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HERMITE'S FORMULAS FOR q-ANALOGUES OF HURWITZ ZETA FUNCTIONS

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Abstract: We treat Hermite's formulas for q-analogues of the Hurwitz zeta function. As their application, we study the classical limit of modified q-analogues of the Hurwitz zeta function. We also treat q-analogues of the Milnor multiple gamma function.

Keywords: Riemann zeta function, Hurwitz zeta function, multiple gamma function, classical limit, q-series.

1. Introduction

We define $\log z = \log|z| + i\arg z$ with $\arg z \in [-\pi, \pi)$, and $z^s = e^{s\log z}$ for $z \in \mathbb{C}\setminus\{0\}$. For $a > 0$ the Hurwitz zeta function is defined by

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \qquad (\mathrm{Re}(s) > 1). \tag{1.1}$$

There are various methods of continuing the Hurwitz zeta function meromorphically to the whole s-plane. One of the methods is to employ Hermite's formula (see [15]):

$$\zeta(s,a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + i\int_0^{\infty} \frac{(a+iy)^{-s} - (a-iy)^{-s}}{e^{2\pi y} - 1}\,dy. \tag{1.2}$$

We consider its q-deformation.

We take $0 < q < 1$ and $a > 0$, and write $[z]_q = \frac{1-q^z}{1-q}$ for $z \in \mathbb{C}$. Then q-analogues of the Hurwitz zeta function are defined as the function of two complex variables s and t by the series

$$\zeta_q(s,t,a) = \sum_{n=0}^{\infty} \frac{q^{(n+a)t}}{[n+a]_q^s} \qquad (\mathrm{Re}(t) > 0). \tag{1.3}$$

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We obtain Hermite's formula for $\zeta_q(s,t,a)$.

Theorem 1.1. *For $s \in \mathbb{C}$, $\mathrm{Re}(t) > 0$ and $|\mathrm{Im}(t)| < \frac{2\pi}{\log(q^{-1})}$ we have*

$$\begin{aligned}\zeta_q(s,t,a) =& \frac{1}{2}\frac{q^{at}}{[a]_q^s} - \frac{(1-q)^s q^{at}}{t\log q} F(s,t;t+1;q^a) \\ &+ i\int_0^\infty \frac{q^{(a+iy)t}[a+iy]_q^{-s} - q^{(a-iy)t}[a-iy]_q^{-s}}{e^{2\pi y}-1}dy,\end{aligned} \tag{1.4}$$

where $F(\alpha,\beta;\gamma;z)$ is Gauss' hypergeometric function defined by

$$F(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!} \qquad (|z|<1) \tag{1.5}$$

with $(s)_n = s(s+1)(s+2)\cdots(s+n-1)$ being the rising factorial. This gives the meromorphic continuation of $\zeta_q(s,t,a)$ to $s \in \mathbb{C}$ and $|\mathrm{Im}(t)| < \frac{2\pi}{\log(q^{-1})}$.

Next, we consider the problem of classical limits. From now on, we take $t := \phi(s)$ and restrict $\phi(s)$ to an s-variable linear function or a constant function as follows:

$$\phi(s) := \begin{cases} \lambda s - \nu, & (\lambda > 0,\ \nu \in \mathbb{C}), \\ \mu, & (\mathrm{Re}(\mu) > 0). \end{cases} \tag{1.6}$$

Under this condition, $\zeta_q(s,\phi(s),a)$ is defined as an s-variable holomorphic function for $\mathrm{Re}(s) > R_\phi := \mathrm{Re}(\nu)/\lambda$, where we take $R_\phi = -\infty$ if $\phi(s) = \mu$ with $\mathrm{Re}(\mu) > 0$.

Restriction on $t = \phi(s)$ can be dropped as long as $\zeta_q(s,\phi(s),a)$ is defined as an s-variable holomorphic function on some s-region and continued meromorphically to a certain proper s-region, but for simplicity we skip the argument in the present paper.

Our problem is that although it follows easily from absolute convergence that

$$\lim_{q\uparrow 1} \zeta_q(s,\phi(s),a) = \zeta(s,a) \qquad (\mathrm{Re}(s) > \max\{1, R_\phi\}), \tag{1.7}$$

it is not trivial whether the classical limit (i.e. $q \uparrow 1$) of $\zeta_q(s,\phi(s),a)$ itself or with certain modification terms offers the original Hurwitz zeta function on the whole s-plane. As a matter of fact, some cases are treated in the preceding papers as follows.

Theorem A.

(i) [14, Corollary 3.8] (cf. [2, Theorem 2]) *For $m \in \mathbb{Z}_{>0}$ we have*

$$\lim_{q\uparrow 1} \zeta_q(s,s-m,a) = \zeta(s,a) \qquad (s \in \mathbb{C}). \tag{1.8}$$

(ii) [13, Theorem 2] *We have*

$$\lim_{q\uparrow 1} \left\{ \zeta_q(s,s,a) + \frac{(1-q)^s}{\log q}\frac{\pi}{\sin(\pi s)} \right\} = \zeta(s,a) \qquad (s \in \mathbb{C}). \tag{1.9}$$

(iii) [12, Lemma 2] *We have*

$$\lim_{q\uparrow 1}\left\{\zeta_q(s,1,a)+\frac{(1-q)^s}{(1-s)\log q}\right\}=\zeta(s,a)\qquad (s\in\mathbb{C}). \tag{1.10}$$

Moreover, Kawagoe, Wakayama and Yamasaki [3] proved the following assertion in 2008.

Theorem B ([3, Theorem 2.1]). *Let $\psi(s)$ be a meromorphic function on $\mathbb{C}$. Then the formula*

$$\lim_{q\uparrow 1}\zeta_q(s,\psi(s),a)=\zeta(s,a)\qquad (s\in\mathbb{C}) \tag{1.11}$$

holds if and only if the function $\psi(s)$ can be written as $\psi(s)=s-m$ for some $m\in\mathbb{Z}_{>0}$.

Therefore our work becomes to construct certain modified q-analogues of the Hurwitz zeta function (like (1.9) or (1.10)) which go to $\zeta(s,a)$ on the whole s-plane by taking their classical limit. Our result is as follows.

Theorem 1.2. *Let $\phi(s)$ be the function defined by* (1.6)*. Then we have*

$$\lim_{q\uparrow 1}\left\{\zeta_q(s,\phi(s),a)+\frac{(1-q)^s}{\log q}B(\phi(s),1-s)\right\}=\zeta(s,a)\qquad (s\in\mathbb{C}), \tag{1.12}$$

where $B(x,y)$ is the beta function.

Since $B(s-m,1-s)=0$ for $m\in\mathbb{Z}_{>0}$, $B(s,1-s)=\frac{\pi}{\sin(\pi s)}$ and $B(1,1-s)=(1-s)^{-1}$, Theorem 1.2 includes the past results (1.8)–(1.10). For convenience, we put $\tilde{\zeta}_q(s,\phi(s),a)=\zeta_q(s,\phi(s),a)+\frac{(1-q)^s}{\log q}B(\phi(s),1-s)$.

Now we consider q-analogues of the Milnor multiple gamma function, i.e. certain special values of the partial derivative of $\tilde{\zeta}_q(s,\phi(s),a)$ with respect to s.

We define q-analogues of the Milnor multiple gamma function by

$$\mathbf{\Gamma}_{q,r}(a)=\exp\left(\frac{\partial}{\partial s}\tilde{\zeta}_q(s,s+r-1,a)\bigg|_{s=1-r}\right) \tag{1.13}$$

for $r\in\mathbb{Z}_{>0}$ and $a>0$. The usual Milnor multiple gamma function (see [4]) is defined by

$$\mathbf{\Gamma}_r(a)=\exp\left(\frac{\partial}{\partial s}\zeta(s,a)\bigg|_{s=1-r}\right).$$

In the case $r=1$, it holds that

$$\mathbf{\Gamma}_1(a)=\frac{\Gamma(a)}{\sqrt{2\pi}}$$

via Lerch's formula [9].

We obtain the infinite product expression for $\mathbf{\Gamma}_{q,r}(a)$.

Theorem 1.3. *Let $r \in \mathbb{Z}_{>0}$ and $a > 0$. Then we have*

$$\mathbf{\Gamma}_{q,r}(a) = \exp\{F_{q,r}(a)\} \prod_{n=0}^{\infty} (1 - q^{a+n})^{-[a+n]_q^{r-1}}, \tag{1.14}$$

where we put

$$F_{q,r}(a) = (1-q)^{-r+1} \Bigg\{ \left(\frac{c(r,1)}{\log q} - B_1(a) \right) \log(1-q) + \left(\frac{c(r,2)}{\log q} - \frac{1}{2} B_2(a) \log q \right) + \sum_{k=1}^{r-1} \binom{r-1}{k} \frac{(-1)^k q^{ka}}{1-q^k} \left(a \log q + \log(1-q) + \frac{q^k \log q}{1-q^k} \right) \Bigg\}$$

with $B_k(a)$ being the Bernoulli polynomial and

$$c(r,1) = \begin{cases} 0, & (r = 1) \\ -\left(1 + \frac{1}{2} + \cdots + \frac{1}{r-1}\right), & (r \geqslant 2) \end{cases},$$

$$c(r,2) = \begin{cases} \frac{\pi^2}{6}, & (r = 1) \\ \frac{\pi^2}{6} + \frac{1}{2}\left\{ \left(1 + \frac{1}{2} + \cdots + \frac{1}{r-1}\right)^2 - \left(1 + \frac{1}{2^2} + \cdots + \frac{1}{(r-1)^2}\right) \right\}, & (r \geqslant 2) \end{cases}.$$

We remark that it follows from Theorem 1.2

$$\lim_{q \uparrow 1} \mathbf{\Gamma}_{q,r}(a) = \mathbf{\Gamma}_r(a) \qquad (a > 0).$$

2. Hermite's formula

For convenience, we put

$$\zeta_{q,0}(s,t,z) = \frac{q^{zt}}{[z]_q^s}. \tag{2.1}$$

Then it holds

$$\zeta_q(s,t,a) = \sum_{n=0}^{\infty} \zeta_{q,0}(s,t,a+n). \tag{2.2}$$

Proof of Theorem 1.1. We use the Abel-Plana summation formula [10]. Let $f(z)$ be a holomorphic function in $\mathrm{Re}(z) \geqslant 0$ and satisfy the following properties:

$$\lim_{y \to \infty} |f(x \pm iy)| \, e^{-2\pi y} = 0 \qquad (\mathrm{Re}(z) = x,\ \mathrm{Im}(z) = y)$$

uniformly for $x \in [0, x_0]$ with any $x_0 > 0$, and

$$\lim_{x \to \infty} \int_0^{\infty} |f(x \pm iy)| \, e^{-2\pi y} dy = 0.$$

Then we have

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2}f(0) + \int_0^{\infty} f(z)dz + i\int_0^{\infty} \frac{f(iy)-f(-iy)}{e^{2\pi y}-1}dy. \tag{2.3}$$

We check that $\zeta_{q,0}(s,t,z)$ satisfy those properties as a z-variable function. For $x, y \geqslant 0$, we have

$$\zeta_{q,0}(s,t,x \pm iy + a) = (1-q)^s \sum_{m=0}^{\infty} \binom{s+m-1}{m} q^{(t+m)(x \pm iy + a)} \tag{2.4}$$

by the binomial theorem, and

$$\begin{aligned}
|\zeta_{q,0}(s,t,x \pm iy + a)| &\leqslant (1-q)^{\mathrm{Re}(s)} \sum_{m=0}^{\infty} \left|\binom{s+m-1}{m}\right| \left|q^{(t+m)(x \pm iy + a)}\right| \\
&\leqslant (1-q)^{\mathrm{Re}(s)} \sum_{m=0}^{\infty} \binom{|s|+m-1}{m} q^{(\mathrm{Re}(t)+m)(x+a) \mp \mathrm{Im}(t)y} \\
&\leqslant (1-q)^{\mathrm{Re}(s)} (1-q^{x+a})^{-|s|} q^{(x+a)\,\mathrm{Re}(t) - |\mathrm{Im}(t)|y} \\
&\leqslant (1-q)^{\mathrm{Re}(s)} (1-q^{a})^{-|s|} q^{(x+a)\,\mathrm{Re}(t) - |\mathrm{Im}(t)|y}.
\end{aligned}$$

Moreover in $s \in \mathbb{C}$, $\mathrm{Re}(t) > 0$ and $|\mathrm{Im}(t)| < \frac{2\pi}{\log(q^{-1})}$ we have

$$\begin{aligned}
&|\zeta_{q,0}(s,t,x \pm iy + a)|\, e^{-2\pi y} \\
&\qquad \leqslant (1-q)^{\mathrm{Re}(s)} (1-q^{a})^{-|s|} q^{-|\mathrm{Im}(t)|y} e^{-2\pi y} \to 0 \qquad (y \to \infty),
\end{aligned} \tag{2.5}$$

In the same region we also have

$$\begin{aligned}
\int_0^{\infty} |\zeta_{q,0}(s,t,x \pm iy)|\, e^{-2\pi y} dy &\leqslant (1-q)^{\mathrm{Re}(s)} (1-q^{a})^{-|s|} q^{(x+a)\,\mathrm{Re}(t)} \\
&\qquad \times \int_0^{\infty} q^{-|\mathrm{Im}(t)|y} e^{-2\pi y} dy \to 0 \qquad (x \to \infty).
\end{aligned} \tag{2.6}$$

Therefore when $s \in \mathbb{C}$, $\mathrm{Re}(t) > 0$ and $|\mathrm{Im}(t)| < \frac{2\pi}{\log(q^{-1})}$, we can apply the Abel-Plana summation formula, and obtain

$$\zeta_q(s,t,a) = \frac{1}{2}\zeta_{q,0}(s,t,a) + \int_0^{\infty} \zeta_{q,0}(s,t,a+x)dx + i\int_0^{\infty} g_q(s,t,a,y)dy, \tag{2.7}$$

where we put

$$g_q(s,t,a,y) = \frac{q^{(a+iy)t}[a+iy]_q^{-s} - q^{(a-iy)t}[a-iy]_q^{-s}}{e^{2\pi y}-1}. \tag{2.8}$$

We notice that $\int_0^{\infty} g_q(s,t,a,y)dy$ is holomorphic in $s \in \mathbb{C}$ and $|\mathrm{Im}(t)| < \frac{2\pi}{\log(q^{-1})}$ by the inequality (2.6). We also have

$$\int_0^{\infty} \zeta_{q,0}(s,t,a+x)dx = -\frac{(1-q)^s q^{at}}{t\log q} F(s,t;t+1;q^a) \qquad (\mathrm{Re}(t) > 0) \tag{2.9}$$

via the integral representation of Gauss' hypergeometric function by Euler

$$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-zu)^{-\alpha}du \tag{2.10}$$

in $\mathrm{Re}(\gamma)>\mathrm{Re}(\beta)>0$ and $|\arg(1-z)|<\pi$. Thus we obtain the equation (1.4).

Moreover, by noting that the right hand side of (2.9) is meromorphic in $(s,t)\in\mathbb{C}^2$ and combining the above results, the right hand side of (1.4) become meromorphic in $s\in\mathbb{C}$ and $|\mathrm{Im}(t)|<\frac{2\pi}{\log(q^{-1})}$, and we conclude the proof of Theorem 1.1. ■

To prove Theorem 1.2, we show the following proposition.

Proposition 2.1. *Let $g_q(s,t,a,y)$ be the one defined by* (2.8). *For $a>0$ and $(s,t)\in\mathbb{C}^2$, we have*

$$\lim_{q\uparrow 1}\int_0^\infty g_q(s,t,a,y)dy=\int_0^\infty \frac{(a+iy)^{-s}-(a-iy)^{-s}}{e^{2\pi y}-1}dy. \tag{2.11}$$

To prove Proposition 2.1, we use the following lemma.

Lemma 2.2. *For $0<q\leqslant 1$ and $z\in\mathbb{C}$, we have*

$$\min\{1,\mathrm{Re}(z)\}\leqslant|[z]_q|\leqslant q^{-|z|}\max\{1,|z|\}. \tag{2.12}$$

Proof of Lemma 2.2. We notice that

$$[z]_q=\left|\frac{1-q^z}{1-q}\right|\geqslant\frac{1-|q^z|}{1-q}=[\mathrm{Re}(z)]_q\geqslant\min\{1,\mathrm{Re}(z)\}.$$

We calculate as

$$\begin{aligned}\left|\frac{1-q^z}{1-q}\right|&=\frac{|1-\{1-(1-q)\}^z|}{1-q}=\left|\sum_{n=1}^{\infty}\binom{z}{n}(-1)^{n+1}(1-q)^{n-1}\right|\\&\leqslant\sum_{n=1}^{\infty}\left|\binom{z}{n}\right|(1-q)^{n-1}\leqslant\sum_{n=1}^{\infty}\binom{|z|+n-1}{n}(1-q)^{n-1}\\&=\frac{q^{-|z|}-1}{1-q}=q^{-|z|}[|z|]_q\leqslant q^{-|z|}\max\{1,|z|\}.\end{aligned}$$

We conclude the proof of Lemma 2.2. ■

Proof of Proposition 2.1. For $q\in[q_0,1]$ and $y\in[y_0,\infty)$ with $q_0>0$ and sufficiently large $y_0>0$, we calculate as

$$\begin{aligned}\left|\frac{q^{(a\pm iy)t}[a\pm iy]_q^{-s}}{e^{2\pi y}-1}\right|&\leqslant M_{y_0}e^{-2\pi y}\left|q^{(a\pm iy)t}[a\pm iy]_q^{-s}\right|\qquad(\exists M_{y_0}\geqslant 0)\\&=M_{y_0}e^{-2\pi y}q^{a\,\mathrm{Re}(t)\mp y\,\mathrm{Im}(t)}\left|[a\pm iy]_q\right|^{-\mathrm{Re}(s)}\\&\quad\times\exp\left(\mathrm{Im}(s)\arg[a\pm iy]_q\right)\\&\leqslant M_{y_0}\left|[a\pm iy]_q\right|^{-\mathrm{Re}(s)}\\&\quad\times\exp\left\{|\mathrm{Im}(s)|\pi-|\mathrm{Re}(t)|a\log q_0-(2\pi+|\mathrm{Im}(t)|\cdot\log q_0)\,y\right\}\end{aligned}$$

Using Lemma 2.2, when $\operatorname{Re}(s) \leqslant 0$, we have

$$\begin{aligned}|[a \pm iy]_q|^{-\operatorname{Re}(s)} &\leqslant |a \pm iy|^{-\operatorname{Re}(s)} q^{|a\pm iy| \operatorname{Re}(s)} \\ &\leqslant (a+y)^{-\operatorname{Re}(s)} q_0^{(a+y)\operatorname{Re}(s)}.\end{aligned}$$

When $\operatorname{Re}(s) \geqslant 0$, we have

$$|[a \pm iy]_q|^{-\operatorname{Re}(s)} \leqslant \max\{1, a^{-\operatorname{Re}(s)}\}.$$

Thus we have

$$\begin{aligned}\left|\frac{q^{(a\pm iy)t}[a \pm iy]_q^{-s}}{e^{2\pi y}-1}\right| &\leqslant M_{y_0} \exp\Big[|\operatorname{Im}(s)|\pi - |\operatorname{Re}(t)| a \log q_0 \\ &\quad - (2\pi + |\operatorname{Im}(t)| \cdot \log q_0)\, y\Big] \cdot H(y, a, s, q_0),\end{aligned} \tag{2.13}$$

where we put $H(y, a, s, q_0) = \max\left\{1, a^{-\operatorname{Re}(s)}, (a+y)^{-\operatorname{Re}(s)} q_0^{(a+y)\operatorname{Re}(s)}\right\}$. We notice that the right hand side of (2.13) is an integrable function on $y \in [\alpha, \infty)$ when q_0 is sufficiently near to 1. We also notice $g_q(s, t, a, y)$ is uniformly bounded on $q \in [q_0, 1]$ and $y \in [0, \alpha]$. Therefore by Lebesgue's convergence theorem we obtain the equation (2.11). This concludes the proof of Proposition 2.1. ■

Remark 2.3. As a corollary of Theorem 1.1 and Proposition 2.1, we prove (1.8). Noting

$$\begin{aligned}\int_0^\infty q^{(a+x)(s-1)}[a+x]_q^{-s} dx &= -\frac{(1-q)^s q^{a(s-1)}}{(s-1)\log q} F(s, s-1; s; q^a) \\ &= \frac{1}{s-1} \frac{q-1}{\log q} \frac{q^{a(s-1)}}{[a]_q^{s-1}}\end{aligned}$$

and applying Theorem 1.1, we notice that it holds "good" q-analogues of Hermite's formula

$$\begin{aligned}\zeta_q(s, s-1, a) =& \frac{1}{2} \frac{q^{a(s-1)}}{[a]_q^s} + \frac{1}{s-1} \frac{q-1}{\log q} \frac{q^{a(s-1)}}{[a]_q^{s-1}} \\ &+ i \int_0^\infty \frac{q^{(a+iy)(s-1)}[a+iy]_q^{-s} - q^{(a-iy)(s-1)}[a-iy]_q^{-s}}{e^{2\pi y}-1} dy\end{aligned} \tag{2.14}$$

for $|\operatorname{Im}(s)| < \frac{2\pi}{\log(q^{-1})}$. From this expression and Proposition 2.1, we obtain

$$\lim_{q \uparrow 1} \zeta_q(s, s-1, a) = \zeta(s, a) \qquad (s \in \mathbb{C}). \tag{2.15}$$

Moreover, using the elementary property

$$\zeta_q(s, s-m-1, a) = \zeta_q(s, s-m, a) + (1-q)\zeta_q(s-1, s-m-1, a), \tag{2.16}$$

we obtain the equation (1.8). We remark that this method of proving (1.8) is much simpler than other ones via the Euler-Maclaurin summation formula used in [2, Proof of Theorem 2][3, Proof of Theorem 1] or the contour integral representation of $\zeta_q(s,t,a)$ in [14, Proof of Corollary 3.8].

Now we prove Theorem 1.2.

Proof of Theorem 1.2. By Proposition 2.1, we only have to consider the term $\frac{(1-q)^s q^{a\phi(s)}}{\phi(s)\log q} F(s,\phi(s);\phi(s)+1;q^a)$.

Using Gauss' linear transformation formula (see e.g. [8])

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta} + \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} F(\alpha,\beta;\alpha+\beta-\gamma+1;1-z) \tag{2.17}$$

in $|\arg(z)|<\pi$ and $|\arg(1-z)|<\pi$, we have

$$\begin{aligned} &-\frac{(1-q)^s q^{a\phi(s)}}{\phi(s)\log q} F(s,\phi(s);\phi(s)+1;q^a) \\ &\quad = -\frac{(1-q)^s q^{a\phi(s)}}{\log q}\left[\frac{(1-q^a)^{1-s}}{s-1} F(\phi(s)+1-s,1;2-s;1-q^a) \right. \\ &\qquad \left. + B(\phi(s),1-s) q^{-a\phi(s)}\right]. \end{aligned} \tag{2.18}$$

Here we notice that

$$F(\phi(s)+1-s,1;2-s;1-q^a) \to 1 \qquad (q\uparrow 1),$$

and

$$-\frac{(1-q)^s q^{a\phi(s)}}{\log q}\frac{(1-q^a)^{1-s}}{s-1} \to \frac{a^{1-s}}{s-1} \qquad (q\uparrow 1).$$

Therefore, combining the results above, we have

$$\begin{aligned} &\lim_{q\uparrow 1}\left(\zeta_q(s,\phi(s),a) + \frac{(1-q)^s}{\log q} B(\phi(s),1-s)\right) \\ &\qquad = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + i\int_0^\infty \frac{(a+iy)^{-s}-(a-iy)^{-s}}{e^{2\pi y}-1}dy \\ &\qquad = \zeta(s,a) \qquad (s\in\mathbb{C}). \end{aligned}$$

This completes the proof of Theorem 1.2. ■

Remark 2.4. The difference between Theorems B and 1.2 is the definitions of $\psi(s)$ and $\phi(s)$. This is caused from the fact Theorems B does not assume that $\zeta(s,\psi(s),a)$ is defined on some region or continued meromorphically to a certain region. Obviously, $\zeta(s,-1,a) = \sum_{n=0}^{\infty} q^{-(a+n)}[a+n]_q^{-s}$ is such an example.

Remark 2.5. We notice that the modification term $\frac{(1-q)^s}{\log q}B(t,1-s)$ is closely related to the "q-Zeta-Raabe formula," which is a certain period integral.

The usual Zeta-Raabe formula is

$$\int_0^1 \zeta(s,x)dx = 0$$

in $\mathrm{Re}(s) < 1$ (e.g. [1, Theorem 2.1]). Therefore the corresponding period integral should be $\int_0^1 \zeta_q(s,t,x)dx$. This can be rewritten as

$$\begin{aligned}\int_0^1 \zeta_q(s,t,x)dx &= (1-q)^s \sum_{n=0}^{\infty} \int_0^1 q^{(x+n)t}(1-q^{x+n})^{-s}dx \\ &= (1-q)^s \int_0^{\infty} q^{xt}(1-q^x)^{-s}dx \\ &= -\frac{(1-q)^s}{\log q}\int_0^1 y^{t-1}(1-y)^{-s}dy \\ &= -\frac{(1-q)^s}{\log q}B(t,1-s)\end{aligned}$$

in $\mathrm{Re}(s) < 1$ and $\mathrm{Re}(t) > 0$. For the topics of generalized Raabe's formulas, see [5, 7].

Now we present some examples of the modified q-analogues of the Hurwitz zeta function. Each example below is easily checked by rewriting $B(\phi(s), 1-s)$ via the reflection formula or the multiplication formula.

Example 2.6. Put $\tilde{\zeta}_q(s,\phi(s),a) := \zeta_q(s,\phi(s),a) + \frac{(1-q)^s}{\log q}B(\phi(s),1-s)$.

(i) For $r \in \mathbb{Z}_{\geqslant 0}$ we have

$$\begin{aligned}\tilde{\zeta}(s,s+r,a) = &\frac{(1-q)^s}{\log q}\binom{s+r-1}{r}\frac{\pi}{\sin(\pi s)} \\ &+ \sum_{n=0}^{\infty}\frac{q^{(n+a)(s+r)}}{[n+a]_q^s} \qquad (\mathrm{Re}(s) > -r).\end{aligned}$$

(ii) [3, Corollary 2.4] For $l \in \mathbb{Z}_{>0}$ we have

$$\tilde{\zeta}(s,l,a) = \frac{(1-q)^s}{\log q}\frac{(l-1)!(-1)^l}{(s-1)(s-2)\cdots(s-l)} + \sum_{n=0}^{\infty}\frac{q^{(n+a)l}}{[n+a]_q^s} \qquad (s \in \mathbb{C}).$$

(iii) We have

$$\tilde{\zeta}\,(2s,s,1) = -\frac{(1-q)^{2s}}{\log q}\frac{\pi}{4^s}\frac{B(s,1/2)}{\cos(\pi s)} + \sum_{n=1}^{\infty}\frac{q^{ns}}{[n]_q^{2s}} \qquad (\mathrm{Re}(s) > 0).$$

Here we notice that $\tilde{\zeta}\,(2s,s,1)$ becomes an analogy of the spectral zeta function $Z(s; SU_q(2))$ associated to the quantum group $SU_q(2)$, which is introduced by Ueno and Nishizawa in [13].

3. q-analogues of Milnor multiple gamma functions

In this section we prove Theorem 1.3.

Proof of Theorem 1.3. Since we have

$$\frac{\partial}{\partial s}\tilde{\zeta}_q(s,s+r,a)\bigg|_{s=-r} = \frac{\partial}{\partial s}\tilde{\zeta}_q(s-r,s,a)\bigg|_{s=0},$$

we calculate the right hand side for $r \in \mathbb{Z}_{\geqslant 0}$.

We primarily notice that

$$\begin{aligned}\tilde{\zeta}_q(s-r,s,a) &= \frac{(1-q)^{s-r}}{\log q}\frac{(-1)^r\pi}{\sin(\pi s)}\binom{s-1}{r} \\ &\quad + (1-q)^{-r}\sum_{k=0}^{r}\binom{r}{k}(-1)^k\zeta_q(s,s+k,a).\end{aligned} \tag{3.1}$$

This is checked as

$$\begin{aligned}\zeta_q(s-r,s,a) &= \sum_{n=0}^{\infty}\frac{[a+n]_q^r q^{(a+n)s}}{[a+n]_q^s} \\ &= (1-q)^{-r}\sum_{n=0}^{\infty}(1-q^{a+n})^r\frac{q^{(a+n)s}}{[a+n]_q^s} \\ &= (1-q)^{-r}\sum_{n=0}^{\infty}\sum_{k=0}^{r}\binom{r}{k}(-1)^k\frac{q^{(a+n)(s+k)}}{[a+n]_q^s} \\ &= (1-q)^{-r}\sum_{k=0}^{r}\binom{r}{k}(-1)^k\zeta_q(s,s+k,a).\end{aligned}$$

On one hand, for $1 \leqslant k \leqslant r$, we calculate as

$$\begin{aligned}\frac{\partial}{\partial s}\zeta_q(s,s+k,a)\bigg|_{s=0} &= \sum_{n=0}^{\infty}\frac{\partial}{\partial s}\frac{q^{(a+n)(s+k)}}{[a+n]_q^s}\bigg|_{s=0} \\ &= \log q\sum_{n=0}^{\infty}(a+n)q^{k(a+n)} - \sum_{n=0}^{\infty}q^{k(a+n)}\log[a+n]_q \\ &= q^{ka}\log q\left(\frac{a}{1-q^k}+\frac{q^k}{(1-q^k)^2}\right) \\ &\quad + \sum_{n=0}^{\infty}q^{k(a+n)}\left\{\log(1-q)-\log(1-q^{a+n})\right\} \\ &= \frac{q^{ka}}{1-q^k}\left(a\log q+\log(1-q)+\frac{q^k\log q}{1-q^k}\right) \\ &\quad - \sum_{n=0}^{\infty}q^{k(a+n)}\log(1-q^{a+n}).\end{aligned}$$

On the other hand, by almost the same calculation as in [6, Theorem 4.1], we have

$$\begin{aligned}\zeta_q(s,s,a) &= (1-q)^s \sum_{n=0}^{\infty} q^{(n+a)s}(1-q^{n+a})^{-s} \\ &= (1-q)^s \sum_{n=0}^{\infty} q^{(n+a)s} \sum_{m=0}^{\infty} \binom{s+m-1}{m} q^{(n+a)m} \\ &= (1-q)^s \sum_{m=0}^{\infty} \binom{s+m-1}{m} \frac{q^{(s+m)a}}{1-q^{s+m}} \\ &= \frac{(1-q)^s q^s}{1-q^s} + \left(\sum_{m=1}^{\infty} \frac{1}{m}\frac{q^{ma}}{1-q^m}\right) s + O(s^2) \\ &= \frac{(1-q)^s q^s}{1-q^s} - \left(\sum_{n=0}^{\infty} \log(1-q^{a+n})\right) s + O(s^2) \qquad (\text{around } s=0).\end{aligned}$$

Putting

$$\frac{(-1)^r \pi}{\sin(\pi s)} \binom{s-1}{r} = \sum_{m=0}^{\infty} c(r+1,m) s^{m-1}, \tag{3.2}$$

and noting

$$\frac{q^s}{1-q^s} = -\sum_{m=0}^{\infty} B_m(a) \frac{(s\log q)^{m-1}}{m!} \qquad (\text{around } s=0) \tag{3.3}$$

and $B_0(a) = c(r+1,0) = 1$, we have

$$\begin{aligned}(1-q)^{s-r}\frac{q^s}{1-q^s} &+ \frac{(1-q)^{s-r}}{\log q}\frac{(-1)^r\pi}{\sin(\pi s)}\binom{s-1}{r} \\ &= (1-q)^{s-r}\sum_{m=1}^{\infty}\left(-B_m(a)\frac{(\log q)^{m-1}}{m!} + \frac{c(r+1,m)}{\log q}\right) s^{m-1}.\end{aligned} \tag{3.4}$$

This equation and

$$(1-q)^{s-r} = (1-q)^{-r} + (1-q)^{-r}\log(1-q)s + O(s^2) \tag{3.5}$$

yield

$$\begin{aligned}\frac{\partial}{\partial s}(1-q)^{s-r}&\left\{\frac{q^s}{1-q^s} + \frac{(-1)^r}{\log q}\frac{\pi}{\sin(\pi s)}\binom{s-1}{r}\right\}\Bigg|_{s=0} \\ &= (1-q)^{-r}\left[\left\{\frac{c(r+1,1)}{\log q} - B_1(a)\right\}\log(1-q)\right. \\ &\qquad \left. + \left(\frac{c(r+1,2)}{\log q} - \frac{1}{2}B_2(a)\log q\right)\right].\end{aligned}$$

We also notice that

$$\begin{aligned}
&-\sum_{n=0}^{\infty}\log(1-q^{a+n})+\sum_{k=1}^{r}\binom{r}{k}(-1)^k\,\frac{\partial}{\partial s}\zeta_q(s,s+k,a)\Big|_{s=0}\\
&\qquad=\sum_{k=1}^{r}\binom{r}{k}\frac{(-1)^k q^{ka}}{1-q^k}\left(a\log q+\log(1-q)+\frac{q^k\log q}{1-q^k}\right)\\
&\qquad\quad-\sum_{n=0}^{\infty}\sum_{k=0}^{r}\binom{r}{k}(-1)^k q^{k(a+n)}\log(1-q^{a+n})\\
&\qquad=\sum_{k=1}^{r}\binom{r}{k}\frac{(-1)^k q^{ka}}{1-q^k}\left(a\log q+\log(1-q)+\frac{q^k\log q}{1-q^k}\right)\\
&\qquad\quad-\sum_{n=0}^{\infty}(1-q^{a+n})^r\log(1-q^{a+n}).
\end{aligned}$$

Combining the results above, we have

$$\begin{aligned}
&\frac{\partial}{\partial s}\tilde{\zeta}_q(s,s+r,a)\Big|_{s=-r}=\frac{\partial}{\partial s}\tilde{\zeta}_q(s-r,s,a)\Big|_{s=0}\\
&\quad=\frac{\partial}{\partial s}\left\{\zeta_q(s-r,s,a)+\frac{(1-q)^{s-r}}{\log q}\frac{(-1)^r\pi}{\sin(\pi s)}\binom{s-1}{r}\right\}\Big|_{s=0}\\
&\quad=\frac{\partial}{\partial s}(1-q)^{s-r}\left\{\frac{q^s}{1-q^s}+\frac{(-1)^r}{\log q}\frac{\pi}{\sin(\pi s)}\binom{s-1}{r}\right\}\Big|_{s=0}\\
&\qquad+(1-q)^{-r}\left[-\sum_{n=0}^{\infty}\log(1-q^{a+k})+\sum_{k=1}^{r}\binom{r}{k}(-1)^k\,\frac{\partial}{\partial s}\zeta_q(s,s+k,a)\Big|_{s=0}\right]\\
&\quad=(1-q)^{-r}\\
&\qquad\times\left\{\left(\frac{c(r+1,1)}{\log q}-B_1(a)\right)\log(1-q)+\left(\frac{c(r+1,2)}{\log q}-\frac{1}{2}B_2(a)\log q\right)\right.\\
&\qquad\left.+\sum_{k=1}^{r}\binom{r}{k}(-1)^k\frac{(-1)^k q^{ka}}{1-q^k}\left(a\log q+\log(1-q)+\frac{q^k\log q}{1-q^k}\right)\right\}\\
&\qquad-\sum_{n=0}^{\infty}[a+n]_q^r\log(1-q^{a+n}).
\end{aligned}$$

Lastly, the explicit values of $c(r+1,1)$ and $c(r+1,2)$ are checked by elementary calculations. This completes the proof of Theorem 1.3. ■

Remark 3.1. In [11] q-analogues of the generalized gamma function (q-Barnes-Milnor type) are treated. In the paper, the author studied their basic properties such as quasi-periodicity and the multiplication formula.

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