}.$$

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Of course, $\Delta(E) \subset \Delta_b(E)$ and the equality holds in case of normed spaces E. But if E is a Fréchet-Montel space which is not Schwartz, then $\Delta(E) = c_0 \subsetneq \Delta_b(E)$ as the following theorems show.

Theorem 1.1 ([4]). *The following properties are equivalent*

(i) *E is Schwartz;*
(ii) *$l_\infty \subset \Delta(E)$;*
(iii) *$c_0 \subsetneq \Delta(E)$;*
(iv) *$\forall V \in \mathcal{V}(E)$, $\exists U \in \mathcal{V}(E)$ with $U \subset V$ such that* $\lim_{n\to+\infty} \delta_n(U,V) = 0$.

and also (the proof of Theorem(1.2) is an adaptation of the proof of Theorem(1.1))

Theorem 1.2. *The following properties are equivalent*

(i) *every bounded subset of E is precompact;*
(ii) *$l_\infty \subset \Delta_b(E)$;*
(iii) *$c_0 \subsetneq \Delta_b(E)$;*
(iv) *$\forall V \in \mathcal{V}(E)$ and for every bounded subset B we have* $\lim_{n\to+\infty} \delta_n(B,V) = 0$.

In the same paper [13], Terzioglu asserts that $\Delta(E) = \Delta_b(E)$ if E is a metrizable quasinormable locally convex space. But the proof is not clear (in the core of the proof, an inclusion implies that if the space is Fréchet Schwartz and has a continuous norm, then it is normed).

However, using the previous theorems (Theorem (1.1),Theorem (1.2)), the problem (of equality or not equality between $\Delta(E)$ and $\Delta_b(E)$) is solved for Fréchet spaces which are not Schwartz. Indeed, we have $\Delta(E) = \Delta_b(E) = c_0$ in case the space is not Montel and $\Delta(E) = c_0 \subsetneq \Delta_b(E)$ in case it is Montel. So, as far as we know, the question is still open for Fréchet Schwartz spaces.

The present paper is organized as follows. In section 2, we give sufficient conditions to have the equality between $\Delta(E)$ and $\Delta_b(E)$ (Theorem (2.2)). In section 3, we apply this theorem for Köthe echelon spaces defined with a regular Köthe matrix, for Köthe echelon spaces with property $(\overline{\Omega})$ and Köthe echelon spaces of type G_∞ (for example, see [6, 13, 10, 11, 8, 1] and references therein for definition and information about these types of Köthe spaces). Remark that these notions play an important role in the study of the structure of nuclear spaces and are closely related to conditions (Ω) and (DN) of D. Vogt's work. In section 4, we construct examples of nuclear (hence Schwartz) non metrizable locally convex spaces for which the equality is false.

In what follows, we will use the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ denotes the set of all strictly positive integers. Moreover, for every $j \in \mathbb{N}_0$, e_j will denote the element of $\mathbb{C}^{\mathbb{N}_0}$ with all components equal to 0 except $(e_j)_j = 1$. And the number of elements of a finite set A will be denoted by $\#A$.

Finally, let us recall a very useful inequality between two Kolmogorov's diameters: if $A \subset A'$ and $B \subset B'$ then

$$\delta_n(A, B') \leqslant \delta_n(A', B), \qquad \forall n \in \mathbb{N}_0.$$

2. A positive general result

Let U_m $(m \in \mathbb{N}_0)$ be a decreasing sequence of absolutely convex 0-neighbourhoods of a locally convex and metrizable space E, which constitute a basis of 0-neighbourhoods.

Remark 2.1. We can assume that

$$\delta_n(U_k, U_m) \neq 0 \qquad \forall m, n, k \in \mathbb{N}_0,\ k \geqslant m.$$

Proof. Indeed, fix $m \in \mathbb{N}_0$; then, if there is $k \geqslant m$ and n such that $\delta_n(U_k, U_m) = 0$, we have of course $\delta_n(B, U_m) = 0$ for every bounded set B. It follows that if there exist infinitely many m with this property, then $\Delta(E) = \Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$. Otherwise, since we keep the same topology if we remove a finite number of U_m, we can assume that $\delta_n(U_k, U_m) \neq 0$, $\forall m, n, k \geqslant m$. ■

The following result gives sufficient conditions to have the equality between the two diametral dimensions which are concerned here.

Theorem 2.2. *Consider the following conditions:*

(i) *c_0 can be replaced by l_∞ in the definition of $\Delta(E)$*
(ii) *for every $m \in \mathbb{N}_0$ and every sequence $(r_k)_{k \geqslant m}$ of strictly positive numbers, there are $M \geqslant m$ and a bounded set B such that*

$$\delta_n(B, U_M) \geqslant \inf_{k \geqslant m} r_k \delta_n(U_k, U_m), \qquad \forall n \in \mathbb{N}_0$$

(iii) *for every $m \in \mathbb{N}_0$, there are $M \geqslant m$ and a bounded set B such that*

$$\delta_n(B, U_M) \geqslant \delta_n(U_M, U_m), \qquad \forall n \in \mathbb{N}_0.$$

If the conditions (i) *and* (ii) *are satisfied or if condition* (iii) *is satisfied, then*

$$\Delta(E) = \Delta_b(E).$$

Remark. It is direct to see that condition (iii) is stronger than condition (ii). Also note that (iii) corresponds to the notion of "prominent bounded set" introduced by Terzioglu in [13].

Proof. A) Assume (i) and (ii) are satisfied and take $\xi \notin \Delta(E)$. Then, there exists $m \in \mathbb{N}_0$ such that for each $j \geqslant m$, there is $n(j) \in \mathbb{N}_0$ such that

$$|\xi_{n(j)}|\ \delta_{n(j)}(U_j, U_m) \geqslant 1.$$

We can of course assume that the sequence $n(j)$ $(j \in \mathbb{N})$ is strictly increasing. From the previous inequalities, one directly gets

$$|\xi_{n(j)}| \delta_{n(j)}(U_k, U_m) \geqslant 1, \qquad \forall k, j \in \mathbb{N}_0, j \geqslant m \text{ and } j \geqslant k$$

because $U_j \subset U_k$. For every $k \geqslant m$, let us define

$$r_k := \sup\left\{1, \sup_{m\leqslant p\leqslant k} \frac{1}{|\xi_{n(p)}|\delta_{n(p)}\left(U_k, U_m\right)}\right\}.$$

We then have

$$r_k|\xi_{n(j)}|\delta_{n(j)}\left(U_k, U_m\right) \geqslant 1, \qquad \forall k \geqslant m \text{ and } j \geqslant m$$

hence also

$$|\xi_{n(j)}| \inf_{k\geqslant m} r_k\delta_{n(j)}\left(U_k, U_m\right) \geqslant 1, \qquad \forall j \geqslant m.$$

Using (ii), it is then clear that ξ can not belong to $\Delta_b(E)$.

B) The condition (iii) trivially implies the inclusion $\Delta_b(E) \subset \Delta(E)$. ■

3. Examples

Let us now give examples of Köthe echelon spaces satisfying the assumptions of Theorem (2.2), hence the equality between the two diametral dimensions (see Corollaries (3.10), (3.11), (3.12)).

First, we recall some definitions and prove a very useful property of Kolmogorov's diameters (Proposition (3.7)).

Definition 3.1. A Banach space $(l, \|.\|_l)$ of complex sequences is said to be *admissible* ([12]) if the following two conditions are verified:

(a) if $\xi \in l_\infty$ and $\eta \in l$, then $\xi\eta \in l$ and $\|\xi\eta\|_l \leqslant \|\xi\|_{l_\infty}\|\eta\|_l$ where the components of the sequence $\xi\eta$ are the products of the respective components of ξ and η;
(b) for every $j \in \mathbb{N}_0$, $e_j \in l$ and $\|e_j\|_l = 1$.

Of course, the spaces l_p (for $p \geqslant 1$), l_∞ and c_0 are some examples of admissible spaces (see also [2, 5]). A useful property of admissible spaces is the following one ([12]).

Proposition 3.2. *Let $(l, \|.\|_l)$ be an admissible space. If $\xi \in l$ and $\eta \in \mathbb{C}^{\mathbb{N}_0}$ are such that $|\eta_n| \leqslant |\xi_n|$ for every $n \in \mathbb{N}_0$, then $\eta \in l$ and $\|\eta\|_l \leqslant \|\xi\|_l$.*

Let us also recall the definition we use for a "Köthe matrix" and an associated Köthe echelon space.

Definition 3.3. A Köthe matrix A is a sequence $(a_k)_{k\in\mathbb{N}_0}$ with $a_k = (a_k(n))_{n\in\mathbb{N}_0}$ and $0 < a_k(n) \leqslant a_{k+1}(n)$ for every $k, n \in \mathbb{N}_0$.

Definition 3.4. If $A = (a_k)_{k\in\mathbb{N}_0}$ is a Köthe matrix, the Köthe echelon space associated to A and l is the linear space

$$\lambda^l(A) := \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall k \in \mathbb{N}_0 \ \ (a_k(n)\xi_n)_{n\in\mathbb{N}_0} \in l\right\},$$

which is endowed with the topology defined by the natural seminorms p_k^l, $k \in \mathbb{N}_0$, where $p_k^l(\xi) := \|(a_k(n)\xi_n)_{n\in\mathbb{N}_0}\|_l$.

We also write $U_k^l := \left\{\xi \in \lambda^l(A) : p_k^l(\xi) \leqslant 1\right\}$ and to simplify the notation, simply U_k if l is fixed.

The spaces $\lambda^l(A)$ are Fréchet locally convex spaces and are Schwartz if and only if for every k, there is $k' > k$ such that $(\delta_n(U_{k'}, U_k))_{n\in\mathbb{N}_0} \in c_0$, or, equivalently, if and only if for every k, there is $k' > k$ such that $(a_k(n)/a_{k'}(n))_{n\in\mathbb{N}_0} \in c_0$ (see for example ([12])). The property of "being Schwartz" is then independent of the admissible space l, as well as the diametral dimension $\Delta(E)$.

Concerning Kolmogorov's diameters, we have the following very useful results.

Proposition 3.5 ([12]). *Let $n \in \mathbb{N}_0$ and J, J' be subsets of $\mathbb{N}_0$ with $\#J = n+1$ and $\#J' \leqslant n$. If $k \geqslant m$, we have*

$$\inf_{j\in J} \frac{a_m(j)}{a_k(j)} \leqslant \delta_n\left(U_k, U_m\right) \leqslant \sup_{j\notin J'} \frac{a_m(j)}{a_k(j)}.$$

Remark. Assume that the sequence $(a_m(n)/a_k(n))_{n\in\mathbb{N}_0}$ converges to 0. Then, if n is fixed and if we define J (resp. J') as the set of the first $n+1$ (resp. n) indices characterizing the $n+1$ (resp. n) greatest elements of the sequence $(a_m(j)/a_k(j))_{j\in\mathbb{N}_0}$, the infimum and the supremum of the previous inequalities coincide and we have

$$\delta_n\left(U_k, U_m\right) = \frac{a_m(n_0)}{a_k(n_0)}$$

where $\{n_0\} = J \setminus J'$, that is to say $\delta_n\left(U_k, U_m\right)$ is the $(n+1)^{th}$ greatest component of the sequence $\left(\frac{a_m(j)}{a_k(j)}\right)_{j\in\mathbb{N}_0}$.

Corollary 3.6. *If $\lambda^l(A)$ is a Schwartz space, then the condition (i) of Theorem 2.2 is satisfied.*

Proof. The "key point" is to prove the following: if $\varepsilon > 0$, if $j, k, m \in \mathbb{N}_0$ are such that $j > k > m$, if $(a_m(n)/a_k(n))_{n\in\mathbb{N}_0} \in c_0$ and if $N \in \mathbb{N}_0$ are such that

$$\frac{a_m(n)}{a_j(n)} \leqslant \varepsilon \frac{a_m(n)}{a_k(n)} \qquad \forall n \geqslant N,$$

then there is $N_0 \geqslant N$ such that

$$\delta_n(U_j, U_m) \leqslant \varepsilon\delta_n(U_k, U_m), \qquad \forall n \geqslant N_0.$$

The proof is given at the end of the paper (appendix).

From this, it is direct to get that l_∞ can replace c_0 in the definition of the diametral dimension. Indeed, let us take $\xi \in \mathbb{C}^{\mathbb{N}_0}$ such that $\forall m \in \mathbb{N}_0$, there is $k > m$ such that $\sup_{n\in\mathbb{N}_0} |\xi_n| \, \delta_n(U_k, U_m) < \infty$. Then fix $m \in \mathbb{N}_0$ and take first $k = k(m) > m$ such that $(a_m(n)/a_k(n))_{n\in\mathbb{N}_0} \in c_0$ and $\sup_{n\in\mathbb{N}_0} |\xi_n| \, \delta_n(U_k, U_m) < \infty$. Next, take $j > k$ such that $(a_k(n)/a_j(n))_{n\in\mathbb{N}_0} \in c_0$. We want to show that $(\xi_n\delta_n(U_j, U_m))_{n\in\mathbb{N}_0} \in c_0$.

If $\varepsilon > 0$, there is $N \in \mathbb{N}_0$ such that $a_k(n)/a_j(n) \leqslant \varepsilon$ for all $n \geqslant N$ hence

$$\frac{a_m(n)}{a_j(n)} \leqslant \varepsilon \frac{a_m(n)}{a_k(n)} \qquad \forall n \geqslant N.$$

Using the "key point", this implies that there is $N_0 \geqslant N$ such that

$$\delta_n(U_j, U_m) \leqslant \varepsilon \delta_n(U_k, U_m), \qquad \forall n \geqslant N_0.$$

The conclusion follows. ■

The proof of the next proposition (Proposition (3.7)) is inspired by the proof ([12]) of Proposition (3.5).

Proposition 3.7. *Let $(r_k)_{k\in\mathbb{N}_0} \in (0,\infty)^{\mathbb{N}_0}$, $m, n \in \mathbb{N}_0$, and $J \subset \mathbb{N}_0$ such that $\#J = n+1$ be given. Then we have*

$$\delta_n\left(\bigcap_{k\geqslant m} r_k U_k, U_m\right) \geqslant \inf_{k\geqslant m}\left(r_k \inf_{j\in J}\left(\frac{a_m(j)}{a_k(j)}\right)\right).$$

Proof. Suppose that the inequality is false; then there exist a real number $\delta > 0$ with $\delta < \delta_0 := \inf_{k\geqslant m}\left(r_k \inf_{j\in J} a_m(j)/a_k(j)\right)$ and $F \in \mathcal{L}_n(\lambda^l(A))$ such that

$$\bigcap_{k\geqslant m} r_k U_k \subset \delta U_m + F.$$

Let us define the projection $P_J : \lambda^l(A) \to \lambda^l(A)\ \xi \mapsto \sum_{j\in J} \xi_j e_j$, and $G := P_J(\lambda^l(A))$.

On one hand, if $\xi \in \lambda^l(A)$, $k \geqslant m$ and $j \in J$, we have

$$|a_k(j)\xi_j| = \frac{a_k(j)}{r_k a_m(j)} r_k |a_m(j)\xi_j| \leqslant \frac{1}{\delta_0} r_k |a_m(j)\xi_j|$$

hence $p_k(\xi) \leqslant (r_k/\delta_0)\, p_m(\xi)$ if $\xi \in G$ (Proposition (3.2)). Therefore

$$U_m \cap G \subset \frac{1}{\delta_0}\left[\left(\bigcap_{k\geqslant m} r_k U_k\right) \cap G\right].$$

On the other hand, from the inclusion

$$\bigcap_{k\geqslant m} r_k U_k \subset \delta U_m + F$$

and using one more time the monotonicity result (Proposition (3.2)), we obtain

$$G \cap \left(\bigcap_{k\geqslant m} r_k U_k\right) \subset \delta P_J(U_m) + P_J(F) \subset (\delta U_m \cap G) + P_J(F).$$

Finally, we have

$$U_m \cap G \subset \frac{\delta}{\delta_0}(U_m \cap G) + P_J(F).$$

Now, let us use the previous inclusion several times. Let $\xi \in U_m \cap G$ be given. We can find $f_0 \in P_J(F)$ such that $p_m(\xi - f_0) \leqslant \delta/\delta_0$. But we also have $(\delta/\delta_0)(U_m \cap G) \subset (\delta/\delta_0)^2(U_m \cap G) + P_J(F)$; hence there exists $f_1 \in P_J(F)$ with $p_m(\xi - f_0 - f_1) \leqslant (\delta/\delta_0)^2$. Recursively, we obtain a sequence $(f_j)_{j\in\mathbb{N}_0}$ of elements of $P_J(F)$ such that, for each $j_0 \in \mathbb{N}_0$,

$$p_m\left(\xi - \sum_{j=0}^{j_0} f_j\right) \leqslant \left(\frac{\delta}{\delta_0}\right)^{j_0+1}.$$

As a consequence, the sequence $\sum_{j=0}^{j_0} f_j$ ($j_0 \in \mathbb{N}$) of elements of $P_J(F)$ converges to ξ in the normed space (G, p_m). Since $P_J(F)$ is finite-dimensional, it is a closed subset of (G, p_m); hence $\xi \in P_J(F)$. From this, we deduce that $U_m \cap G \subset P_J(F)$. As G and $P_J(F)$ are linear spaces, we get $G \subset P_J(F)$ and therefore $G = P_J(F)$. But this is not possible since $\dim G = n + 1 > n \geqslant \dim(P_J(F))$. ∎

We will also consider special types of Köthe matrices A.

Definition 3.8. We say that A is *regular* if, for every $k \in \mathbb{N}_0$, the sequence $\left(\frac{a_k(n)}{a_{k+1}(n)}\right)_{n\in\mathbb{N}_0}$ is decreasing.

Nice properties of Köthe spaces defined with a regular matrix are the following: if $\lambda^l(A)$ is regular, then it is Schwartz or it is isomorphic to l ([12]) and

$$\delta_n(U_k, U_m) = a_m(n)/a_k(n) \tag{1}$$

for any $k, m, n \in \mathbb{N}_0$, $k \geqslant m$ ([4, 12]).

In that case, from Proposition (3.7), we immediately obtain:

Proposition 3.9. *Suppose that A is regular. Then for every $(r_k)_{k\in\mathbb{N}_0} \in (0,\infty)^{\mathbb{N}_0}$ and $m, n \in \mathbb{N}_0$, we have*

$$\delta_n\left(\bigcap_{k\geqslant m} r_k U_k, U_m\right) = \inf_{k\geqslant m}\left(r_k \frac{a_m(n)}{a_k(n)}\right) = \inf_{k\geqslant m} r_k \delta_n(U_k, U_m).$$

Proof. Basic properties of Kolmogorov's diameters ([4]) give $\delta_n\left(\cap_{k\geqslant m} r_k U_k, U_m\right) \leqslant \inf_{k\geqslant m}\left(r_k \delta_n(U_k, U_m)\right) = \inf_{k\geqslant m}\left(r_k a_m(n)/a_k(n)\right)$. From Proposition (3.7), since A is regular, we get

$$\delta_n\left(\bigcap_{k\geqslant m} r_k U_k, U_m\right) \geqslant \inf_{k\geqslant m} \inf_{j\in\{0,\dots,n\}}\left(r_k \frac{a_m(j)}{a_k(j)}\right) = \inf_{k\geqslant m}\left(r_k \frac{a_m(n)}{a_k(n)}\right). \qquad ∎$$

It follows from Theorem (2.2) that we have a first concrete case for which the equality between the two diametral dimensions holds:

Corollary 3.10. *Suppose that A is regular. Then we have*

$$\Delta\left(\lambda^l(A)\right) = \Delta_b\left(\lambda^l(A)\right).$$

Proof. If $\lambda^l(A)$ is not Schwartz, then it is isomorphic to l ([12]), hence normed and not Montel. We already know that this implies the equality of the two diametral dimensions.

If $\lambda^l(A)$ is Schwartz, then the assumption (i) of Theorem (2.2) is satisfied (here this is straightforward because of regularity). Since (ii) of Theorem (2.2) is also trivially satisfied in case A is regular (see Proposition (3.9)), we finally obtain the announced equality. ■

Here is a second example of spaces for which the equality between the diametral dimensions is also true.

Let us recall the definition of the condition

$$(\overline{\Omega})$$

for a Köthe matrix A: $\forall m \in \mathbb{N}_0$, there is $k > m$ such that for every $j \in \mathbb{N}_0$, there $C > 0$ such that

$$a_k^2(n) \ \geqslant \ C\ a_m(n)a_j(n), \quad \forall n \in \mathbb{N}_0$$

Corollary 3.11. *Suppose that the Köthe matrix A satisfies the condition $\left(\overline{\Omega}\right)$ and that the Köthe echelon space is Schwartz. Then we have*

$$\Delta\left(\lambda^l(A)\right) = \Delta_b\left(\lambda^l(A)\right).$$

Proof. Let us fix $m \in \mathbb{N}_0$. By assumption, there exists $k_m \in \mathbb{N}_0,\ k_m > m$ such that, for every $j \in \mathbb{N}_0$, there exists $r_j^{(k_m)} > 0$ verifying

$$r_j^{(k_m)} a_{k_m}^2(n) \geqslant a_m(n)a_j(n) \qquad \forall n \in \mathbb{N}_0.$$

Hence, for every n, using Proposition (3.7) with J giving the precise value of $\delta_n(U_{k_m}, U_m)$ (since we can take $k_m > m$ such that $(a_m(n)/a_{k_m}(n))_{n\in\mathbb{N}_0} \in c_0$), we get

$$\begin{aligned}\delta_n\left(\bigcap_{j\geqslant k_m} r_j^{(k_m)} U_j, U_{k_m}\right) &\geqslant \inf_{j\geqslant k_m}\left(r_j^{(k_m)} \inf_{p\in J}\left(\frac{a_{k_m}(p)}{a_j(p)}\right)\right)\\ &\geqslant \inf_{p\in J}\frac{a_m(p)}{a_{k_m}(p)} = \delta_n(U_{k_m}, U_m).\end{aligned}$$

Hence condition (iii) of Theorem (2.2) is satisfied and we are done. ■

And finally, here is the third announced example.

First, let us recall the condition satisfied by a Köthe matrix A to say that the space $\lambda^l(A)$ is of type

$$G_\infty :$$

$\forall k \in \mathbb{N}_0$, the sequence $(a_k(n))_{n\in\mathbb{N}_0}$ is increasing and there are $j \in \mathbb{N}_0, C > 0$ such that

$$a_k^2(n) \leqslant C\, a_j(n), \quad \forall n \in \mathbb{N}_0$$

(see [13]). Notice that to require that the space is Schwartz amounts to saying that the sequences $(a_k(n))_{n\in\mathbb{N}_0}$ are increasing to ∞.

Corollary 3.12. *For every Schwartz space $\lambda^l(A)$ of type G_∞, we have*

$$\Delta\left(\lambda^l(A)\right) = \Delta_b\left(\lambda^l(A)\right).$$

Proof. Let us fix $m \in \mathbb{N}_0$. Using the assumption, for every $k \geqslant m$, one can find $j(k) \geqslant k$ and $s_k > 0$ such that

$$\frac{1}{a_k(n)} \geqslant s_k \frac{a_k(n)}{a_{j(k)}(n)}, \qquad \forall n.$$

It follows that for every n and every sequence $z_k > 0$, we have

$$\begin{aligned}
\delta_n\left(\bigcap_{k\geqslant m} z_k U_k, U_m\right) &\geqslant \inf_{k\geqslant m} z_k \inf_{p\leqslant n} \frac{a_m(p)}{a_k(p)} \\
&\geqslant a_m(0) \inf_{k\geqslant m} z_k \frac{1}{a_k(n)} \\
&= a_m(0) \inf_{k\geqslant m} z_k \sup_{p\geqslant n} \frac{1}{a_k(p)} \\
&\geqslant a_m(0) \inf_{k\geqslant m} z_k s_k \sup_{p\geqslant n} \frac{a_k(p)}{a_{j(k)}(p)} \\
&\geqslant a_m(0) \inf_{k\geqslant m} z_k s_k \sup_{p\geqslant n} \frac{a_m(p)}{a_{j(k)}(p)} \\
&\geqslant a_m(0) \inf_{k\geqslant m} z_k s_k \delta_n\left(U_{j(k)}, U_m\right).
\end{aligned}$$

Now, if $r_k > 0$ $(k \geqslant m)$ is also given, we have

$$\inf_{k\geqslant m} r_k \delta_n\left(U_k, U_m\right) \leqslant \inf_{k\geqslant m} r_{j(k)} \delta_n\left(U_{j(k)}, U_m\right)$$

hence if we take

$$z_k := \frac{r_{j(k)}}{s_k a_m(0)}$$

condition (ii) of Theorem (2.2) is satisfied. ∎

4. Counterexamples

In what follows, we construct Schwartz locally convex spaces E such that

$$\Delta(E) \subsetneq \Delta_b(E).$$

In fact, since $\Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$ if E has only finite-dimensional bounded sets, it suffices to give examples of such Schwartz locally convex spaces for which $\Delta(E) \neq \mathbb{C}^{\mathbb{N}_0}$.

In this context, a first natural question is to know when a locally convex space has only finite-dimensional bounded sets. It is explained in the next result, quoted in [15] in the case of Hausdorff locally convex spaces. We will use the following notations: E^* is the algebraic dual of a locally convex space E and E^b is the set of all linear bounded functionals on E (i.e. linear functionals which maps bounded sets of E on bounded sets of $\mathbb{C}$).

Proposition 4.1. *Let E be a locally convex space. If $E^b = E^*$, then every bounded set of E is finite-dimensional.*

Conversely, if E is a Hausdorff space and if each bounded set of E is finite-dimensional, then $E^b = E^*$.

Proof. Let us suppose that $E^b = E^*$ and let B be a bounded set of E. If B is not finite-dimensional, there exists a sequence $(x_n)_{n\in\mathbb{N}_0}$ of linearly independent elements of B. We can then define a linear map $x^* : E \to \mathbb{C}$ such that $x^*(x_n) = n$ for every $n \in \mathbb{N}_0$. Since x^* is bounded, $x^*(B)$ is bounded, which is clearly not possible.

Now, we suppose that E is a Hausdorff space and that all the bounded subsets of E are finite-dimensional. Let $x^* : E \to \mathbb{C}$ be a linear map and B be a bounded set of E. We have to prove that $x^*(B)$ is bounded in $\mathbb{C}$.

By hypothesis, there exist $J \in \mathbb{N}$ and $x_1, ..., x_J \in E$, linearly independent, such that $B \subset F := \mathrm{span}(x_1, ..., x_J)$. Since E is a Hausdorff space, we know that F is isomorphic to $\mathbb{C}^J$. Thus, for every $j \in \{1, ..., J\}$ the canonical projection $\pi_j : F \to \mathbb{C}$ related to x_j is continuous. Therefore, for every $j = 1, \ldots, J$, there is $C_j > 0$ such that $\sup_{b\in B} |\pi_j(b)| \leqslant C_j$, hence $\sup_{b\in B} |x^*(b)| \leqslant \sum_{j=1}^{J} C_j |x^*(x_j)|$ and we are done. ■

As a consequence of the previous result, we see that for any linear space E, every bounded set of $(E, \sigma(E, E^*))$ is finite-dimensional. Remark also that the topology $\sigma(E, E^*)$ is nuclear, since it is a weak topology ([6]).

Unfortunately, the weak topology $\sigma(E, E^*)$ does not give a counterexample to our problem, as the next proposition shows.

Proposition 4.2. *If E is a locally convex space endowed with a weak topology, then $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$. In particular, $\Delta(E) = \Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$.*

Proof. Let us take a finite subset M of E^*. We define

$$p_M : E \to [0, \infty) : x \mapsto \sup_{y^*\in M} |y^*(x)|$$

and $U_{p_M} := \{x \in E : p_M(x) \leqslant 1\}$. Since the linear space $\ker(p_M)$ has a finite codimension N, there exists a subspace F of E with dimension N such that $E = \ker(p_M) \oplus F$. It follows that for every $\delta > 0$, we have $E \subset \delta U_{p_M} + F$ and then $\delta_n(E, U_{p_M}) = 0$ for every $n \geqslant N$. Hence the conclusion. ∎

So, we have to work a little bit more and we recall the definition of nearly open maps and a property involving these maps and diametral dimension.

Definition 4.3. If E and F are two locally convex spaces, a linear map $T : E \to F$ is nearly open if, for every 0-neighbourhood $V$of E, the closure $\overline{T(V)}$ of $T(V)$ in F is a 0-neighbourhood in F.

Of course, T is nearly open in case it is open or in case it is surjective and F is barrelled ([4]). A nearly open map can be used to compare the diametral dimension of two spaces (cf. [2, 4]).

Proposition 4.4. *Let E, F be locally convex spaces. If there exists a linear, continuous and nearly open map $T : E \to F$, then $\Delta(E) \subset \Delta(F)$.*

In particular, if $\mathcal{T}_1$ and $\mathcal{T}_2$ are two locally convex topologies on the linear space E, if $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ and if $(E, \mathcal{T}_1)$ is barrelled, then

$$\Delta(E, \mathcal{T}_2) \subset \Delta(E, \mathcal{T}_1).$$

We are now ready to give counterexamples.

Let $(E, \mathcal{T}_1)$ be a locally convex space which is barrelled and for which $\Delta(E, \mathcal{T}_1) \neq \mathbb{C}^{\mathbb{N}_0}$ (remark that there are a lot of such spaces-even Schwartz or nuclear ones-see for example [2, 4, 9, 10, 12]).

Proposition 4.5. *Let $\mathcal{T}$ be a locally convex topology on E for which every bounded subset of $(E, \mathcal{T})$ is finite-dimensional (for example $\mathcal{T}$ can be the weak topology $\sigma(E, E^*)$) and let $\mathcal{T}_2$ be the locally convex topology whose $0-$ neighbourhoods are just the intersections of those of $(E, \mathcal{T}_1)$ and $(E, \mathcal{T})$. Then the space $(E, \mathcal{T}_2)$ verifies*

$$\Delta(E, \mathcal{T}_2) \subsetneq \Delta_b(E, \mathcal{T}_2).$$

In particular, if $\mathcal{T}_1$ and $\mathcal{T}$ are Schwartz (resp. nuclear), then $\mathcal{T}_2$ is also Schwartz (resp. nuclear).

Proof. Since $\mathcal{T}_2$ is finer than $\mathcal{T}$, the bounded sets of $(E, \mathcal{T}_2)$ are all finite-dimensional. Moreover, as $(E, \mathcal{T}_1)$ is barrelled, Proposition (4.4) gives $\Delta(E, \mathcal{T}_2) \subset \Delta(E, \mathcal{T}_1)$. It follows that

$$\Delta(E, \mathcal{T}_2) \subset \Delta(E, \mathcal{T}_1) \subsetneq \mathbb{C}^{\mathbb{N}_0} = \Delta_b(E, \mathcal{T}_2).$$

The particular case is straightforward. ∎

We have given examples of Schwartz (even nuclear) spaces E such that $\Delta(E) \neq \Delta_b(E)$; to achieve this, we used spaces with finite dimensional bounded sets. As such infinite dimensional spaces can not be metrizable, the question about the equality (or not) of the two diametral dimensions of a general metrizable locally convex Schwartz space remains open.

5. Appendix

Let us prove the following auxiliary result.

Lemma 5.1. *If $\varepsilon > 0$, if $j, k, m \in \mathbb{N}_0$ are such that $j > k > m$, if $(a_m(n)/a_k(n))_{n\in\mathbb{N}_0} \in c_0$ and if $N \in \mathbb{N}_0$ are such that*

$$\frac{a_m(n)}{a_j(n)} \leqslant \varepsilon \frac{a_m(n)}{a_k(n)} \qquad \forall n \geqslant N,$$

then there is $N_0 \geqslant N$ such that

$$\delta_n(U_j, U_m) \leqslant \varepsilon \delta_n(U_k, U_m), \qquad \forall n \geqslant N_0.$$

Proof. Let us first consider the left hand side of the inequalities. Since $(a_m(n)/a_j(n))_{n\in\mathbb{N}_0} \in c_0$, there is $N_1 > N$ such that (*)

$$\frac{a_m(n)}{a_j(n)} < \inf_{0\leqslant p\leqslant N} \frac{a_m(p)}{a_j(p)}, \qquad \forall n \geqslant N_1.$$

When we rearrange the elements of the sequence $(a_m(n)/a_j(n))_{n\in\mathbb{N}_0}$ to obtain a decreasing sequence (in order to obtain the exact value of the Kolmogorov's diameters), it may happen that for some n, the element $a_m(n)/a_j(n)$ has to be placed before or between the elements $a_m(p)/a_j(p)$ $(p = 0, \ldots, N)$. If this is the case, then necessarily $n < N_1$ (see (*)). If

$$J := \left\{ l \in \mathbb{N}_0 : \inf_{0\leqslant p\leqslant N} \frac{a_m(p)}{a_j(p)} \leqslant \frac{a_m(l)}{a_j(l)} \right\}$$

then $\{0, \ldots, N\} \subset J$, $l < N_1$ for all $l \in J$ and $\#J \leqslant N_1$. In fact, the elements of J give the exact values of $\delta_n(U_m, U_j)$ for $0 \leqslant n < \#J$. It follows that

$$n \geqslant N_1 \;\Rightarrow\; \delta_n(U_j, U_m) = a_m(n_0)/a_j(n_0) \;\text{ for some } n_0 > N.$$

Similarly, there is $N_2 > N$ such that

$$n \geqslant N_2 \;\Rightarrow\; \delta_n(U_k, U_m) = a_m(m_0)/a_k(m_0) \;\text{ for some } m_0 > N.$$

Now, if $n \geqslant \sup\{N_1, N_2\}$ and if we define J'_j (resp. J'_k) as the set of the indexes of the sequence $(a_m(l)/a_j(l))_{l\in\mathbb{N}_0}$ (resp. $(a_m(l)/a_k(l))_{l\in\mathbb{N}_0}$) used to have the exact value of $\delta_l(U_j, U_m)$ (resp. $\delta_l(U_k, U_m)$) for $l = 0, \ldots, n-1$, two situations can occur:

(i) $J'_j = J'_k (:= J')$; in this case, we have

$$\delta_n(U_j, U_m) = \sup_{p\notin J'} \frac{a_m(p)}{a_j(p)} \leqslant \varepsilon \sup_{p\notin J'} \frac{a_m(p)}{a_k(p)} = \varepsilon \delta_n(U_k, U_m)$$

since J' contains the indexes $0, \ldots N$.

(ii) $J'_j \neq J'_k$ (remind that they both have the same finite cardinality n). In that case, by construction, there is $l \in J'_j \setminus J'_k$ and $l \geqslant N$. It follows that

$$\delta_n(U_j, U_m) \leqslant \frac{a_m(l)}{a_j(l)} \leqslant \varepsilon \frac{a_m(l)}{a_k(l)} \leqslant \varepsilon \sup_{p\notin J'_k} \frac{a_m(p)}{a_k(p)} = \varepsilon \delta_n(U_k, U_m)$$

and we conclude. ∎

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