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ON SOME COMPLEX EXPLICIT FORMULAE CONNECTED WITH THE EULER'S φ FUNCTION. I

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Abstract: Following [4] and [1] we describe the analytic character of some function $f(z)$ connected with the Euler's φ function being for $\operatorname{Im} z > 0$ a series over all non - trivial zeros of the Riemann zeta - function.

Keywords: Euler's φ function, Riemann zeta - function.

1. In this note we describe basic analytic properties of a function $f(z)$ defined for $\operatorname{Im} z > 0$ as follows

$$f(z) = \lim_{n\to\infty} \sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<T_n}} \frac{e^{\varrho z}\zeta(\varrho-1)}{\zeta'(\varrho)}. \tag{1.1}$$

The summation is over non - trivial zeros the Riemann zeta - function with a positive imaginary part. For simplicity we assume here that the zeros are simple. Throuought the paper, however, we treat the general case, where the corresponding term in (1.1) is replaced by an appropriate residue. T_n denotes a sequence of real numbers yields appropriate grouping of the zeros.

First we show that $f(z)$ is a holomorphic function for $\operatorname{Im} z > 0$. Next we continue analytically the function $f(z)$ to a meromorphic function on the whole complex plane, which satisfies a certain functional equation. The functional equation for $f(z)$ connects the values of the function f at the points z and $\overline{z}$. Finally we describe all singularities of $f(z)$.

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2. Lemma 1. *There exists a positive constant c such that for all $n \in \mathbb{N}$ exists T_n, where $n \leq T_n \leq n+1$ for which*

$$\left|\frac{\zeta(\sigma-1+iT_n)}{\zeta(\sigma+iT_n)}\right| \ll T_n^c, \quad \text{for} \quad -\frac{1}{4} \leq \sigma \leq \frac{5}{2} \tag{2.1}$$

Proof. By theorem 9.7 in [5] there is a constant c_1 such that each interval $(n, n+1)$ contains a value of T_n for which $\left|\frac{1}{\zeta(\sigma+iT_n)}\right| \leq T_n^{c_1}$. Hence the lemma follows trom a well - known estimates for the Riemann zeta - function on the vertical strip. ■

3. Let l denote a smooth curve $\tau : [0,1] \longrightarrow C$ such that $\tau(0) = -\frac{1}{4}$, $\tau(1) = \frac{5}{2}$ and $0 < \operatorname{Im} \tau < 1$ for $t \in (0,1)$.

Moreover, let L denote the contour consisting of line segments

$$[\frac{5}{2}, \frac{5}{2} + iT_n], [\frac{5}{2} + iT_n, -\frac{1}{4} + iT_n], [-\frac{1}{4} + iT_n, -\frac{1}{4}],$$

and l, where T_n have the same meaning as before. We consider the following contour integral round L:

$$\int_L \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds. \tag{3.1}$$

The integral along the upper side of contour tends to 0 as n tends to ∞, because for $y > 0$

$$\left|\int_{-\frac{1}{4}+iT_n}^{\frac{5}{2}+iT_n} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds\right| \ll \int_{-\frac{1}{4}}^{\frac{5}{2}} T_n^c e^{\sigma x - T_n y} d\sigma \underset{n\to\infty}{\longrightarrow} \to 0$$

By Cauchy's theorem of residues we have

$$\begin{aligned} &\int_{-\frac{1}{4}+i\infty}^{-\frac{1}{4}} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds + \int_{l(-\frac{1}{4},\frac{5}{2})} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds + \\ &+ \int_{\frac{5}{2}}^{\frac{5}{2}+i\infty} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds = 2\pi i f(z) \end{aligned} \tag{3.2}$$

where for $\operatorname{Im} z > 0$

$$f(z) = \lim_{n\to\infty} \sum_{\substack{\varrho \\ 0<\operatorname{Im} \varrho<T_n}} \frac{1}{(k_\varrho - 1)!} \frac{d^{k_\varrho - 1}}{ds^{k_\varrho - 1}} \left[(s-\varrho)^{k_\varrho} \frac{e^{sz}\zeta(s-1)}{\zeta(s)}\right]_{s=\varrho} \tag{3.3}$$

and k_ϱ denotes the order of multiplicity of the complex zero ϱ of the zeta - function. Observe that if there are no multiple zeros of the zeta function (3.3) reduces to (1.1)

$$f(z)=\lim_{n\to\infty}\sum_{\substack{\varrho\\ 0<\operatorname{Im}\varrho<T_n}}\frac{e^{\varrho z}\zeta(\varrho-1)}{\zeta'(\varrho)}$$

The analytic character of f-function is described by the following theorems:

Theorem 1. *The function $f(z)$ is holomorphic on the upper half - plane H and for $z\in H$ we have*

$$2\pi i f(z)=f_1(z)+f_2(z)-e^{\frac{5}{2}z}\sum_{n=1}^{\infty}\frac{\varphi(n)}{n^{\frac{5}{2}}(z-\log n)} \tag{3.4}$$

where the last term on the right is the meromorphic function on the whole complex plane with the poles at $z=\log n$, $n=1,2,\ldots$

$$f_1(z)=\int_{-\frac{1}{4}+i\infty}^{-\frac{1}{4}}\frac{\zeta(s-1)}{\zeta(s)}e^{sz}ds \tag{3.5}$$

is analytic on H and

$$f_2(z)=\int_{l(-\frac{1}{4},\frac{5}{2})}\frac{\zeta(s-1)}{\zeta(s)}e^{sz}ds \tag{3.6}$$

is regular on the whole complex plane.

Theorem 2. *The function $f(z)$ can be continued analytically to a meromorphic function on the whole complex plane, which satisfies the functional equation*

$$f(z)+\overline{f(\overline{z})}=B(z) \tag{3.7}$$

and

$$\begin{aligned}B(z)=&-\frac{6}{\pi^2}e^{2z}+\\&+\frac{1}{2\pi^2}\sum_{k,n=1}^{\infty}\frac{\mu(k)}{n^2k}\left[\frac{1}{(nke^z-1)^2}+\frac{2}{nke^z-1}+\frac{1}{(nke^z+1)^2}-\frac{2}{nke^z+1}\right]\end{aligned} \tag{3.8}$$

where $B(z)$ is the meromorphic function on the whole complex plane with the poles of the second order at $z=-\log nk$, $n,k=1,2,\ldots$.

The only singularities of $f(z)$ are simple poles at the points $z=\log n$ ($n=1,2,\ldots$) on the real axis with residue

$$\operatorname*{res}_{z=\log n}f(z)=-\frac{\varphi(n)}{2\pi i},$$

and the poles of the second order at $z = -\log m \quad (m = 1, 2, \ldots)$ *with residue*

$$\operatorname*{res}_{z=-\log m} f(z) = \frac{1}{4\pi^2 m^2} \sum_{l|m} \mu(l) \cdot l.$$

We have for $\operatorname{Im} z > 0$

$$\begin{aligned} f(z) =& \frac{1}{2\pi i} \int\limits_{-\frac{1}{4}+i\infty}^{-\frac{1}{4}} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds + \frac{1}{2\pi i} \int\limits_{l(-\frac{1}{4},\frac{5}{2})} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds - \\ & - \frac{1}{2\pi i} e^{\frac{5}{2}z} \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^{\frac{5}{2}}(z - \log n)} \end{aligned} \tag{3.9}$$

for $\operatorname{Im} z < 0$

$$f(z) = -\overline{f(\overline{z})} + B(z)$$

and for $|\, Im\ z \,| < \pi$

$$\begin{aligned} f(z) =& \frac{1}{8\pi^3 i} \int\limits_{-\frac{1}{4}}^{-\frac{1}{4}+i\infty} \frac{\zeta(2-s)}{\zeta(1-s)} e^{is\pi} \Gamma(2-s)\Gamma(s) e^{sz} ds + \\ & + \frac{1}{4\pi^3 i} \int\limits_{-\frac{1}{4}}^{-\frac{1}{4}+i\infty} \frac{\zeta(2-s)}{\zeta(1-s)} \Gamma(2-s)\Gamma(s) e^{sz} ds + \\ & + \frac{1}{4\pi^2} \sum_{k,n=1}^{\infty} \frac{\mu(k)}{n^2 k} \left[\frac{2}{nke^z - 1} + \frac{1}{(nke^z - 1)^2} \right] - \\ & - \frac{1}{8\pi^3 i} \int\limits_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}} \frac{\zeta(2-s)}{\zeta(1-s)} e^{-is\pi} \Gamma(2-s)\Gamma(s) e^{sz} ds + \\ & + \frac{1}{2\pi i} \int\limits_{l(-\frac{1}{4},\frac{5}{2})} \frac{\zeta(2-s)}{\zeta(1-s)} e^{sz} ds - \frac{1}{2\pi i} e^{\frac{5}{2}z} \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^{\frac{5}{2}}(z - \log n)} \end{aligned} \tag{3.10}$$

4. Proof of Theorem 1. For z from the upper half - plane H we have

$$2\pi i f(z) = f_1(z) + f_2(z) + f_3(z),$$

say, where $f_1(z)$, $f_2(z)$, $f_3(z)$ denote corresponding integrals in (3.2).

Since $\operatorname{Re} s = \frac{5}{2} > 2$ and $\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}$

$$f_3(z) = \sum_{n=1}^{\infty} \varphi(n) \int_{\frac{5}{2}}^{\frac{5}{2}+i\infty} e^{sz - s\log n} ds = -e^{\frac{5}{2}z} \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^{\frac{5}{2}}(z - \log n)} .$$

The inversion of the order of integration and summation is justified for $z \in H$ by the uniform convergence of the integral and the series.

The second term

$$f_2(z) = \int_{l(-\frac{1}{4},\frac{5}{2})} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds$$

is regular on the whole complex plane.

By Stirling's formula and the functional equation for $\zeta(s)$ we have

$$\left| \frac{\zeta(-\frac{5}{4} + it)}{\zeta(-\frac{1}{4} + it)} \right| \ll t$$

and consequently $f_1(z)$ is absolutely convergent for $y = \operatorname{Im} z > 0$.

5. We shall first prove that $f(z)$ $(z = x + iy)$ has a meromorphic continuation to $y > -\pi$.

Consider the integral

$$f_1(z) = -\int_{-\frac{1}{4}}^{-\frac{1}{4}+i\infty} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds$$

convergent for $y > 0$. By the functional equation for $\zeta(s)$ we get

$$\zeta(s-1) = 2^{s-1}\pi^{s-2} \sin\frac{(s-1)\pi}{2} \Gamma(2-s)\zeta(2-s) ,$$

$$\frac{1}{\zeta(s)} = \frac{2^{1-s}\pi^{-s} \cos\frac{s\pi}{2}\Gamma(s)}{\zeta(1-s)}$$

and consequently

$$f_1(z) = \frac{i}{4\pi^2} \int_{-\frac{1}{4}}^{-\frac{1}{4}+i\infty} \frac{\zeta(2-s)}{\zeta(1-s)} (e^{i\frac{(s-1)\pi}{2}} - e^{-i\frac{(s-1)\pi}{2}})(e^{i\frac{s\pi}{2}} + e^{-i\frac{s\pi}{2}})\Gamma(2-s)\Gamma(s)e^{sz} ds =$$

$$= \frac{1}{4\pi^2} \int_{-\frac{1}{4}}^{-\frac{1}{4}+i\infty} \frac{\zeta(2-s)}{\zeta(1-s)} (e^{is\pi} + 2 + e^{-is\pi})\Gamma(2-s)\Gamma(s)e^{sz} ds =$$

$$= f_{11}(z) + f_{12}(z) + f_{13}(z) \tag{5.1}$$

Since $\Gamma(s) \ll t^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}|t|}$, $f_{11}(z)$ is regular for $y > -2\pi$, $f_{12}(z)$ for $y > -\pi$, $f_{13}(z)$ for $y > 0$. It means, that we have to continue $f_{13}(z)$ to a meromorphic function for $y > -\pi$.

We have

$$f_{13}(z) = \frac{1}{4\pi^2}\left(\int\limits_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} - \int\limits_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}}\right) e^{-is\pi}\frac{\zeta(2-s)}{\zeta(1-s)}\Gamma(2-s)\Gamma(s)e^{sz}ds = \tag{5.2}$$
$$= I_1(z) + I_2(z)$$

It is easy to verify that the integral $I_2(z)$ is convergent for $y < 2\pi$. Since $f_{13}(z)$ was regular for $y > 0$, the integral $I_1(z)$ is convergent for $0 < y < \pi$.

We have formally

$$I_1(z) = \frac{1}{4\pi^2}\sum_{k,n=1}^{\infty}\frac{\mu(k)}{n^2k}\int\limits_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} e^{s(\log nk - i\pi + z)}\Gamma(2-s)\Gamma(s)ds\,. \tag{5.3}$$

To justify the inversion of the order of summation and integration for $0 < y < \pi$ we will see that the integral and the series converge uniformly.

Since for $0 < y < \pi$ integrals along the upper and the lower side of the contour tend to 0 we get by Cauchy's theorem of residues

$$\begin{aligned}\int\limits_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} & e^{s(\log nk - i\pi + z)}\Gamma(2-s)\Gamma(s)ds = \\ &= \int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} e^{s(\log nk - i\pi + z)}\Gamma(2-s)\Gamma(s)ds - 2\pi i = \\ &= I_1^*(z) - 2\pi i\,,\end{aligned} \tag{5.4}$$

say. Putting $s = -w$ in $I_1^*(z)$ we have

$$I_1^*(z) = \int\limits_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} e^{-w(\log nk - i\pi + z)}\Gamma(2+w)\Gamma(-w)dw$$

Using now for this Mellin - Barnes integral formula p. 256 in [3], for $0 < y < \pi$, we get

$$I_1^*(z) = 2\pi i\Gamma(2)(1 + e^{-\log nk + i\pi - z})^{-2}\,. \tag{5.5}$$

By (5.3), (5.4) and (5.5) we have

$$\begin{aligned} I_1(z) &= \frac{1}{4\pi^2}\sum_{k,n=1}^{\infty}\frac{\mu(k)}{n^2k}[2\pi i(1+\mathrm{e}^{-\log nk+i\pi-z})^{-2}-2\pi i] = \\ &= \frac{i}{2\pi}\sum_{k,n=1}^{\infty}\frac{\mu(k)}{n^2k}\left[\frac{2}{nke^z-1}+\frac{1}{(nke^z-1)^2}\right] \end{aligned} \tag{5.6}$$

and $I_1(z)$ has the poles of the second order at $z=-\log nk$, $n,k=1,2,\dots$.

Finally by (5.1), (5.2) and (5.6) we obtain the following analytic continuation of $f_1(z)$ for $y>-\pi$.

For $|\,y\,|<\pi$

$$\begin{aligned} f_1(z) &= \frac{1}{4\pi^2}\int_{-\frac14}^{-\frac14+i\infty}\frac{\zeta(2-s)}{\zeta(1-s)}e^{is\pi}\Gamma(2-s)\Gamma(s)e^{sz}\,ds+ \\ &+\frac{1}{2\pi^2}\int_{-\frac14}^{-\frac14+i\infty}\frac{\zeta(2-s)}{\zeta(1-s)}\Gamma(2-s)\Gamma(s)e^{sz}\,ds+ \\ &+\frac{i}{2\pi}\sum_{k,n=1}^{\infty}\frac{\mu(k)}{n^2k}\left[\frac{2}{nke^z-1}+\frac{1}{(nke^z-1)^2}\right]- \\ &-\frac{1}{4\pi^2}\int_{-\frac14-i\infty}^{-\frac14}\frac{\zeta(2-s)}{\zeta(1-s)}e^{-is\pi}\Gamma(2-s)\Gamma(s)e^{sz}\,ds \end{aligned} \tag{5.7}$$

where the first is holomorphic for $y>-2\pi$, the second for $y>-\pi$, the third is meromorphic on the whole complex plane and the next is holomorphic for $y<2\pi$.

In accordance with Theorem 1, (5.7) completes the continuation of $f(z)$ to the region $y>-\pi$.

6. Let us consider the function

$$f^-(z)=\lim_{n\to\infty}\sum_{\substack{\varrho \\ -T_n<\mathrm{Im}\ \varrho<0}}\frac{1}{(k_\varrho-1)!}\frac{d^{k_\varrho-1}}{ds^{k_\varrho-1}}\left[(s-\varrho)^{k_\varrho}\frac{e^{sz}\zeta(s-1)}{\zeta(s)}\right]_{s=\varrho} \tag{6.1}$$

where k_ϱ denotes the order of multiplicity of the complex zero ϱ of $\zeta(s)$, defined for z belonging to

$$H^-=\{z\in\mathbb{C}:\ \mathrm{Im}\ z<0\} \tag{6.2}$$

Since $|\,\zeta(\bar{s})\,|=|\,\zeta(s)\,|$ by (2.1) we choose $T_n(n\le T_n\le n+1)$ such that

$$\left|\frac{\zeta(\sigma-1-iT_n)}{\zeta(\sigma-iT_n)}\right|\le T_n^c \qquad \text{for} \quad -\frac14\le\sigma\le\frac52 \tag{6.3}$$

If $\zeta(s)$ has only simples zeros, then

$$f(z) = \lim_{n\to\infty} \sum_{\substack{\varrho \\ -T_n<\operatorname{Im}\,\varrho<0}} \frac{e^{\varrho z}\zeta(\varrho-1)}{\zeta'(\varrho)} \tag{6.4}$$

Now taking the integral

$$\int \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds$$

round the contour symmetrical upon the real axis to L in (3.1).

Then the integral along the lower side of the contour tends to 0 as n tends to ∞ for $z \in H^-$ and we have for n tending to ∞ by Cauchy's residue theorem

$$2\pi i f^-(z) = f_1^-(z) + f_2^-(z) + f_3^-(z) \tag{6.5}$$

where

$$f_1^-(z) = -\int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds \tag{6.6}$$

is regular on H^- (the proof similar to this for $f_1(z)$),

$$f_2^-(z) = \int_{\overline{l(-\frac{1}{4},\frac{5}{2})}} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds \tag{6.7}$$

is regular on the whole complex plane and

$$\begin{aligned} f_3^-(z) &= \int_{\frac{5}{2}-i\infty}^{\frac{5}{2}} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds = \sum_{n=1}^{\infty} \varphi(n) \int_{\frac{5}{2}-i\infty}^{\frac{5}{2}} e^{s(z-\log n)} ds = \\ &= e^{\frac{5}{2}z} \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^{\frac{5}{2}}(z-\log n)} \end{aligned} \tag{6.8}$$

is meromorphic on the whole complex plane.

The inversion of the order of integration and summation is justified for $z \in H^-$ by the uniform convergence of the integral and the series.

Now $f_1^-(z)$ analytic for $y < 0$ we have to continue to $y < \pi$ just as $f_1(z)$ in §5. We have by functional equation for $\zeta(s)$

$$f_1^-(z) = f_{11}^-(z) + f_{12}^-(z) + f_{13}^-(z) \tag{6.9}$$

where

$$f_{11}^-(z) = \frac{1}{4\pi^2}\int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}} \frac{\zeta(2-s)}{\zeta(1-s)} e^{is\pi+sz}\Gamma(2-s)\Gamma(s)ds \tag{6.10}$$

is absolutely convergent for $y < 0$,

$$f_{12}^-(z) = \frac{1}{2\pi^2}\int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}} \frac{\zeta(2-s)}{\zeta(1-s)} e^{sz}\Gamma(2-s)\Gamma(s)ds \tag{6.11}$$

is regular for $y < \pi$,

$$f_{13}^-(z) = \frac{1}{4\pi^2}\int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}} \frac{\zeta(2-s)}{\zeta(1-s)} e^{-is\pi+sz}\Gamma(2-s)\Gamma(s)ds \tag{6.12}$$

is absolutely convergent for $y < 2\pi$.

So, we have to continued $f_{11}^-(z)$ to $y < \pi$.

We have

$$\begin{aligned} f_{11}^-(z) &= \frac{1}{4\pi^2}\int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} \frac{\zeta(2-s)}{\zeta(1-s)} e^{is\pi+sz}\Gamma(2-s)\Gamma(s)ds- \\ &- \frac{1}{4\pi^2}\int_{-\frac{1}{4}}^{-\frac{1}{4}+i\infty} \frac{\zeta(2-s)}{\zeta(1-s)} e^{is\pi+sz}\Gamma(2-s)\Gamma(s)ds = \\ &= I_1^-(z) + I_2^-(z) \end{aligned} \tag{6.13}$$

It is easy to verify that the integral $I_2^-(z)$ is convergent for $y > -2\pi$. Since $f_{11}^-(z)$ is regular for $y < 0$, the integral $I_1^-(z)$ is convergent for $-2\pi < y < 0$ and we can apply formula p. 256 in [3] to $I_1^-(z)$ for $-\pi < y < 0$ in the similar way as to $I_1(z)$ (see (5.3) - (5.6)). We get

$$I_1^-(z) = \frac{1}{4\pi^2}\sum_{n,k=1}^{\infty} \frac{\mu(k)}{n^2 k}\left[\frac{2}{nke^z - 1} + \frac{1}{(nke^z-1)^2}\right] \tag{6.14}$$

and $I_1^-(z)$ has the poles of the second order at $z = -\log nk$, $n, k = 1, 2, \ldots$.

Finally by (6.9), (6.10), (6.11), (6.12), (6.13) and (6.14) we obtain the following continuation of $f_1^-(z)$ to $y < \pi$.

For $|y|<\pi$

$$\begin{aligned}
f_1^-(z) = & \frac{1}{2\pi}\sum_{k,n=1}^{\infty}\frac{\mu(k)}{n^2k}\left[\frac{2}{nke^z-1}+\frac{1}{(nke^z-1)^2}\right]- \\
& -\frac{1}{4\pi^2}\int_{-\frac{1}{4}}^{-\frac{1}{4}+i\infty}\frac{\zeta(2-s)}{\zeta(1-s)}e^{is\pi}\Gamma(2-s)\Gamma(s)e^{sz}ds+ \\
& +\frac{1}{2\pi^2}\int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}}\frac{\zeta(2-s)}{\zeta(1-s)}\Gamma(2-s)\Gamma(s)e^{sz}ds+ \\
& +\frac{1}{4\pi^2}\int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}}\frac{\zeta(2-s)}{\zeta(1-s)}e^{-is\pi}\Gamma(2-s)\Gamma(s)e^{sz}ds
\end{aligned} \tag{6.15}$$

what completes the continuation of $f^-(z)$ analytic for $y<0$ to the halfplane $y<\pi$.

7. Proof of Theorem 2. By (5.7) and (6.15) for $|y|<\pi$

$$\begin{aligned}
f_1(z)+f_1^-(z) = & \frac{i}{\pi}\sum_{k,n=1}^{\infty}\frac{\mu(k)}{n^2k}\left[\frac{2}{nke^z-1}+\frac{1}{(nke^z-1)^2}\right]+ \\
& +\frac{1}{2\pi^2}\int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty}\frac{\zeta(2-s)}{\zeta(1-s)}\Gamma(2-s)\Gamma(s)e^{sz}ds= \\
& =\frac{1}{\pi}\sum_{k,n=1}^{\infty}\frac{\mu(k)}{n^2k}\left[\frac{2}{nke^z-1}+\frac{1}{(nke^z-1)^2}\right]+A_1
\end{aligned}$$

The integrals A_1 is convergent for $|y|<\pi$ and we can apply formula p.256 in [3] in the similar way as to $I_1(z)$ and $I_1^-(z)$ (see (5.3) - (5.6)).

We get

$$A_1(z)=\frac{i}{\pi}\sum_{k,n=1}^{\infty}\frac{\mu(k)}{n^2k}\left[\frac{1}{(nke^z+1)^2}-\frac{2}{nke^z+1}\right].$$

Finally for $|y|<\pi$

$$\begin{aligned}
& f_1(z)+f_1^-(z)= \\
& =\frac{i}{\pi}\sum_{k,n=1}^{\infty}\frac{\mu(k)}{n^2k}\left[\frac{1}{(nke^z-1)^2}+\frac{2}{nke^z-1}+\frac{1}{(nke^z+1)^2}-\frac{2}{nke^z+1}\right]
\end{aligned} \tag{7.1}$$

By theorem of residue using (3.6) and (6.7) for all y

$$f_2(z) + f_2^-(z) = \int\limits_{l(-\frac{1}{4},\frac{5}{2})} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds - \int\limits_{\overline{l(-\frac{1}{4},\frac{5}{4})}} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds = \tag{7.2}$$
$$= -2\pi i \operatorname*{res}_{s=2} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} = -2\pi i \cdot \frac{6e^{2z}}{\pi^2}$$

and by Theorem 1 and (6.8)

$$f_3(z) + f_3^-(z) = 0 \tag{7.3}$$

Thus for $|y| < \pi$ by (7.1), (7.2) and (7.3) we have

$$2\pi i(f(z) + f^-(z)) =$$
$$= \frac{i}{\pi} \sum_{k,n=1}^{\infty} \frac{\mu(k)}{n^2 k} \left[\frac{1}{(nke^z - 1)^2} + \frac{2}{nke^z - 1} + \frac{1}{(nke^z + 1)^2} - \frac{2}{nke^z + 1} \right] + \tag{7.4}$$
$$- \frac{12i}{\pi} e^{2z} = 2\pi i B(z)$$

Hence according to theorem 1 for all $y < \pi$

$$f(z) = -f^-(z) + B(z)$$

by the principle of analytic continuation and similarly for $y > -\pi$

$$f^-(z) = -f(z) + B(z)\ .$$

This implies that $f(z)$ and $f^-(z)$ can be continued analytically over the whole complex plane as a meromorphic function and for all z

$$f(z) + f^-(z) = B(z) \tag{7.5}$$

To prove the functional equation (3.7) observe that if ϱ is a complex zero of $\zeta(s)$ than so is $\overline{\varrho}$.

For $z \in H$ we have

$$f(z) = \lim_{n\to\infty} \overline{\sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<T_n}} \frac{1}{(k_\varrho - 1)!} \overline{\frac{d^{k_\varrho - 1}}{ds^{k_\varrho - 1}}} \left[(s-\varrho)^{k_\varrho} \frac{e^{sz}\zeta(s-1)}{\zeta(s)} \right]_{s=\varrho}}$$

and further denoting $s = \sigma + i\tau$

$$f(z) = \lim_{n\to\infty} \overline{\sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<T_n}} \frac{1}{(k_\varrho - 1)!} \overline{\frac{\partial^{k_\varrho - 1}}{\partial\sigma^{k_\varrho - 1}}} \left[(s-\varrho)^{k_\varrho} \frac{e^{sz}\zeta(s-1)}{\zeta(s)} \right]_{s=\varrho}}\ .$$

Now since $\zeta(\overline{s}) = \overline{\zeta(s)}$ we get

$$f(z) = \lim_{n\to\infty} \overline{\sum_{\substack{\varrho \\ 0<\mathrm{Im}\,\varrho<T_n}} \frac{1}{(k_\varrho - 1)!} \frac{\partial^{k_\varrho - 1}}{\partial\sigma^{k_\varrho - 1}} \left[(\overline{s} - \overline{\varrho})^{k_\varrho} \frac{e^{\overline{s}\overline{z}} \zeta(\overline{s} - 1)}{\zeta(\overline{s})} \right]_{s=\varrho}}$$

and finally

$$\begin{aligned} f(z) &= \lim_{n\to\infty} \overline{\sum_{\substack{\varrho \\ -T_n \mathrm{Im}\,\varrho<0}} \frac{1}{(k_\varrho - 1)!} \frac{d^{k_\varrho - 1}}{ds^{k_\varrho - 1}} \left[(s - \varrho)^{k_\varrho} \frac{e^{s\overline{z}} \zeta(s - 1)}{\zeta(s)} \right]_{s=\varrho}} = \\ &= \overline{f^{-}(\overline{z})} \end{aligned} \tag{7.6}$$

Next using (6.5) we have for $z \in H$

$$f(z) = \overline{f^{-}(\overline{z})} = \overline{-f(\overline{z})} + \overline{B(\overline{z})} = -\overline{f(\overline{z})} + B(z)$$

and by complex conjugation for $z \in H^{-}$ and by the principle of analytic continuation for z with $\mathrm{Im}\, z = 0$. This proves (3.7).

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