Functiones et Approximatio
XXVIII (2000), 75–81

To Professor Włodzimierz Staś on his
75th birthday

ON THE DISTRIBUTION IN THE ARITHMETIC PROGRESSIONS OF REDUCIBLE QUADRATIC POLYNOMIALS IN SHORT INTERVALS

G. Coppola and S. Salerno

Abstract: In this paper we study the distribution in the arithmetic progressions (modulo a product of two primes) of reducible quadratic polynomials $(an+b)(cn+d)$ in short intervals, i.e. when $n \in [x, x+H]$, $H = o(x)$; here $H = x^{\vartheta}$, with $\vartheta \in]3/4, 1[$. Using Large Sieve techniques we get results beyond the classical level ϑ, reaching $3\vartheta - 3/2$; these also improve the results of Salerno and Vitolo [6] in "large" intervals ($\vartheta = 1$) obtaining level $3/2$ instead of $4/3$.

1. Introduction and statement of the results

In this paper we study the distribution in the arithmetic progressions of the polynomial sequence $n(n+2)$ or, also, of sequences of reducible quadratic polynomials, in short intervals.

We expect that, on average over the odd square-free moduli d, the following estimate holds

$$\sum_{\substack{x<n\leq x+h \\ n(n+2)\equiv 0 (mod\ d)}} 1 = \frac{h}{d} \sum_{\substack{t|d \\ d/(x+h)<t<x+h}} 1 + R_d(x,h)$$

where $h = x^{\vartheta}$ is the length of the short interval and $R_d(x,h)$ is a "good" error term, on average over $d \sim D$, i.e. over $D < d \leq 2D$.

When this happens with $D = x^{\alpha-\varepsilon}$ ($\varepsilon \in]0, \alpha[$), we say that the **level of distribution** of our sequence (here $n(n+2)$), in the arithmetic progressions, is (at least) α .

For example, by standard arguments the distribution level of $n(n+2)$ when $n \in [x, x+x^{\vartheta}]$ is at least ϑ. For instance, we refer to [2], where classical sieve methods are used.

A higher distribution level of a sequence in the arithmetic progressions is usually reached through the introduction of bilinear forms.

1991 Mathematics Subject Classification: 11N25

In the bilinear form we consider, we will use the bounded coefficients γ_q and δ_r (with $q \sim Q$ and $r \sim R$).

Here the level of distribution is obviously $\log(QR)/\log(x)$.

An important example of non-trivial treatment of the bilinear form of the error we quote is

$$\sum_{q\sim Q}\sum_{r\sim R}\gamma_q\delta_r \sum_{\substack{n\le x\\ n^2+1\equiv 0(qr)}} 1.$$

Iwaniec in 1978 [4] showed that the level of distribution in this case is 16/15. Using this, he was able to prove that $n^2+1 = P_2$ for infinitely many n (here P_k denotes an integer with at most k prime factors), whilst the trivial level of distribution, that is 1, enables one to obtain only P_3.

In our case, we study the bilinear form restricted to prime moduli, that is

$$\sum_{q\sim Q}\sum_{r\sim R}\gamma_q\delta_r \sum_{\substack{x<n\le x+h\\ n(n+2)\equiv 0(qr)}} 1,$$

where q and r are distinct primes (assuming $QR > 2h+2$).

This restriction is connected with the arithmetical nature of the problem and is essential in our proof, as well as in the result on this problem we quote in the sequel.

We hope in the future to manage also the case of general moduli.

We expect that the following estimate holds on average

$$\sum_{\substack{q\sim Q\\ r\sim R}}\gamma_q\delta_r \sum_{\substack{x<n\le x+h\\ n(n+2)\equiv 0(qr)}} 1 = \sum_{\substack{q\sim Q\\ r\sim R}}\gamma_q\delta_r\left(\frac{2h}{qr} + \sum_{\substack{x<n\le x+h\\ n\equiv 0(qr)}} 1 + \sum_{\substack{x<n\le x+h\\ n\equiv -2(qr)}} 1\right),$$

with a "good" error term, that is $\mathcal{O}(h^{1-\varepsilon})$.

In [6] Salerno and Vitolo obtain the distribution level 4/3 for long intervals, using Weil estimates of Kloosterman sums (see Lemma 3 of [3]).

In this paper we use the Large Sieve to improve the distribution level from 4/3 to 3/2 (when $h = x$) and, also, to generalize the results of [6] to short intervals.

The distribution level reached in this way for short intervals of the kind $[x, x+x^{\vartheta}]$ is $3\vartheta - 3/2$ and, hence, is better than the classical level (i.e. ϑ) when $\vartheta > 3/4$ (we include the case $\vartheta = 1$ of "long" intervals).

We plan to study the same problem by means of different techniques, namely the Dispersion method, in order to reach some improvement on the level of distribution.

Our results are as follows.

Theorem 1.1. *Let $x > 4$, $3/4 < \vartheta \le 1$, $x^{\vartheta} \le h \le x$, $0 < \varepsilon < (4\vartheta - 3)/10$; let $Q, R \in [1, h/2[$ and γ_q, δ_r be bounded arithmetical functions with support on the* *primes in the intervals* $]Q,2Q]$, $]R,2R]$, *respectively. Then, for* $(q,r)=1$

$$\begin{aligned}\sum_{\substack{q\sim Q\\ r\sim R}}\gamma_q\delta_r \sum_{\substack{x<n\le x+h\\ n(n+2)\equiv 0(qr)}} 1 &= \sum_{\substack{q\sim Q\\ r\sim R}}\gamma_q\delta_r\left(\frac{2h}{qr}+\sum_{\substack{x<n\le x+h\\ n\equiv 0(qr)}}1+\sum_{\substack{x<n\le x+h\\ n\equiv -2(qr)}}1\right)+O(h^{1-\varepsilon}),\end{aligned} \tag{1.1}$$

provided that $R\le h^2/x^{1+3\varepsilon}$, $Q\le h/x^{1/2+2\varepsilon}$, $QR>2h+2$.

As an application we get now the level of distribution $3(\vartheta-1/2)$:

Corollary 1.2. *Let* x, ϑ, ε, q,r, γ_q,δ_r *be as above; let* $h=x^{\vartheta}$. *Then the estimate* (1.1) *of Theorem 1.1 holds for* $QR=x^{3\vartheta-3/2-5\varepsilon}$.

As in [6] we can easily generalize the results of Theorem 1.1 and of Corollary 1.2 to reducible quadratic polynomials, getting

Corollary 1.3. *Let* $(an+b)(cn+d)$ *be a polynomial without fixed divisors. Let* x, ϑ, h, ε, q, r, γ_q *and* δ_r *be as above, with* qr *coprime with* $[a, c, ad-bc]$. *Then the same conclusion of Theorem 1.1 and of Corollary 1.2 holds true with* $(an+b)(cn+d)$ *in place of* $n(n+2)$.

The paper is organized as follows:

- in section 2 we give the proof of Theorem 1.1, based on the Large Sieve;
- in section 3 we give in fact the proof of the variant of the Large Sieve involving reciprocals (*mod* q), that we use in section 2 (Lemma 3.2).

The authors wish to thank Professor H. Iwaniec for helpful comments and discussions.

2. Proof of Theorem 1.1 (and of the Corollaries)

Let q and r be distinct primes (as in the sequel) and define

$$I\equiv\sum_{q\sim Q}\sum_{r\sim R}\gamma_q\delta_r\sum_{\substack{x<n\le x+h\\ n(n+2)\equiv 0(qr)}}1.$$

Since q and r are distinct primes we get

$$I=\sum_{\substack{q\sim Q\\ r\sim R}}\gamma_q\delta_r\left(\sum_{\substack{x<n\le x+h\\ n\equiv 0(r)\\ n\equiv -2(q)}}1+\sum_{\substack{x<n\le x+h\\ n\equiv 0(q)\\ n\equiv -2(r)}}1+\sum_{\substack{x<n\le x+h\\ n\equiv 0(qr)}}1+\sum_{\substack{x<n\le x+h\\ n\equiv -2(qr)}}1\right).$$

Thus to prove (1.1) it will suffice to prove

$$\sum_{\substack{q\sim Q\\ r\sim R}}\gamma_q\delta_r\sum_{\substack{x<n\le x+h\\ n\equiv 0(r)\\ n\equiv -2(q)}}1=h\sum_{\substack{q\sim Q\\ r\sim R}}\frac{\gamma_q\delta_r}{qr}+\mathcal{O}(h^{1-\varepsilon}). \tag{2.1}$$

Clearly, it is sufficient to show that

$$\Sigma \equiv \sum_{\substack{q\sim Q\\ r\sim R}} \gamma_q\delta_r \sum_{\substack{x<n\le x+h\\ n\equiv 0(r)\\ n\equiv -2(q)}} 1 - \sum_{\substack{q\sim Q\\ r\sim R}} \gamma_q\delta_r \sum_{\substack{x<n\le x+h\\ n\equiv 0(r)}} \frac{1}{q} \ll h^{1-\varepsilon}; \tag{2.2}$$

in fact the difference between the main terms of (2.1) and (2.2) is negligible, because

$$\sum_{\substack{q\sim Q\\ r\sim R}} \gamma_q\delta_r \sum_{\substack{x<n\le x+h\\ n\equiv 0(r)}} \frac{1}{q} - h \sum_{\substack{q\sim Q\\ r\sim R}} \frac{\gamma_q\delta_r}{qr} = \sum_{\substack{q\sim Q\\ r\sim R}} \gamma_q\delta_r \mathcal{O}(1/q) \ \ll \ R$$

and $R \ll h^{1-\varepsilon}$ by hypothesis.

We have, by the orthogonality of the additive characters (see [7], Chapter 1, Lemma 5)

$$\Sigma = \sum_{r\sim R} \delta_r \sum_{q\sim Q} \frac{\gamma_q}{q} \sum_{j<q} \sum_{x/r<m\le (x+h)/r} e_q(j(m+2\overline{r})). \tag{2.3}$$

We use Mellin's transform in order to detect the integers m belonging to $]x/r,(x+h)/r]$ (i.e. to express the characteristic function of this interval). For this purpose we use a suitable kernel $K_r(\tau)$ with the property:

$$\int_{\mathbf{R}} |K_r(\tau)|d\tau \ll \log x;$$

in this way, letting $m \simeq x/R$ mean $m \in]x/2R, 2x/R]$ we get from (2.3)

$$\begin{aligned}\Sigma &= \sum_{r\sim R} \delta_r \sum_{q\sim Q} \frac{\gamma_q}{q} \sum_{j<q} \sum_{m\simeq x/R} e_q(j(m+2\overline{r})) \int_{\mathbf{R}} K_r(\tau) m^{i\tau} d\tau \\ &= \sum_{r\sim R} \delta_r \int_{\mathbf{R}} K_r(\tau) \sum_{q\sim Q} \frac{\gamma_q}{q} \sum_{j<q} e_q(2j\overline{r}) \sum_{m\simeq x/R} m^{i\tau} e_q(jm) d\tau;\end{aligned}$$

hence, by Hölder inequality (recalling that $\gamma_q,\delta_r \ll 1$) we get (for a certain $\tau \in \mathbf{R}$)

$$\begin{aligned}\Sigma &\ll (\log x) \sum_{r\sim R} \left| \sum_{q\sim Q} \frac{\gamma_q}{q} \sum_{j<q} \left(\sum_{m\simeq x/R} m^{i\tau} e_q(jm) \right) e_q(2j\overline{r}) \right| \\ &\ll (\log x) \sum_{r\sim R} \left| \sum_{q\sim Q} \frac{\gamma_q}{q} \sum_{j<q} \left(\sum_{m\simeq x/R} m^{i\tau} e_q(\overline{2}jm) \right) e_q(j\overline{r}) \right|.\end{aligned}$$

Thus, applying Cauchy inequality we obtain (since q is prime and hence the sum over j is over reduced classes $\bmod q$)

$$\Sigma \ll (\log x)\sqrt{R} \sqrt{\sum_{r\sim R} \left| \sum_{q\sim Q} \frac{\gamma_q}{q} \sideset{}{^*}\sum_{j\le q} \left(\sum_{m\simeq x/R} m^{i\tau} e_q(\overline{2}jm) \right) e_q(j\overline{r}) \right|^2},$$

whence, by Lemma 3.2 (see next section)

$$\begin{aligned}\Sigma &\ll (\log x)\sqrt{R}\sqrt{x^{2\delta}(R+Q^2)\sum_{q\sim Q}\frac{1}{q^2}\sum_{j\leq q}\Big|\sum_{m\simeq x/R} m^{i\tau}e_q(jm)\Big|^2} \\ &\ll x^{2\delta}\sqrt{R}\sqrt{(R+Q^2)\sum_{q\sim Q}\frac{1}{q}\frac{x}{R}} \ll x^{2\delta}\sqrt{(R+Q^2)x}.\end{aligned} \tag{2.4}$$

In order to make $\Sigma \ll h^{1-\varepsilon}$ it suffices, by (2.2), to choose in (2.4). for example, $\delta = \varepsilon/4$ and

$$R \leq (h^2/x)x^{-3\varepsilon}, \quad Q \leq (h/x^{1/2})x^{-2\varepsilon},$$

which give Theorem 2. (Therefore, this approach sets no limits for ϑ, apart from the trivial, i.e. $\vartheta > 1/2$; however, the level of distribution becomes greater than ϑ, as already stated, only for $\vartheta > 3/4$). ■

As regards the proof of Corollary 1.2, it is immediate from Theorem 1.1, choosing $R = (h^2/x)x^{-3\varepsilon}$, $Q = (h/x^{1/2})x^{-2\varepsilon}$. Corollary 1.3 can be proved along the same lines of the proof of Corollary 1.2 in [6]. ■

3. Lemmas (first and second version of the Large Sieve)

Lemma 3.1. *Let Q and N be natural numbers and $\{a_n\}$ a sequence of complex numbers; then*

$$\sum_{q\leq Q}\sum_{a\leq q}{}^{*}\Big|\sum_{n\leq N} a_n e_q(an)\Big|^2 \ll (N+Q^2)\sum_{n\leq N}|a_n|^2, \tag{3.1}$$

where the $*$ means (as usual) that the summation is over reduced residue classes $(\mathrm{mod}\, q)$.

This is the well known Large Sieve inequality (for a proof see, for instance, [1], p. 13).

From this Lemma we derive, in a quite elementary way, a Large Sieve in which we deal with inverse residue classes:

Lemma 3.2. *Let Q and N be natural numbers and $\lambda_{a,q}$ be complex numbers* $(\forall a, q \in N)$; *then $\forall \delta > 0$ we have*

$$\sum_{n\leq N}\Big|\sum_{\substack{q\leq Q\\(q,n)=1}}\sum_{a\leq q}{}^{*}\lambda_{a,q}e_q(a\overline{n})\Big|^2 \ll (QN)^{\delta}(N+Q^2)\sum_{q\leq Q}\sum_{a\leq q}{}^{*}|\lambda_{a,q}|^2. \tag{3.2}$$

Proof. We prove that

$$\sum_{q\le Q}\sum_{a\le q}{}^{*}\left|\sum_{\substack{n\le N\\(n,q)=1}} a_n e_q(a\overline{n})\right|^2 \ll (QN)^{\delta}(N+Q^2)\sum_{n\le N}|a_n|^2, \tag{3.3}$$

since (3.2) will follow by the Duality Principle (see for instance [5], p. 134).

In order to do this we expand the square in (3.3) obtaining

$$\sum_{q\le Q}\sum_{a\le q}{}^{*}\sum_{\substack{n_1\le N\\(n_1,q)=1}}\sum_{\substack{n_2\le N\\(n_2,q)=1}} a_{n_1}\overline{a_{n_2}}e_q(a(\overline{n_1}-\overline{n_2}))$$

whence, setting $b=a\overline{n_1 n_2}$ we get (since $(n_1,q)=(n_2,q)=1$)

$$\sum_{q\le Q}\sum_{b\le q}{}^{*}\left|\sum_{\substack{n\le N\\(n,q)=1}} a_n e_q(bn)\right|^2,$$

i.e. the left hand side of (3.3); hence it will suffice to prove the following variant of the Large Sieve:

$$\sum_{q\le Q}\sum_{a\le q}{}^{*}\left|\sum_{\substack{n\le N\\(n,q)=1}} a_n e_q(an)\right|^2 \ll (QN)^{\delta}(N+Q^2)\sum_{n\le N}|a_n|^2. \tag{3.4}$$

The left hand side here equals

$$\sum_{q\le Q}\sum_{a\le q}{}^{*}\left|\sum_{n\le N}\sum_{\substack{d\mid n\\d\mid q}}\mu(d)a_n e_q(an)\right|^2=\sum_{q\le Q}\sum_{a\le q}{}^{*}\left|\sum_{\substack{d\le N\\d\mid q}}\mu(d)\sum_{m\le N/d}a_{md}e_q(amd)\right|^2$$

whence (by Cauchy inequality and the estimate for the divisor function $\tau(q)\ll q^{\delta}$)

$$\sum_{q\le Q}\sum_{a\le q}{}^{*}\left|\sum_{\substack{n\le N\\(n,q)=1}} a_n e_q(bn)\right|^2 \ll Q^{\delta}\sum_{d\le N} d\sum_{g\le \frac{Q}{d}}\sum_{c\le g}{}^{*}\left|\sum_{m\le N/d} a_{md}e_g(cm)\right|^2.$$

Then, applying the Large Sieve, i.e. by (3.1), we get

$$\begin{aligned}\sum_{q\le Q}\sum_{a\le q}{}^{*}\left|\sum_{\substack{n\le N\\(n,q)=1}} a_n e_q(bn)\right|^2 &\ll Q^{\delta}\sum_{d\le N} d\left(\frac{N}{d}+\frac{Q^2}{d^2}\right)\sum_{m\le N/d}|a_{md}|^2\\
&\ll Q^{\delta}\left(N\sum_{n\le N}|a_n|^2\sum_{\substack{d\mid n\\d\le N}}1+Q^2\sum_{n\le N}|a_n|^2\sum_{\substack{d\mid n\\d\le N}}\frac{1}{d}\right)\\
&\ll Q^{\delta}(N+Q^2)N^{\delta}\sum_{n\le N}|a_n|^2,\end{aligned}$$

whence the estimate (3.4), which completes the proof of Lemma 3.2. ■

References

[1] E. Bombieri, *Le Grande Crible dans la theorie analitique des nombres*, Asterisque 18, Societè mathematique de France 1974.

[2] H. Halberstam, H. E. Richert, *Sieve Methods*, Academic Press, London 1974.

[3] C. Hooley, *On the greatest prime factor of a quadratic polynomial*, Acta Math. **117** 281–299, 1967.

[4] H. Iwaniec, *Almost-primes Represented by Quadratic Polynomials*, Inventiones Math. Vol. 47, 171–188, 1978.

[5] H. L. Montgomery, *Ten Lectures on the Interface between Analytic Number Theory and Harmonic Analysis*, CBMS n. 84, AMS ed.

[6] S. Salerno, A. Vitolo, *On the distribution in the arithmetic progressions of reducible quadratic polynomials*, Izwestiya Rossiyskoy AN, Math. Series Vol. 58 n. 4, 211–223, 1994.

[7] I. M. Vinogradov, *The Method of Trigonometrical Sums in the Theory of Numbers*, Interscience Publishers LTD., London 1954.

Received: 29 Nov 1999