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SOME CONDITIONAL RESULTS ON PRIMES BETWEEN CONSECUTIVE SQUARES

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Abstract: A well-known conjecture about the distribution of primes asserts that between two consecutive squares there is always at least one prime number. The proof of this conjecture is quite out of reach at present, even under the assumption of the Riemann Hypothesis. The aim of this paper is to provide the upper bounds for the exceptional set for this conjecture under the assumption of some heuristic hypotheses.

Keywords: distribution of prime numbers, primes between squares.

1. Introduction

A well known conjecture about the distribution of primes asserts that for every positive integer n, the interval $[n^2,(n+1)^2]$ contains at least one prime. The proof of this conjecture is quite out of reach at present, even under the assumption of the Riemann Hypothesis. However it is not difficult to prove unconditionally that the conjecture holds for almost all positive integers n. Indeed, we can prove that almost all intervals of the type $[n^2,(n+1)^2]$ contain the expected number of primes.

This paper is concerned with the exceptional set for the distribution of primes between two consecutive squares, under the assumption of some unproved heuristic hypotheses. The basic idea was to connect the exceptional set for the distribution of primes in intervals of the type $[n^2,(n+1)^2]$ to the exceptional set of the asymptotic formula for the distribution of primes in short intervals. The properties of the latter set, see D. Bazzanella and A. Perelli [2], were thus used to obtain the desired results.

In a previous paper, see D. Bazzanella [1], the author proved that each of the intervals $[n^2,(n+1)^2]\subset[1,N]$, with at most $O(N^{1/4+\varepsilon})$ exceptions, contained the expected number of primes, for every constant $\varepsilon>0$. Under the assumption of the Riemann Hypothesis, the author also proved that each of the intervals $[n^2,(n+1)^2]\subset[1,N]$, with at most $O(f(N)\log^2 N)$ exceptions, contained the expected number of primes, for every real valued function $f(x)\to\infty$ arbitrarily slowly.

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Under the assumption of a stronger hypotheses than the Riemann Hypothesis, we would expect to obtain smaller bounds for the exceptional set. Infact, assuming a strong form of the Montgomery Conjecture, the author proved that, with n sufficiently large, each of the intervals $[n^2,(n+1)^2]$ contained the expected number of primes without exceptions, see [1]. On the other hand, under the assumption of a weaker hypotheses than the Riemann Hypothesis, we would expect to achieve upper bounds of sizes between $O(f(N)\log^2 N)$ and $O(N^{1/4+\varepsilon})$.

In order to estimate some sums which arose in our argument we employed the counting functions $N(\sigma,T)$ and $N^*(\sigma,T)$. The former is defined as the number of zeros $\rho=\beta+i\gamma$ of the Riemann zeta function which satisfy $\sigma\leqslant\beta\leqslant 1$ and $|\gamma|\leqslant T$, while $N^*(\sigma,T)$ is defined as the number of ordered sets of zeros $\rho_j=\beta_j+i\gamma_j$ $(1\leqslant j\leqslant 4)$, each counted by $N(\sigma,T)$, for which $|\gamma_1+\gamma_2-\gamma_3-\gamma_4|\leqslant 1$. As D. Bazzanella and A. Perelli [2] we considered the heuristic assumption that there exists a constant T_0 such that

$$N^*(\sigma,T)\ll\frac{N(\sigma,T)^4}{T}T^{\varepsilon} \tag{1.1}$$

for every $T\geqslant T_0$ and arbitrarily small $\varepsilon>0$, which is close to being the best possible, in view of the trivial estimate

$$N^*(\sigma,T)\gg\frac{N(\sigma,T)^4}{T}.$$

Recall that the Ingham–Huxley density estimate [5, Theorem 11.1] implies

$$N(\sigma,T)\ll\begin{cases}T^{3(1-\sigma)/(2-\sigma)+\nu} & 1/2\leqslant\sigma\leqslant 3/4\\ T^{3(1-\sigma)/(3\sigma-1)+\nu} & 3/4\leqslant\sigma\leqslant 1\end{cases}, \tag{1.2}$$

for arbitrarily $\nu>0$. Thus from (1.1) it follows that

$$N^*(\sigma,T)\ll\begin{cases}T^{(10-11\sigma)/(2-\sigma)+\varepsilon} & 1/2\leqslant\sigma\leqslant 3/4\\ T^{(13-15\sigma)/(3\sigma-1)+\varepsilon} & 3/4\leqslant\sigma\leqslant 13/15\end{cases} \tag{1.3}$$

Remark. The estimate of $N^*(\sigma,T)$ proved by D. R. Heath-Brown, see [5, Lemma 12.7], implies that the upper bound (1.3) holds for $1/2\leqslant\sigma\leqslant 2/3$.

Assuming that (1.3) holds also for $\sigma>2/3$ we obtain the first result.

Theorem 1.1. *Let $\varepsilon>0$ be arbitrarily small and assume (1.3). Then each of the intervals $[n^2,(n+1)^2]\subset[1,N]$, with at most $O(N^{1/5+\varepsilon})$ exceptions, contains the expected number of primes.*

If we assume the Lindelöf hypothesis, which states that, for every $\eta>0$, the Riemann Zeta-function satisfies

$$\zeta(\sigma+it)\ll t^{\eta}\qquad(1/2\leqslant\sigma\leqslant 1, t\geqslant 2),$$

we can get the following stronger result.

Theorem 1.2. *Let $\varepsilon > 0$ be arbitrarily small and assume the Lindelöf hypothesis. Then each of the intervals $[n^2, (n+1)^2] \subset [1, N]$, with at most $O(N^\varepsilon)$ exceptions, contains the expected number of primes.*

Finally we assume the Density Hypothesis, which states that for every $\eta > 0$ the counting function $N(\sigma, T)$ satisfies

$$N(\sigma, T) \ll T^{2(1-\sigma)+\eta} \qquad (1/2 \leqslant \sigma \leqslant 1),$$

and we thus obtain our last result.

Theorem 1.3. *Let $\varepsilon > 0$ be arbitrarily small, assume the Density Hypothesis and (1.1). Then each of the intervals $[n^2, (n+1)^2] \subset [1, N]$, with at most $O(N^\varepsilon)$ exceptions, contains the expected number of primes.*

2. Definitions and preliminary lemmas

We will always assume that n, x and N are sufficiently large as prescribed by the various statements, and $\varepsilon > 0$ is arbitrarily small and not necessarily the same at each occurrence. As in [2] we define a set related to the asymptotic formula

$$\psi(x + h(x)) - \psi(x) \sim h(x) \qquad (x \to \infty) \tag{2.1}$$

as

$$E_\delta(N, h) = \{N \leqslant x \leqslant 2N : |\psi(x + h(x)) - \psi(x) - h(x)| \geqslant \delta h(x)\},$$

where $h(x)$ is an increasing function such that $x^\varepsilon \leqslant h(x) \leqslant x$ for some $\varepsilon > 0$. It is clear that (2.1) holds if and only if for every $\delta > 0$ there exists $N_0(\delta)$ such that $E_\delta(N, h) = \emptyset$ for every $N \geqslant N_0(\delta)$. Hence for small $\delta > 0$, N tending to ∞ and $h(x)$ which is suitably small with respect to x, the set $E_\delta(N, h)$ contains the exceptions, if any, to the expected asymptotic formula for the number of primes in short intervals. Our first lemma is concerned with the structure of the exceptional set above.

Lemma 2.1. *Let $\delta > 0$, $x^\varepsilon \leqslant h(x) \leqslant x$, for some $\varepsilon > 0$ and N be sufficiently large depending on the function $h(x)$. If $x_0 \in E_\delta(N, h)$ then there exists an effective constant c, depending on δ and $h(x)$, such that*

$$[x_0, x_0 + c \cdot h(N)] \cap [N, 2N] \subset E_{\delta/2}(N, h).$$

Lemma 2.1 may be proved along the same lines as Theorem 1 of [2], and essentially says that if we have a single exception in $E_\delta(N, h)$, with a fixed δ, then we necessarily have an interval of exceptions in $E_{\delta/2}(N, h)$. Moreover we define a set related to the asymptotic formula

$$\psi((n+1)^2) - \psi(n^2) \sim 2n \qquad (n \to \infty) \tag{2.2}$$

as

$$A_\delta(N) = \{\sqrt{N} \leqslant n \leqslant \sqrt{2N} : |\psi((n+1)^2) - \psi(n^2) - (2n+1)| \geqslant \delta(2n+1)\},$$

that contains the exceptions, if any, to the expected asymptotic formula for the number of primes in intervals of the type $[n^2,(n+1)^2]\subset[N,2N]$. The main tool of the proofs is the following lemma.

Lemma 2.2. *For $h(x)=2\sqrt{x}+1$ and every $\delta>0$ we have*

$$|A_\delta(N)|\ll_\delta \frac{|E_{\delta/2}(N,h)|}{\sqrt{N}}+1.$$

Proof. Let $n\in A_\delta(N)$ and put $x=n^2\in[N,2N]$. From the definition of the set $A_\delta(N)$ we get

$$|\psi((n+1)^2)-\psi(n^2)-(2n+1)|\geqslant\delta(2n+1)$$

and thus

$$|\psi(x+h(x))-\psi(x)-h(x)|\geqslant\delta h(x),$$

which implies $x\in E_\delta(N,h)$. From Lemma 2.1 follows that there exists an effective constant c such that

$$[x,x+c\cdot h(x)]\cap[N,2N]\subset E_{\delta/2}(N,h).$$

Let $m\in A_\delta(N)$, $m>n$. As before we can define $y=m^2\in[N,2N]$ and obtain, again by Lemma 2.1, that

$$[y,y+c\cdot h(y)]\cap[N,2N]\subset E_{\delta/2}(N,h).$$

If we choose $c<1$, we get

$$y-x=m^2-n^2\geqslant(n+1)^2-n^2=2n+1=2\sqrt{x}+1>ch(x)$$

and thus

$$[x,x+c\cdot h(x)]\cap[y,y+c\cdot h(y)]=\emptyset.$$

Hence the lemma is proved, since for every $n\in A_\delta(N)$ and $x=n^2$, with at most one exception, we have

$$[x,x+c\cdot h(x)]\subset[N,2N]. \qquad \blacksquare$$

The next lemma concerns the conditional estimate for the fourth power mean value of the function $\psi(y)$ in short intervals.

Lemma 2.3. *Assume the Lindelöf hypothesis and let $\varepsilon>0$ be arbitrarily small. Then there exists a function $\Sigma(y,T)$ such that for every $\varepsilon>0$ we have*

$$\int_N^{2N}\left|\psi\left(y+\frac{y}{T}\right)-\psi(y)-\frac{y}{T}+\Sigma(y,T)\right|^4 \mathrm{d}y\ll N^{4+\varepsilon}T^{-3} \tag{2.3}$$

and

$$\Sigma(y,T) \ll \frac{y}{T\log y}, \tag{2.4}$$

uniformly for $N \geqslant 2, 1 \leqslant T \leqslant N$ *and* $N \leqslant y \leqslant 2N$.

Lemma 2.3 is due to G. Yu [7, Lemma B]. The interesting consequence of this lemma is that it allows to obtain the following conditional bound for the exceptional set $E_\delta(N,h)$, with $h(x) = 2\sqrt{x}+1$.

Lemma 2.4. *Assume the Lindelöf hypothesis, let* $h(x) = 2\sqrt{x}+1$ *and let* $\varepsilon > 0$ *be arbitrarily small. Then*

$$|E_\delta(N,h)| \ll N^{1/2+\varepsilon}. \tag{2.5}$$

Proof. In order to prove the lemma, we subdivide $[N,2N]$ into $O(\log^2 N)$ intervals of type $I_j = [N_j, N_j+Y]$ with

$$N \leqslant N_j < 2N \qquad \text{and} \qquad Y \ll \frac{N}{\log^2 N}.$$

For every $y \in E_\delta(N,h)$ we have

$$|\psi(y+h(y)) - \psi(y) - h(y)| \gg N^{1/2}$$

and then

$$|E_\delta(N,h)|N^2 \ll \int\limits_{E_\delta(N,h)} |\psi(y+h(y)) - \psi(y) - h(y)|^4 \,\mathrm{d}y \tag{2.6}$$

$$\ll \sum_j \int\limits_{E_\delta^j(N,h)} |\psi(y+h(y)) - \psi(y) - h(y)|^4 \,\mathrm{d}y,$$

where $E_\delta^j(N,h) = E_\delta(N,h) \cap [N_j, N_j+Y]$.

If we choose $T_j = N_j^{1/2}/2$ we have, by Lemma 2.3, that there exist functions $\Sigma_j(y,T_j)$ satisfy the conditions (2.3) and (2.4), for all values of j. From the Brun–Titchmarsh theorem, see H. L. Montgomery and R. C. Vaughan [6], we can deduce

$$\Big(\psi(y+h(y)) - \psi(y) - h(y)\Big) - \Big(\psi(y+\frac{y}{T_j}) - \psi(y) - \frac{y}{T_j} + \Sigma_j(y,T_j)\Big) \ll \frac{y}{T_j\log y},$$

for every j and every $y \in E_\delta^j(X_n,\theta)$.

Then from (2.6) it follows that

$$
\begin{aligned}
|E_\delta(N,h)|N^2 \ll & \sum_j \int_{E_\delta^j(N,h)} \left|\psi(y+\frac{y}{T_j})-\psi(y)-\frac{y}{T_j}+\Sigma_j(y,T_j)\right|^4 \mathrm{d}y \\
& + \sum_j \int_{E_\delta^j(N,h)} \left|\frac{y}{T_j\log y}\right|^4 \mathrm{d}y \\
\ll & \sum_j \int_N^{2N} \left|\psi(y+\frac{y}{T_j})-\psi(y)-\frac{y}{T_j}+\Sigma_j(y,T_j)\right|^4 \mathrm{d}y \\
& + |E_\delta(N,h)|\frac{N^2}{\log^4 N}
\end{aligned}
$$

which implies

$$
|E_\delta(N,h)|N^2 \ll \sum_j \int_N^{2N} \left|\psi(y+\frac{y}{T_j})-\psi(y)-\frac{y}{T_j}+\Sigma_j(y,T_j)\right|^4 \mathrm{d}y. \tag{2.7}
$$

At this point we use Lemma 2.3 to get

$$
\begin{aligned}
|E_\delta(N,h)|N^2 & \ll \sum_j \int_N^{2N} \left|\psi(y+\frac{y}{T_j})-\psi(y)-\frac{y}{T_j}+\Sigma_j(y,T_j)\right|^4 \mathrm{d}y \\
& \ll \sum_j N^{4+\varepsilon}T_j^{-3} \\
& \ll N^{5/2+\varepsilon}
\end{aligned}
$$

and this leads to (2.5). ■

3. Proof of the theorems

Let $h(x) = 2\sqrt{x}+1$ and use the classical explicit formula, see H. Davenport [3, chapter 17], to write

$$
\psi(x+h(x)) - \psi(x) - h(x) = -\sum_{|\gamma|\leqslant T} x^\rho c_\rho(x) + O\left(\frac{N\log^2 N}{T}\right), \tag{3.1}
$$

uniformly for $N \leqslant x \leqslant 2N$, where $10 \leqslant T \leqslant N$, $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$ and

$$
c_\rho(x) = \frac{(1+h(x)/x)^\rho - 1}{\rho}.
$$

Let

$$T = \sqrt{N} \log^3 N \tag{3.2}$$

and note that

$$c_\rho(x) \ll \min\left(\frac{1}{\sqrt{N}}, \frac{1}{|\gamma|}\right). \tag{3.3}$$

If we follow the method of D. R. Heath-Brown we can find a constant $0 < u < 1$ such that

$$\sum_{|\gamma| \leqslant T,\ \beta > u} x^\rho c_\rho(x) \ll \frac{\sqrt{N}}{\log N},$$

see [5, page 319].

Remark. More precisely from (3.2) it follows that we can choose $5/6 \leqslant u < 1$. Note that under the assumption of the Density Hypothesis we have the weaker condition $11/14 \leqslant u < 1$.

Thus we have

$$\psi(x + h(x)) - \psi(x) - h(x) = - \sum_{|\gamma| \leqslant T,\ \beta \leqslant u} x^\rho c_\rho(x) + O\left(\frac{\sqrt{N}}{\log N}\right)$$

and then

$$|E_\delta(N, h)| N^2 \ll \int_N^{2N} \left| \sum_{|\gamma| \leqslant T,\ \beta \leqslant u} x^\rho c_\rho(x) \right|^4 \mathrm{d}x. \tag{3.4}$$

To estimate the fourth power integral we divide the interval $[0, u]$ into $O(\log N)$ subintervals I_k of the form

$$I_k = \left[\frac{k}{\log N}, \frac{k+1}{\log N}\right].$$

By the Hölder inequality we have

$$\left| \sum_{|\gamma| \leqslant T,\ \beta \leqslant u} x^\rho c_\rho(x) \right|^4 \ll \log^3 N \sum_k \left| \sum_{|\gamma| \leqslant T,\ \beta \in I_k} x^\rho c_\rho(x) \right|^4.$$

Following again the method of D. R. Heath-Brown, we can get

$$\int_N^{2N} \left| \sum_{|\gamma| \leqslant T,\ \beta \leqslant u} x^\rho c_\rho(x) \right|^4 \mathrm{d}x \ll N^{-1+\varepsilon} \max_{\sigma \leqslant u} N^{4\sigma} M(\sigma, T), \tag{3.5}$$

where

$$M(\sigma, T) = \sum_{\substack{\beta_1, \ldots, \beta_4 \geqslant \sigma \\ |\gamma_1| \leqslant T, \ldots, |\gamma_4| \leqslant T}} \frac{1}{1 + |\gamma_1 + \gamma_2 - \gamma_3 - \gamma_4|}$$

and

$$M(\sigma,T) \ll N^*(\sigma,T)\log N, \tag{3.6}$$

see [5, p. 336]. From (3.4), (3.5) and (3.6) it follows that

$$|E_\delta(N,h)| \ll N^{-3+\varepsilon} \max_{\sigma\leqslant u} N^{4\sigma} N^*(\sigma,T). \tag{3.7}$$

To prove Theorem 1.1 we assume (1.3). Then we write

$$|E_\delta(N,h)| \ll \begin{cases} N^{-3+\varepsilon} \max\limits_{\sigma\leqslant u} N^{4\sigma} T^{(10-11\sigma)/(2-\sigma)+\varepsilon} & 1/2 \leqslant \sigma \leqslant 3/4 \\ N^{-3+\varepsilon} \max\limits_{\sigma\leqslant u} N^{4\sigma} T^{(13-15\sigma)/(3\sigma-1)+\varepsilon} & 3/4 \leqslant \sigma \leqslant 13/15 \end{cases}.$$

The above upper bound for the exceptional set attains its maximum at $\sigma = 3/4$, so we get

$$|E_\delta(N,h)| \ll N^{7/10+\varepsilon},$$

for every $\delta > 0$. From Lemma 2.2 we conclude

$$|A_\delta(N)| \ll_\delta \frac{|E_{\delta/2}(N,h)|}{\sqrt{N}} + 1 \ll N^{1/5+\varepsilon},$$

for every $\delta > 0$. Then the proof of Theorem 1.1 is complete.

Remark. In Theorem 1.1 we need to assume the upper bound (1.3) only for $(139 - \sqrt{761})/160 \leqslant \sigma \leqslant 5/6$. For smaller values of σ the estimate of $N^*(\sigma,T)$ proved by D. R. Heath-Brown, see [5, Lemma 12.7], is enough and the bigger values are not involved in (3.7).

To prove Theorem 1.2 we assume the Lindelöf hypothesis. By Lemma 2.4 we have (2.5) and then, again by Lemma 2.2, we obtain that

$$|A_\delta(N)| \ll_\delta \frac{|E_{\delta/2}(N,h)|}{\sqrt{N}} + 1 \ll N^{\varepsilon},$$

for every $\delta > 0$. Then Theorem 1.2 is proved.

In order to prove Theorem 1.3 we follow the proof of Theorem 1.1 until the equation (3.7). Then we use (1.1) and the Density Hypothesis to have

$$\begin{aligned} |E_\delta(N,h)| &\ll N^{-7/2+\varepsilon} \left(\max_{\sigma\leqslant u} N^{\sigma} N(\sigma,T) \right)^4 \\ &\ll N^{-7/2+\varepsilon} \left(\max_{\sigma\leqslant u} N^{\sigma} T^{2(1-\sigma)} \right)^4 \\ &\ll N^{1/2+\varepsilon}. \end{aligned}$$

By Lemma 2.2, we can conclude that

$$|A_\delta(N)| \ll_\delta \frac{|E_{\delta/2}(N,h)|}{\sqrt{N}} + 1 \ll N^{\varepsilon},$$

for every $\delta > 0$, and this complete the proof of Theorem 1.3.

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