Functiones et Approximatio
XXXII (2004), 57–66

THE WALSH TRANSFORM OF WAVELET TYPE SYSTEMS: DIVERGENCE ALMOST EVERYWHERE

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Abstract: The main result of the paper is the following: for the Walsh transform of a wavelet type system on $[0,1]$, there is an integrable function whose Fourier expansion with respect to the transformed system is divergent almost everywhere on $[0,1]$. This is an extension of the result by K. S. Kazarian and A. S. Sargsian [9] for the bounded Ciesielski system, i.e. the Walsh transform of the Franklin system.
Keywords: Walsh transform, wavelet, divergence a.e.

1. Introduction

In 1975 S.V. Bočkariev [1] has proved that for any uniformly bounded ONS $\{f_n\}_{n\in\mathbb{N}}$ on $[0,1]$ there is a function $f \in L_1(0,1)$ whose Fourier series in the system $\{f_n\}_{n\in\mathbb{N}}$ diverges unboundedly at every point of a set $E \subset [0,1]$ of positive measure. In general, it is not possible to assert the existence of a function $f \in L_1(0,1)$ whose Fourier series in the system $\{f_n\}_{n\in\mathbb{N}}$ diverges a.e. on $[0,1]$ – this follows from K.S. Kazarian [8], where he has proved that for any set $G \subset [0,1]$, $0 < |G| < 1$, there is a uniformly bounded CONS $\{f_n\}_{n\in\mathbb{N}}$ with the property that for any function $f \in L_1(0,1)$ the Fourier series of f in $\{f_n\}_{n\in\mathbb{N}}$ converges to f a.e. on G.

However, for certain uniformly bounded ONS one can find a function whose Fourier series diverges unboundedly a.e. For the trigonometric system, the existence of such a function is a classical result by A.N. Kolmogorov [11]. An analogous result for Walsh system has been obtained by E.M. Stein [17] (see also [16]). Moreover, K.S. Kazarian and A.S. Sargsian [9] have proved the same for the bounded Ciesielski system – here, by the bounded Ciesielski system we mean the bounded system of polygonals which arises from Franklin system in the same way as Walsh system arises from Haar system (this system was introduced by Ciesielski in [4]).

2001 Mathematics Subject Classification: primary 40A30, 42C15, secondary 42C40.

In this paper we extend the result of K.S. Kazarian and A.S. Sargsian to the Walsh transform of wavelet type systems on $[0,1]$. By *a biorthogonal wavelet type system* on $[0,1]$ we mean a system $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ of functions in $L_2(0,1)$, biorthogonal with respect to the scalar product in $L_2(0,1)$, where $N \geqslant -1$ is given, and such that for any $x \in [0,1]$

$$\begin{cases} |\psi_n(x)|, |\phi_n(x)| \leqslant C, & -N \leqslant n \leqslant 1, \\ |\psi_{2^j+k}(x)|, |\phi_{2^j+k}(x)| \leqslant 2^{\frac{j}{2}} S(2^j|x - \frac{k}{2^j}|), & j \geqslant 0, \quad 1 \leqslant k \leqslant 2^j, \end{cases} \tag{1}$$

where $S : [0,\infty) \to \mathbb{R}$ is a nonincreasing function satisfing some kind of integral condition. In this paper we assume that

$$\int_0^{\infty} \log(1+x)S(x)dx < +\infty. \tag{2}$$

In our main result – Theorem 1.1 below – we assume condition (2), the linear density of $\{\psi_n\}_{n=-N}^{\infty}$ in $L_2(0,1)$ and the Riesz system property in $L_2(0,1)$ for $\{\psi_n\}_{n=-N}^{\infty}$. These assumptions guarantee in particular that for any $f \in L_1(0,1)$

$$\{P_j f\}_{j\in\mathbb{N}} \text{ converges to } f \text{ in } L_1 - \text{norm}, \tag{3}$$

where P_j denote the partial sum operators on $L_1(0,1)$, i.e. $P_j f(x) = \sum_{n=1}^{2^j} (f, \phi_n)\psi_n$. Indeed, one can show for the kernels $K_j(x,y) = \sum_{n=1}^{2^j} \phi_n(x)\psi_n(y)$ for almost all $x \in [0,1]$ there is a set $I_x \subset [0,1]$ with $|I_x| = 1$ such that for $y \in I_x$

$$|K_j(x,y)| \leqslant C 2^j R(2^j|x-y|), \quad \text{where} \quad R(t) = \sum_{i=0}^{\infty} 2^i S(2^{i-1}\max(t,1)),$$

and condition (2) implies that $\int_0^{\infty} R(t)dt < \infty$, so consequently $\|P_j\|_1 \leqslant C$ for all $j \geqslant 0$, and the system $\{\psi_n\}_{n=-N}^{\infty}$ is a basis in $L_1(0,1)$ (more details can be found in [18] or [10], which contains an earlier analogous result for wavelets on R). Moreover, under these assumptions it follows that $\{\phi_n\}_{n=-N}^{\infty}$ is also a Riesz system in $L_2(0,1)$.

Let us note that if we know (3) in advance, then it is enough to assume

$$\int_0^{\infty} S(x)dx < +\infty, \tag{4}$$

and moreover, the assumption on linear density of $\{\psi_n\}_{n=-N}^{\infty}$ in $L_2(0,1)$ is not needed (however, then we need to assume the Riesz system property for $\{\phi_n\}_{n=-N}^{\infty}$).

The current paper should be considered as a complement to B. Wolnik [19], where the convergence a.e. of Fourier series with respect to Walsh transform of wavelet type systems is studied. The main results of [19] are the convergence a.e. of the Fourier series with respect to the Walsh transforms of wavelet type systems for $f \in L_p(0,1)$, $p > 1$, and Cesàro summability of such expansions for $f \in L_1(0,1)$.

Let us mention some examples of systems satisfying assumptions of Theorem 1.1, other than Haar system or the Franklin system. This list includes e.g. both orthonormal and biorthogonal spline systems, see e.g. [5], [6], Carleson's system (see [3] for the construction, and [13] for the estimates for the biorthogonal system), the orthogonal system of trigonometric conjugates to the Franklin functions (see [2], and [18] for estimates (1)), various adaptations of Daubechies' wavelets to the interval $[0,1]$ (see e.g. [12]) or biorthogonal systems consisting of rational functions of uniformly bounded degrees [14].

We consider uniformly bounded systems which arise from $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ in the same way as the Walsh functions are obtained from the Haar functions.

Let $\{\chi_n\}_{n\in\mathbb{N}}$ and $\{w_n\}_{n\in\mathbb{N}}$ denote the Haar and Walsh functions, respectively. For any $j \geqslant 0$ we define the matrix

$$A_{k,l}^{(j)} = (w_{2^j+k}, \chi_{2^j+l}) = 2^{-\frac{j}{2}} w_k\left(\frac{2l-1}{2^{j+1}}\right), \quad k,l = 1,2,\dots,2^j \tag{5}$$

which is orthogonal and symmetric (see [4] or [16]).

Now, the Walsh transform of the wavelet type system $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ is defined by the formulae

$$\psi_n^b = \psi_n, \quad \phi_n^b = \phi_n \quad \text{for} \quad n = -N, \dots, 1$$

and for $j \geqslant 0$, $1 \leqslant k \leqslant 2^j$

$$\psi_{2^j+k}^b(x) = \sum_{l=1}^{2^j} A_{k,l}^{(j)} \psi_{2^j+l}(x), \quad \phi_{2^j+k}^b(x) = \sum_{l=1}^{2^j} A_{k,l}^{(j)} \phi_{2^j+l}(x).$$

Recall that if $\{\psi_n\}_{n\in\mathbb{N}}$ is Franklin system then $\{\psi_n^b\}_{n\in\mathbb{N}}$ is Ciesielski bounded system. One should remark that some questions concerning the Walsh transforms of spline systems of higher order have been first discussed in [15].

Let us recall that the series $\sum_{n=1}^{\infty} a_n$ is said to diverge unboundedly if

$$\limsup_{N\to\infty} \Big|\sum_{n=1}^{N} a_n\Big| = \infty.$$

The main result of this paper is the following:

Theorem 1.1. *Let the biorthogonal wavelet type system $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ satisfy conditions (1) and (2). If $\{\psi_n\}_{n=-N}^{\infty}$ is both linearly dense and a Riesz system in $L_2(0,1)$, then there is a function $f \in L_1(0,1)$ whose Fourier series in the system $\{\psi_n^b\}_{n=-N}^{\infty}$ unboundedly diverges a.e. on $[0,1]$.*

Let us recall that by a Riesz system in $L_2(0,1)$ we mean a system of functions $\{x_n\}_{n\in\mathbb{N}}$ such that for any sequence of coefficients $\{a_n\}_{n\in\mathbb{N}}$ we have $\frac{1}{A}\left(\sum_{n\in\mathbb{N}} a_n^2\right)^{\frac{1}{2}} \leqslant \|\sum_{n\in\mathbb{N}} a_n x_n\|_2 \leqslant A\left(\sum_{n\in\mathbb{N}} a_n^2\right)^{\frac{1}{2}}$, with the constant A independent of the coefficients.

1.1. Idea of the proof of Theorem 1.1

In [9] K.S.Kazarian and A.S.Sargsian have proved that there is a function from $L_1(0,1)$ whose Fourier series in Ciesielski bounded system diverges a.e. on $[0,1]$. One of the crucial steps in their proof is the following lemma concerning Franklin system:

Lemma 1.2. *Let $\{f_n\}_{n\in\mathbb{N}}$ be the Franklin system. There is a constant $\alpha > 0$ such that for all $m \in \mathbb{N}$ and $x \in [0,1]$*

$$\sum_{n=2^m+1}^{2^{m+1}} f_n^2(x) \geqslant \alpha \cdot 2^m.$$

Their proof of this lemma depends heavily on the concrete formulae for Franklin functions and is rather technical.

In general case we do not have such precise informations on $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$. However, in the general situation, we can prove weaker version of Lemma 1.2 (see Lemma 2.3 below) which is also sufficient for the proof of Theorem 1.1. The method we use is simpler and transparent.

Once we get Lemma 2.3, the rest of the proof is the same as in [9] but we sketch it for the reader's convenience.

2. Proof of Theorem 1.1

2.1. Auxiliary results

We introduce some notation. By $I_{j,k}$ we will denote the interval $\left[\frac{k-1}{2^j}, \frac{k}{2^j}\right]$ and for $n \in \mathbb{N}$ we define $n * I_{j,k}$ as the set $\{x \in [0,1] : \ |x - \frac{k}{2^j}| \leqslant \frac{n}{2^j}\}$. The ball $B(x_0, h)$ is defined similarly, i.e. $B(x_0,h) = \{x \in [0,1] : \ |x - x_0| \leqslant h\}$.

We start from an easy fact concerning systems $\{\psi_n^b, \phi_n^b\}_{n=-N}^{\infty}$:

Lemma 2.1. *Let $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ satisfy conditions (1), (2). Then $\{\psi_n^b, \phi_n^b\}_{n=-N}^{\infty}$ is a uniformly bounded biorthogonal system.*

If $\{\phi_n\}_{n=-N}^{\infty}$, $\{\psi_n^b\}_{n=-N}^{\infty}$ are Riesz systems in $L_2(0,1)$, then $\{\phi_n^b\}_{n=-N}^{\infty}$, $\{\psi_n^b\}_{n=-N}^{\infty}$ are also Riesz systems in $L_2(0,1)$.

Proof. Using (5), the fact that $|w_n(x)| = 1$ and the properties of S we get

$$|\psi^b_{2^j+k}(x) = |\sum_{l=1}^{2^j} 2^{-\frac{j}{2}} w_k\left(\frac{2l-1}{2^{j+1}}\right)\psi_{2^j+l}(x)| \leqslant |\sum_{l=1}^{2^j} 2^{-\frac{j}{2}} 2^{\frac{j}{2}} S(2^j|x-\frac{l}{2^j}|) \leqslant C.$$

The biorthogonality of $\{\psi^b_n, \phi^b_n\}_{n=-N}^{\infty}$ and the property of being a Riesz system follows by the orthogonality of the Walsh matrix. ■

The next two results concern the systems $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$.

Lemma 2.2. *Let $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ satisfy conditions (1), (2). There are constants $\alpha > 0$ and $\gamma > 0$ such that for each $0 < h < \frac{1}{2}$ there exists $j_h \in \mathbb{N}$ such that for any $x_0 \in [0,1]$ and $j > j_h$*

$$|\{x \in B(x_0,h) : \sum_{k=1}^{2^j} \psi^2_{2^j+k}(x) \geqslant \alpha 2^j\}| \geqslant \gamma |B(x_0,h)|.$$

Proof. Conditions (1) and (2) imply that there is a constant $A > 0$ such that $\|\phi_n\|_2, \|\psi_n\|_2 \leqslant A$, $n \geqslant -N$. In turn, this and the biorthogonality condition gives

$$1 = (\psi_n, \phi_n) \leqslant \|\psi_n\|_2 \|\phi_n\|_2 \leqslant A\|\psi_n\|_2 \quad \text{and} \quad \frac{1}{A} \leqslant \|\psi_n\|_2 \leqslant A.$$

Let j_0 be any fixed natural number for which we have

$$\int_{2^{j_0}}^{\infty} S^2(x)dx \leqslant \frac{1}{4A}, \tag{6}$$

then

$$\int_{2^{j_0} * I_{j,k}} \psi^2_{2^j+k}(x)dx \geqslant \frac{1}{2A}.$$

If $\alpha = \frac{1}{8 \cdot 2^{j_0} A}$ and $\gamma = \frac{1}{16 \cdot 2^{j_0} \cdot S^2(0) A}$ then for $D_{j,k} = \{x \in 2^{j_0} * I_{j,k} : \psi^2_{2^j+k}(x) \geqslant \geqslant \alpha 2^j\}$ we can write

$$\frac{1}{2A} \leqslant \int_{2^{j_0} * I_{j,k}} \psi^2_{2^j+k}(x)dx = \int_{D_{j,k}} \psi^2_{2^j+k}(x)dx + \int_{2^{j_0} * I_{j,k} \setminus D_{j,k}} \psi^2_{2^j+k}(x)dx \leqslant$$
$$\leqslant |D_{j,k}| 2^j S^2(0) + |2^{j_0} * I_{j,k}| \alpha 2^j \leqslant |D_{j,k}| 2^j S^2(0) + \frac{1}{4A}.$$

Thus

$$|D_{j,k}| \geqslant \frac{1}{4S^2(0) 2^j A} \geqslant 2\gamma |2^{j_0} * I_{j,k}|.$$

Now we fix $h \in (0, \frac{1}{2})$ and $x_0 \in [0,1]$. Let j_h be such that $h > 2 \cdot 2^{j_0 - j_h}$, where j_0 is as in (6). Now let $j > j_h$. We divide the interval $[0,1]$ into 2^{j-j_0-1} equal parts. Each interval of this partition has the length 2^{j_0+1-j} and is equal to the set $2^{j_0} * I_{j,k}$ for some k. Since $h > 2 \cdot 2^{j_0 - j_h} \geqslant |2^{j_0} * I_{j,k}|$, so there is $s \in \mathbb{N}$ such that the set $B(x_0, h)$ contains s disjoint intervals of the form $2^{j_0} * I_{j,k}$ and can be covered by at most $s+2$ such disjoint intervals. Thus

$$
\begin{aligned}
&|\{x \in B(x_0,h) : \sum_{k=1}^{2^j} \psi^2_{2^j+k}(x) \geqslant \alpha 2^j\}| \geqslant \\
&\geqslant \sum_{i=1}^{s} |\{x \in 2^{j_0} * I_{j,k_i} : \psi^2_{2^j+k_i}(x) \geqslant \alpha 2^j\}| = \\
&= \sum_{i=1}^{s} |D_{j,k_i}| \geqslant \sum_{i=1}^{s} 2\gamma |2^{j_0} * I_{j,k_i}| \geqslant \frac{s}{s+2} 2\gamma |B(x_0,h)| \geqslant \gamma |B(x_0,h)|. \qquad \blacksquare
\end{aligned}
$$

Using the above lemma we get an analogue of Lemma 1.2:

Lemma 2.3. *Let $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ satisfy the decay conditions (1), (2). There is $\alpha > 0$ such that for a.e. $x \in [0,1]$ the inequality*

$$
\sum_{k=1}^{2^j} \psi^2_{2^j+k}(x) \geqslant \alpha 2^j \tag{7}
$$

holds for infinitely many j.

Proof. Let α be such as in Lemma 2.2. Denote:

$$
\begin{aligned}
A_j &= \{x \in [0,1] : \sum_{k=1}^{2^j} \psi^2_{2^j+k}(x) \geqslant \alpha 2^j\}, \quad j \in \mathbb{N} \\
B_m &= \bigcup_{j \geqslant m} A_j, \quad m \in \mathbb{N} \\
H &= \bigcap_{m=1}^{\infty} B_m = \limsup_{j \to \infty} A_j.
\end{aligned}
$$

We need to show that $|H| = 1$.

From the fact that $\mathbb{1}_H \in L_1(0,1)$ we have

$$
\lim_{h \to 0} \frac{1}{2h} \int_{x_0-h}^{x_0+h} \mathbb{1}_H(x)dx = \mathbb{1}_H(x_0) \tag{8}
$$

for each Lebesgue point of $\mathbb{1}_H$, and consequently - for almost every $x_0 \in (0,1)$. Let x_0 be any Lebesgue point of $\mathbb{1}_H$. By Lemma 2.2 for any $h \in (0, x_0 \wedge (1-x_0))$ there is $j_h \in \mathbb{N}$ such that for $j > j_h$

$$
|A_j \cap B(x_0,h)| \geqslant \gamma |B(x_0,h)| \geqslant \gamma h,
$$

which implies that for all $m \geqslant 1$

$$|B_m \cap B(x_0, h)| \geqslant \gamma h.$$

The last statment is equivalent to

$$\frac{1}{2h} \int_{x_0-h}^{x_0+h} \mathbb{1}_{B_m}(x)dx \geqslant \frac{1}{2}\gamma$$

for all $m \in \mathbb{N}$. Since $\mathbb{1}_{B_m} \searrow \mathbb{1}_H$, we have

$$\frac{1}{2h} \int_{x_0-h}^{x_0+h} \mathbb{1}_H(x)dx = \frac{1}{2h} \int_{x_0-h}^{x_0+h} \lim_{m\to\infty} \mathbb{1}_{B_m}(x)dx = \\ = \lim_{m\to\infty} \frac{1}{2h} \int_{x_0-h}^{x_0+h} \mathbb{1}_{B_m}(x)dx \geqslant \frac{1}{2}\gamma.$$

By (8), this means that $\mathbb{1}_H(x_0) \geqslant \frac{\gamma}{2} > 0$, and consequently $\mathbb{1}_H(x_0) = 1$. Therefore $|H| = 1$. ■

2.2. Final part of the proof of Theorem 1.1

Now we are ready to complete the proof of the main theorem. As we have said at the begining, this part repeats the arguments of Kazarian-Sargsian's paper [9], but we present it for the sake of completeness.

Proof of Theorem 1.1: Note that for each $g \in L_1(0,1)$ and $K \geqslant 0$ we have

$$\sum_{n=2^K+1}^{2^{K+1}} (g, \phi_n)\psi_n = \sum_{n=2^K+1}^{2^{K+1}} (g, \phi_n^b)\psi_n^b,$$

thus from (3) we get

$$\sup_{m>K} \| \sum_{n=2^K+1}^{2^m} (g, \phi_n^b)\psi_n^b\|_1 \to 0 \quad \text{as} \quad K \to \infty.$$

Therefore, it is enough to show that for any $\varepsilon > 0$ there is a function $g_\varepsilon \in L_1(0,1)$ such that the Fourier series of g_ε is unboundedly divergent on a set E_ε with $|E_\varepsilon| \geqslant 1 - \varepsilon$.

Suppose that for some $\varepsilon > 0$ and for any $f \in L_1(0,1)$ the set

$$\{x \in [0,1] : \limsup_{N\to\infty} |\sum_{n=1}^{N} (f, \phi_n^b)\psi_n^b(x)| < \infty\}$$

has measure at least ε.

Then it follows from Saks Theorem (see e.g. [7], p. 26) that there is a set $T \subset [0,1]$ such that

$$\text{(A)} \quad \forall f \in L_1(0,1) \quad \limsup_{N\to\infty} \Big|\sum_{n=1}^{N}(f,\phi_n^b)\psi_n^b(x)\Big| < \infty \text{ a.e.on } T,$$

$$\text{(B)} \quad \forall f \in L_1(0,1)\setminus F \quad \limsup_{N\to\infty} \Big|\sum_{n=1}^{N}(f,\phi_n^b)\psi_n^b(x)\Big| = \infty \text{ a.e.on } T^c.$$

where F is a subset of $L_1(0,1)$ of first category. What more, $|T| \geqslant \varepsilon$.

Now, we refer to S.V. Bočkariev [1]. In [1], he has proved that for any uniformly bounded ONS on $[0,1]$, there is an intergrable function whose Fourier series in this system diverges unboundedly on a set of positive measure. Analyzing his proof, one finds that his argument can be applied also to a uniformly bounded biorthogonal system $\{f_n, g_n\}$ satisfying an additional condition that both $\{f_n\}$, $\{g_n\}$ are Riesz systems. Thus, the following version of Bočkariev's result is true:

Theorem 2.4. *Let $\{f_n, g_n\}_{n\in\mathbb{N}}$ be an uniformly bounded biorthogonal system of functions on $[0,1]$. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ are the Riesz systems. Let T be a set of positive measure, and suppose that there is $\beta > 0$ such that for a.e. $x \in T$ and for every $N \in \mathbb{N}$ the inequality*

$$\sum_{n=2^N p+1}^{2^N(p+1)} f_n^2(x) \geqslant \beta 2^N \tag{9}$$

holds for infinitely many p. Then there is $f \in L_1(0,1)$ and a set of positive measure $T_1 \subset T$ such that the series $\sum_{n=1}^{\infty}(f,g_n)f_n(x)$ unboundedly diverges on T_1.

(Moreover, for $\{f_n, g_n\}$ as in Theorem 2.4, it is not hard to prove the existence of set T satisfying conditions of Theorem 2.4 – this can be done by arguments analogous to those used in [1] for a bounded orthonormal system; therefore for any uniformly bounded biorthogonal system $\{f_n, g_n\}_{n\in\mathbb{N}}$ functions on $[0,1]$ such that $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ are the Riesz systems there exists integrable function whose Fourier series in $\{f_n\}_{n\in\mathbb{N}}$ is unboundedly divergent on a set of positive measure.)

It follows by Lemma 2.3 that there is $\alpha > 0$ such that for a.e. $x \in [0,1]$ and for all $N \in \mathbb{N}$ the inequality

$$\sum_{n=2^N p+1}^{2^N(p+1)} (\psi_n^b(x))^2 \geqslant \alpha 2^N \tag{10}$$

[1] The main change needed to treat the biorthogonal case is that one should analyse the kernels $\sum_k \phi_k \otimes \psi_k$ instead of $\sum_k \varphi_k \otimes \varphi_k$ from the orthogonal case.

holds for infinitely many p. Thus, the set T and the system $\{\psi_n^b, \phi_n^b\}_{n=-N}^{\infty}$ satisfy the assumptions of Theorem 2.4, so we conclude that there is a function $f_0 \in L(0,1)$ such that the Fourier series of f_0 is unboundedly divergent on some $T_1 \subset T$ with $|T_1| > 0$, which contradicts (A). ■

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Received: 8 December 2003