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RANK TWO MATRICES WITH ELEMENTS OF NORM 1

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Abstract: If the determinant of a 3×3 matrix vanishes and its entries are unimodular complex numbers then two rows or two columns of the matrix are linearly dependent. The proof is remarkably easy. Generalizations include estimates for subdeterminants if the determinant is small and the moduli of the entries are close to 1.
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1. Introduction

A paper of Győry and Schinzel [1] contains the lemma that a 3×3 matrix whose elements are roots of unity and whose determinant vanishes must contain two rows or two columns that are linearly dependent. On the other hand in Schlickewei and Wirsing [3] one finds an identity which in particular implies that a 3×3 matrix A of complex elements a_{ij} which all have modulus 1 and whose determinant vanishes contains at least one 2×2 submatrix A' with determinant zero, and which moreover allows to estimate $\det A' \ll \delta^{1/4}$ if the assumption $|a_{ij}| = 1$ is replaced by $1 - \delta \leqslant |a_{ij}| \leqslant 1 + \delta$.

The two statements are closely related and in both cases the proofs are fairly complicated. It is the objective of the present note to give a short proof of the following common generalization of these results and also to generalize and sharpen them.

Theorem 1. *Let A be a 3×3 matrix of unimodular complex numbers a_{ij} such that* $\det A = 0$. *Then either two rows or two columns of A are linearly dependent.*

The quantitative aspect and remarks on related identities we defer to the last sections. It might be an interesting question if there are more such identities, particularly with matrices of higher dimension.

While the mentioned identity from [3] relates the norm of a 3×3 determinant to a product of four subdeterminants or conjugates thereof, the proofs of

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the present paper can be linked to similar identities refering, however, to five subdeterminants. It seems an interesting question what further relations of this type might hold, in particular for matrices of higher dimension.

2. Proof of Theorem 1

Without loss of generality let A be of the form

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & u_1 & u_2 \\ 1 & u_3 & u_4 \end{pmatrix} \qquad \text{with} \quad |u_\nu| = 1 \text{ for } \nu = 1, \ldots, 4 ; \tag{1}$$

otherwise divide rows and columns by their first elements. We may write the assumption $\det A = 0$ as

$$(u_1 - 1)(u_4 - 1) = (u_2 - 1)(u_3 - 1) . \tag{2}$$

If any factor $u_i - 1$ vanishes then one factor from the other side of the equation also does. All these cases produce matrices A with two parallel lines of ones, which would prove the theorem. We may therefore assume that all $u_i \neq 1$. Applying $u - 1 = (1 - \overline{u})u$ to all u_i, and combining it with (2) and its conjugate equation yields $u_1u_4 = u_2u_3$. In combination with (2) also $u_1 + u_4 = u_2 + u_3$ follows. Now $\{u_1, u_4\} = \{u_2, u_3\}$, since both sets contain the zeros of the same quadratic equation ('Vieta's Theorem'). If $u_1 = u_2$, $u_4 = u_3$ then columns 2 and 3, otherwise rows 2 and 3 are equal . ■

3. Remarks and Generalizations

Short as our proof is it can be made more transparent by using the following lemma, which is an easy exercise in linear algebra. This, incidentally, is the only part of the proof that generalizes in an obvious way to higher dimension.

Lemma. *Let A be an $n \times n$ matrix over some field and A' an $(n-1) \times (n-1)$ submatrix and assume that $\det A = \det A' = 0$. Then either the rows or the columns of A that pass through A' are linearly dependent.*

Proof of the Lemma. Let

$$A = \begin{pmatrix} A' & c \\ r & a \end{pmatrix}$$

and assume that the rows of $\begin{pmatrix} A' & c \end{pmatrix}$ are linearly independent. Then, since $\operatorname{rk} A \leqslant n-1$, the row $\begin{pmatrix} r & a \end{pmatrix}$ is a linear combination of the rows of $\begin{pmatrix} A' & c \end{pmatrix}$. In particular r is a linear combination of the rows of A'. Hence

$$\operatorname{rk} \begin{pmatrix} A' \\ r \end{pmatrix} = \operatorname{rk} A' \leqslant n - 2 . \qquad ■$$

[1] This proof is already published and attributed to J. Browkin in A. Schinzel [2].

In the proof of Theorem 1 the lemma not only replaces the appeal to Vieta but also the discussion about the linear factors $u_i - 1$, since these, like $u_1u_4 - u_2u_3$, are subdeterminants of A. All that is needed is the demonstration that (2) implies $u_1u_4 = u_2u_3$ unless one of the u_i equals 1.

Also the lemma presents Theorem 1 as an immediate consequence of the result of Schlickewei and Wirsing, although with a more complicated overall proof.

Algebraically the proof needs rather little of the properties of modulus or norm. Thus without changing the proof we obtain

Theorem 2. *Let $\mathcal{K}$ be any field and σ any automorphism of it. Define $\nu(x) := x\sigma(x)$ for $x \in \mathcal{K}$. If A is a 3×3 matrix with elements from the group $\mathcal{G}$ of u with $\nu(u) = 1$ and if $\det A = 0$ then either two rows or two columns of A are linearly dependent.*

Remark. If $\mathcal{G}$ is not simply $\{1, -1\}$ then it consists of the norm-one elements of some quadratic field extension. Indeed, from $\sigma(u) = u^{-1}$ one gets immediately $\sigma^2(u) = u$. So if the fixed fields $\mathcal{F}_1$, $\mathcal{F}_2$ of σ, σ^2 respectively are distinct then $\mathcal{G} \subset \mathcal{F}_2$, $[\mathcal{F}_2 : \mathcal{F}_1] = 2$ and the restriction of ν to $\mathcal{F}_2$ coincides with the relative norm of this field over $\mathcal{F}_1$.

A slight formal extension of Theorem 2 makes the assumption invariant against multipying rows or columns with constants. Since matrices of the form $B = (\alpha_i\beta_j)$ are characterized by the property $\operatorname{rk} B \leqslant 1$ we may replace the requirement that all $\nu(a_{ij}) = 1$ by $\operatorname{rk}\big(\nu(a_{ij})\big) \leqslant 1$.

Our last generalization of Theorem 2 is to matrices of any size and shape, but with a rather restrictive assumption.

Theorem 3. *Let $A = (a_{ij})$ be an $m \times n$ matrix over any field $\mathcal{K}$, σ an automorphism of $\mathcal{K}$ and $\nu(x) := x\sigma(x)$. Assume $\operatorname{rk} A \leqslant 2$ and $\operatorname{rk}\big(\nu(a_{ij})\big) \leqslant 1$. Then either the nonvanishing rows or the nonvanishing columns of A fall into at most two classes of pairwise linearly dependent ones.*

Proof. All is trivial if there are fewer than three rows or columns, so let $m \geqslant 3$, $n \geqslant 3$. Zeros occur only in complete rows or columns. Therefore we can assume without loss of generality that all elements of A are nonzero. As in the proof of Theorem 1 we can normalize in such a way that all rows and columns begin with ones. Two rows or columns are now linearly dependent precisely if they are equal.

If all entries are 1, the proposition is true; so assume they are not. Again without loss of generality we assume that $a_{22} := u \neq 1$.

Thus the first two rows and the first two columns of A are distinct. Hence every row is a linear combination of the first two ones, and similarly for columns.

Suppose now that A has three distinct rows and three distinct columns. This will imply a contradiction.

The three distinct rows and columns shall be the first three ones each. By Theorem 2 the upper left 3×3 submatrix S has two equal rows or two equal columns. Let it be rows, transposing if necessary.

Thus the first two columns of S are

$$\begin{pmatrix} 1 & 1 \\ 1 & u \\ 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 \\ 1 & u \\ 1 & u \end{pmatrix}.$$

Since every column of A is a linear combination of the first two columns, we see that the third row of A equals the first or the second row. Contradiction. ∎

4. A Quantitative Variant

Finally we show how our proof of Theorem 1 can be modified to improve the quoted estimate of Schlickewei and Wirsing. We do not even need the vanishing of the determinant; having it small will suffice. Moreover we do not estimate just one subdeterminant but give a bound for the deviation of two rows or two columns of the matrix from being parallel.

Let D_{ij} denote the 2×2 subdeterminant of A that is the coefficient of a_{ij} in some Laplace expansion of $\det A$.

Theorem 4. *Let A be a complex 3×3 matrix. If for sufficiently small $\delta > 0$ we have*

$$1 - \delta \leqslant |a_{ij}| \leqslant 1 + \delta \qquad \text{for all} \quad i, j, \tag{3}$$

and

$$|\det A| \leqslant \delta^{3/2} \tag{4}$$

then there are either two rows or two columns in A such that $D_{ij} \ll \delta^{1/2}$ for all three subdeterminants built from these rows or columns.

Remark. The exponents $1/2$ and $3/2$ are best possible, as can be seen from the following examples. Concerning the $1/2$ take $\varepsilon = \delta^{1/2}, u_1 = e^{4\varepsilon i}$, $u_2 = u_3 = e^{2\varepsilon i}$, and determine u_4 from $\det A = 0$. One finds $u_4 = 1 + i \tan \varepsilon$. So all $|u_i| = 1$ or $= 1 + \varepsilon^2/2 + O(\varepsilon^3)$ but all 2×2 subdeterminants are found to be $\gg \varepsilon$. If secondly u_1, u_2, u_3 are as before but $u_4 = e^{\varepsilon i}$ then all $|u_\nu| = 1$, $|\det A| = 4 \sin \varepsilon (1 - \cos \varepsilon) \ll \varepsilon^3$, and again all 2×2 determinants are $\gg \varepsilon$. Now (3) holds with any $\delta > 0$ and if we take it small compared to ε^2 the proposition of the theorem no longer holds. So the exponent $3/2$ in (4) too cannot be improved.

Proof of the Theorem. We begin by showing that there is at least one subdeterminant that is as small as claimed. After dividing all rows and columns by their first elements the matrix takes the shape (1), where now $|u_i| = 1 + O(\delta)$. The moduli of the subdeterminants $\tilde{D}_{ij}$ of the normalized matrix differ from the original D_{ij} only by insignificant factors $1 + O(\delta)$.

The assertion is true if one of the subdeterminants $u_i - 1$ is small, so we assume

$$|u_i - 1| \geqslant \delta^{1/2} \qquad \text{for all } i. \tag{5}$$

We shall estimate $\tilde{D}_{11} = u_1u_4 - u_2u_3$. Define r_i and R by

$$\frac{u_i - 1}{1 - \overline{u}_i} = u_i + r_i\,, \qquad \det A = (u_2 - 1)(u_3 - 1)R\,.$$

Then (5) and (4) imply

$$|r_i| = \frac{\big||u_i| - 1\big|\big(|u_i| + 1\big)}{|u_i - 1|} \ll \delta^{1/2} \qquad \text{and} \qquad |R| \ll \delta^{1/2}\,.$$

From $(u_1 - 1)(u_4 - 1) = (u_2 - 1)(u_3 - 1) + \det A$ we see

$$\begin{aligned} &\left|\frac{(u_1 - 1)(u_4 - 1)}{(\overline{u}_1 - 1)(\overline{u}_4 - 1)} - \frac{(u_2 - 1)(u_3 - 1)}{(\overline{u}_2 - 1)(\overline{u}_3 - 1)}\right| \\ &\qquad = \left|\frac{(u_2 - 1)(u_3 - 1)(1 + R)}{(\overline{u}_2 - 1)(\overline{u}_3 - 1)(1 + \overline{R})} - \frac{(u_2 - 1)(u_3 - 1)}{(\overline{u}_2 - 1)(\overline{u}_3 - 1)}\right| \\ &\qquad = \left|\frac{1 + R}{1 + \overline{R}} - 1\right| = \left|\frac{R - \overline{R}}{1 + \overline{R}}\right| \\ &\qquad \ll \delta^{1/2}\,. \end{aligned}$$

Inserting the definition of r_i gives

$$(u_1 + r_1)(u_4 + r_4) - (u_2 + r_2)(u_3 + r_3) \ll \delta^{1/2}\,,$$

$$\begin{aligned} \tilde{D}_{11} &= u_2r_3 + u_3r_2 + r_2r_3 - u_1r_4 - u_4r_1 - r_1r_4 + O(\delta^{1/2}) \\ &\ll \delta^{1/2}\,. \end{aligned}$$

We have shown so far that one out of the five subdeterminants is 'small' that correspond to the positions marked with '$*$' in

$$\begin{pmatrix} * & & \\ & * & * \\ & * & * \end{pmatrix}. \tag{6}$$

Permuting rows and columns of A rearranges rows and columns of (D_{ij}) in the corresponding way (apart from the appropriate sign changes). Thus the one-out-of-five estimate holds for each one of the patterns obtained from (6) by such rearrangments. For each position there are such patterns that do not contain it. Therefore there are at least two positions (i, j) where $D_{ij} \ll \delta^{1/2}$.

If the two are in one line, horizontally or vertically, then the estimate carries over to the third one in that line by the Laplace formula, e.g.

$$\sum_j a_{ij} D_{ij} = \det A \ll \delta^{3/2}\,,$$

which would finish the proof. If the two are not in a line then we can assume without loss of generality that

$$D_{33} = u_1 - 1 \ll \delta^{1/2} \quad \text{and} \quad D_{22} = u_4 - 1 \ll \delta^{1/2},$$

and find

$$D_{32}D_{23} = (u_2 - 1)(u_3 - 1) = (u_1 - 1)(u_4 - 1) - \det A \ll \delta,$$
$$D_{32} \ll \delta^{1/2} \quad \text{or} \quad D_{23} \ll \delta^{1/2}.$$

So there are again two small D_{ij} with positions in one line and the proof ends as before. ∎

5. Identities

The mentioned identity from [3] is proved under the assumption $\det A = 0$. It is not difficult to drop this condition at the expense of a more complicated formula. But basic algebraic terminology allows to suppress unimportant details.

Assume that A is given in the normalized form (1). Let R be the polynomial ring $\mathfrak{F}[u_1, \ldots, u_4, \overline{u}_1, \ldots, \overline{u}_4]$, where $\mathfrak{F}$ is any field and $u_1, \ldots, u_4, \overline{u}_1, \ldots, \overline{u}_4$ are considered independent indeterminates. Let overlining denote the automorphism of R that for all i exchanges u_i and $\overline{u}_i$. For abbreviation call

$$D := \det A, \quad d := u_1u_4 - u_2u_3, \quad v_i := u_i - 1, \quad r_i := u_i\overline{u}_i - 1.$$

If now $\mathfrak{J}$ is the ideal in R generated by $r_1, \ldots, r_4, D, \overline{D}$, then the identity from [3], generalized as indicated, gives

$$(u_2 - 1)(u_3 - 1)(\overline{u}_2 - \overline{u}_4)(u_3 - u_4) \equiv 0 \pmod{\mathfrak{J}}. \tag{7}$$

If $\mathfrak{F} = \mathbb{C}$ and '$-$' is complex conjugation then the assumption $r_i \ll \delta$ is equivalent to $|u_i| - 1 \ll \delta$. Since this implies that all polynomials are bounded and since (7) is just short for

$$\text{LHS} = \sum_{i=1}^{4} r_i q_i + q_5 D + q_6 \overline{D}$$

with polynomials $q_1, \ldots, q_6$, it follows that

$$\begin{gathered} \text{If} \quad |r_i| \leqslant \delta \quad \text{for} \quad i = 1, \ldots, 4 \quad \text{and} \quad |D| \leqslant \delta \quad \text{then} \\ (u_2 - 1)(u_3 - 1)(u_2 - u_4)(u_3 - u_4) \ll \delta; \end{gathered} \tag{8}$$

in particular one of the factors will be $\ll \delta^{1/4}$.

The first part of our proof of Theorem 1 can be reformulated using congruences mod $\mathfrak{J}$. It turns out that with a bit of care one needs only three subdeterminants as factors. Note that $u_i\overline{v}_i = r_i - v_i \equiv -v_i$, $v_1v_4 \equiv v_2v_3$, $\overline{v}_1\overline{v}_4 \equiv \overline{v}_2\overline{v}_3 \pmod{\mathfrak{J}}$. Hence

$$\begin{aligned}\overline{v}_1\overline{v}_4 d &= \overline{v}_1\overline{v}_4(u_1u_4 - u_2u_3)\\ &\equiv \overline{v}_1\overline{v}_4u_1u_4 - \overline{v}_2\overline{v}_3u_2u_3\\ &\equiv v_1v_4 - v_2v_3\\ &\equiv 0 \pmod{\mathfrak{J}}.\end{aligned}$$

Again, the quantitative version is immediate: $v_1v_4d \ll \delta$. Removing the normalization and translating the result to the original form of the matrix we obtain

Theorem 5. *Let A be a complex 3×3 matrix. If with some $\delta \in (0,1)$ we have*

$$1-\delta \leqslant |a_{ij}| \leqslant 1+\delta \qquad \textit{for all} \quad i,j,$$

and

$$|\det A| \leqslant \delta$$

then

$$D_{11}D_{22}D_{33} \ll \delta\,;$$

and the same holds, of course, after permuting rows or columns.

It is not clear if a similar congruence exists that would yield Theorem 4 (as far as estimating one subdeterminant goes). Because of the unequal exponents the ideal $\mathfrak{J}$ would not be suitable. Rather, if $\mathfrak{R}$ is generated by the r_i and $\mathfrak{D}$ by D and $\overline{D}$, then the ideal $(\mathfrak{R}^3, \mathfrak{D}^2)$ might be the one involved. But then in addition to v_1, v_2, v_3, v_4, d, or their conjugates some further factor would seem to be missing on the left hand side to balance the exponents.

In fact the proof of Theorem 4 given above can be reduced to an identity but this is of a rather different nature:

$$d = u_4\frac{r_1}{\overline{v}_1} + u_1\frac{r_4}{\overline{v}_4} - u_3\frac{r_2}{\overline{v}_2} - u_2\frac{r_3}{\overline{v}_3} - \frac{r_1r_4}{\overline{v}_1\overline{v}_4} + \frac{r_2r_3}{\overline{v}_2\overline{v}_3} + \frac{D}{\overline{v}_1\overline{v}_4} - \frac{v_2v_3}{\overline{v}_2\overline{v}_3}\frac{\overline{D}}{\overline{v}_1\overline{v}_4}. \tag{9}$$

This does indeed imply Theorem 4 since if all $|r_i| \leqslant \delta$ and $|D| \leqslant \delta^{3/2}$ then either one of the v_i is $\leqslant \delta^{1/2}$ or all are $\geqslant d^{1/2}$ and then (9) gives $d \ll \delta^{1/2}$.

Identity (9) is not hard to prove. One can start observing that

$$\begin{aligned}\Delta := \frac{v_1v_4}{\overline{v}_1\overline{v}_4} - \frac{v_2v_3}{\overline{v}_2\overline{v}_3} &= \frac{D + v_2v_3}{\overline{v}_1\overline{v}_4} - \frac{v_2v_3}{\overline{v}_2\overline{v}_3}\\ &= \frac{D}{\overline{v}_1\overline{v}_4} - \frac{v_2v_3}{\overline{v}_2\overline{v}_3}\frac{\overline{D}}{\overline{v}_1\overline{v}_4}\end{aligned}$$

and also

$$\begin{aligned}\Delta &= \left(u_1 - \frac{r_1}{\overline{v}_1}\right)\left(u_4 - \frac{r_4}{\overline{v}_4}\right) - \left(u_2 - \frac{r_2}{\overline{v}_2}\right)\left(u_3 - \frac{r_3}{\overline{v}_3}\right)\\ &= d - u_4\frac{r_1}{\overline{v}_1} + \dots\,.\end{aligned}$$

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