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A REMARK ON SOLUTIONS OF FUNCTIONAL EQUATIONS OF RIEMANN'S TYPE

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Abstract: It is proved that a general functional equation of the Riemann type with multiple gamma factors has non-trivial solutions in the space of generalized Dirichlet series. Moreover, for a fixed functional equation, the space of such solutions has uncountable basis. The proof is based on Hecke's theory of Dirichlet series associated with modular forms for the groups $G(\lambda)$. This is in constrast with the situation in the extended Selberg class where there exist functional equations without non-trivial solutions. Presumably this holds for non-integer degrees d, but up to date was confirmed only for $0 \leqslant d < 5/3$. In the case of $d = 0$ or $d = 1$ the space of solutions belonging to the extended Selberg class, through non-trivial, is finite dimensional.
Keywords: Riemann's functional equation, Selberg class, Dirichlet series, Hecke modular forms.

1. Introduction

A well known conjecture about the Selberg class $\mathcal{S}$ (see the survey papers [3] and [2] for definitions and basic properties) asserts that the degree d of any function in $\mathcal{S}$ is a non-negative integer. This conjecture, called the degree conjecture, reflects the arithmetic nature of the Selberg class, and is at present known in the range $0 \leqslant d < 5/3$, see [4]. However, the same conjecture is expected to hold for the extended Selberg class $\mathcal{S}^\sharp$ as well (see again [3] and [2]), and in fact it is known to hold in the same range $0 \leqslant d < 5/3$, see [4]. Note that $\mathcal{S}^\sharp$ is obtained from $\mathcal{S}$ by omitting the Ramanujan conjecture and the Euler product axioms, which are of distinct arithmetic nature. Such a situation suggests that the axioms defining $\mathcal{S}^\sharp$ still carry some arithmetic information. Inspection of such axioms shows that the only reasonable trace of arithmeticity comes from the requirement that functions in $\mathcal{S}^\sharp$ are ordinary Dirichlet series, *i.e.* the coefficients are supported on integers. Roughly speaking, in this paper we show that this is in fact the case, in the sense that the degree conjecture is false if one allows generalized Dirichlet series in $\mathcal{S}^\sharp$.

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We denote by $\mathcal{C}$ the class of generalized Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{l_n^s}$$

absolutely convergent for σ sufficiently large, where (l_n) is an increasing sequence of positive real numbers tending to ∞, such that $F(s)$ is an entire function of finite order. Moreover, given $\omega \in \mathbb{C}$ with $|\omega| = 1$, $Q > 0$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ with $\lambda_j > 0$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r)$ with $\Re\mu_j \geqslant 0$, we consider the functional equation

$$\Phi(s) = \omega\overline{\Phi}(1 - s), \tag{1}$$

where $\overline{f}(s) = \overline{f(\overline{s})}$ and

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s).$$

Further, we denote by

$$D(\boldsymbol{\lambda}, \boldsymbol{\mu}, Q, \omega)$$

the real vector space of solutions $F \in \mathcal{C}$ of functional equation (1).

Theorem. *Let $\boldsymbol{\lambda}, \boldsymbol{\mu}, Q, \omega$ be as above. Then $D(\boldsymbol{\lambda}, \boldsymbol{\mu}, Q, \omega)$ has an uncountable basis.*

The proof is based on Hecke's theory of Dirichlet series associated with modular forms for the groups $G(\lambda)$ with $\lambda > 2$, see Chapter II of Hecke [1].

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2. Proof

We first recall an important result from Hecke's theory [1]. Let $\lambda, k > 0$ and $\gamma \in \{-1, 1\}$ be real numbers, and let

$$\phi(s) = \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s}$$

be a Dirichlet series which converges somewhere. According to Hecke [1], $\phi(s)$ has signature (λ, k, γ) if $(s - k)\phi(s)$ is an entire function of finite order and

$$\left(\frac{\lambda}{2\pi}\right)^s \Gamma(s)\phi(s) = \gamma\left(\frac{\lambda}{2\pi}\right)^{k-s} \Gamma(k - s)\phi(k - s).$$

Lemma. [Hecke] *For every* $\lambda > 2$ *and* $k > 0$ *there exists a non-constant entire Dirichlet series of signature* $(\lambda, k, 1)$ *with real coefficients.*

Proof. This is a weak version of Theorem II of Chapter II of Hecke [1]. In fact, such a theorem asserts that for every $\lambda > 2$ and $k > 0$, the vector space of Dirichlet series with signature $(\lambda, k, 1)$ has infinite dimension. Hence, by a suitable linear combination, there exists a non-constant entire $\phi(s)$ of signature $(\lambda, k, 1)$. Therefore $\overline{\phi}(s)$ has signature $(\lambda, k, 1)$ as well, thus both $\phi(s)+\overline{\phi}(s)$ and $(\phi(s)-\overline{\phi}(s))/i$ have signature $(\lambda, k, 1)$ and real coefficients, and at least one is non-constant. ■

Let $\phi(s)$ be as in the Lemma and consider the normalization

$$f(s) = \phi(s + \frac{k-1}{2}),$$

so that $f(s)$ is entire of finite order and functional equation takes the form

$$(\frac{\lambda}{2\pi})^s \Gamma(s + \frac{k-1}{2}) f(s) = (\frac{\lambda}{2\pi})^{1-s} \Gamma(1 - s + \frac{k-1}{2}) f(1-s). \tag{2}$$

Using (2), we construct a solution of a rather general functional equation with one Γ-factor. We first consider real numbers $\alpha > 0, \eta \geqslant 0$ and $A > \pi^{-\alpha}$, we choose $\lambda = 2\pi A^{1/\alpha}$ and $k = 2\eta + \alpha$ in the Lemma and write

$$g(s) = f(\alpha s + \frac{1-\alpha}{2}).$$

Using the substitution $s \to \alpha s + \frac{1-\alpha}{2}$, a simple computation based on (2) and on identity

$$1 - (\alpha s + \frac{1-\alpha}{2}) = \alpha(1-s) + \frac{1-\alpha}{2}$$

shows that $g(s)$ satisfies

$$A^s \Gamma(\alpha s + \eta) g(s) = A^{1-s} \Gamma(\alpha(1-s) + \eta) g(1-s). \tag{3}$$

Let now α, η and A be as before, and let $\beta = \eta + i\theta$ with $\theta \in \mathbb{R}$. Writing

$$h(s) = g(s + i\frac{\theta}{\alpha}),$$

from (3) we deduce that $h(s)$ satisfies

$$\Psi(s) = \delta \overline{\Psi}(1-s), \tag{4}$$

where $\delta = A^{-2i\theta/\alpha}$ and

$$\Psi(s) = A^s \Gamma(\alpha s + \beta) h(s).$$

Moreover, it is easy to see that $h(s)$ is a generalized Dirichlet series of type

$$h(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^{\alpha s}} \tag{5}$$

which converges absolutely for σ sufficiently large, and that $h(s)$ is a non-constant entire function of finite order. We therefore have a non-trivial solution of the required type for the general functional equation with one Γ-factor, with the restrictions that $A > \pi^{-\alpha}$ and $\delta = A^{-2i\theta/\alpha}$.

Let now $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ be as in the Theorem and write

$$d = 2\sum_{j-1}^{r} \lambda_j.$$

For $j = 1, \dots, r$ choose $\alpha = \alpha_j = \lambda_j, \beta = \beta_j = \mu_j$ and $A = A_j > \pi^{-\lambda_j}$ in (4), write $\delta_j = A_j^{-2i\Im\mu_j/\lambda_j}$ and denote by $h_j(s)$ a corresponding solution satisfying the above properties. Writing

$$Q_0 = \prod_{j=1}^{r} A_j > \pi^{-d/2} \qquad \omega_0 = \prod_{j=1}^{r} \delta_j \qquad F_0(s) = \prod_{j=1}^{r} h_j(s) \tag{6}$$

and multiplying (4) for $j = 1, \dots, r$, we see that $F_0(s)$ is a non-trivial member of the class $\mathcal{C}$ satisfying the functional equation

$$\Phi_0(s) = \omega_o \overline{\Phi_0}(1-s), \tag{7}$$

where

$$\Phi_0(s) = Q_0^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F_0(s).$$

In order to remove the restrictions on Q_0 and ω_0 in (6), given $Q > 0$ and $\omega \in \mathbb{C}$ with $|\omega| = 1$ let $q \in \mathbb{R}$ and $\epsilon \in \mathbb{C}$ be such that

$$0 < q < Q\pi^{d/2} \qquad \omega_0 \epsilon^2 = \omega.$$

Writing $Q_0 = Q/q$ and

$$l(s) = \epsilon q^{-s}$$

we have that $Q_0 > \pi^{-d/2}$ and $l(s)$ satisfies

$$(\frac{Q}{Q_0})^s l(s) = \frac{\omega}{\omega_0}(\frac{Q}{Q_0})^{1-s} \bar{l}(1-s), \tag{8}$$

since $|\epsilon| = 1$. Therefore, by (7) and (8) the function

$$F(s) = l(s) F_0(s) \tag{9}$$

is a non-constant element of the vector space $D(\boldsymbol{\lambda}, \boldsymbol{\mu}, Q, \omega)$.

Finally, in order to show that $D(\boldsymbol{\lambda}, \boldsymbol{\mu}, Q, \omega)$ has uncountably many linearly independent elements, we observe that by (5), (6) and (9) the above function $F \in D(\boldsymbol{\lambda}, \boldsymbol{\mu}, Q, \omega)$ is a generalized Dirichlet series with coefficients supported on real numbers l of type

$$l = qn_1^{\lambda_1} \cdots n_r^{\lambda_r}, \qquad n_j \in \mathbb{N} \text{ and } 0 < q < Q\pi^{d/2}.$$

Moreover, for every $0 < q < Q\pi^{d/2}$ we get a non-trivial function $F_q \in$ $\in D(\boldsymbol{\lambda}, \boldsymbol{\mu}, Q, \omega)$, say. Denoting by G the countable semigroup of real numbers of the form $n_1^{\lambda_1} \cdots n_r^{\lambda_r}, n_j \in \mathbb{N}$, by the uniqueness theorem for generalized Dirichlet series we have that

$$\sum_{i=1}^{N} c_i F_{q_i}(s) = 0 \Rightarrow c_1 = \ldots = c_N = 0$$

provided $q_i/q_j \notin G$ for $i \neq j$. The result follows since clearly there are uncountably many $q \in (0, Q\pi^{d/2})$ non-equivalent modulo G.

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