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ON MULTIPLE EXPONENTIAL SUMS AND THEIR APPLICATIONS

H.-Q. Liu

Abstract: We prove new estimates for the remainder terms in the known asymptotic formulas for three famous problems, by using the contemporary bounds for triple exponential sums.

Keywords: multiple exponential sums, direct factors of abelian groups, squarefree divisor problem, primitive circle problem.

1. Introduction

The van der Corput's method and its two-dimensional extensions for estimating exponential sums always play importants roles in studying many problems of number theory. In the last decades some new methods were introduced in the field of estimating multiple exponential sums. We mention them as follows.

(a) The double large sieve inequality of [BI], which was used in [FI] for exponential sums with monomials(see also [SW],[RS] for the later development). This new method allows the simultaneous varying of many variables.
(b) The discrete Weyl's shift technique of [HB] for bounding triple exponential sums with monomials(see Lemma 2.1 of [W3] for the generalized version).
(c) Jia's technique of [J](see its Lemma 13) for estimating double exponential sums with bilinear complex coefficients(for the generalized version see (2.1) of [W2]).

On the basis of these novel methods, in this paper we shall obtain new results for the following three problems.

Let $t(G)$be the number of direct factors of a finite abelian group G(for an explanation, see [C]), and

$$T(x) = \sum t(G),$$

where the summation is taken over all abelian groups of order not exceeding x, x is a sufficiently large positive number. The asymptotic formula of $T(x)$ has been

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investigated in many papers. In particular, in [K] it was shown that

$$T(x) = \text{main terms} + \Delta(x), \Delta(x) = O\left(x^{\theta+\varepsilon}\right),$$

if, for $d(1,1,2,2;n) = \left|\{(n_1,n_2,n_3,n_4) : n_1n_2n_3^2n_4^2 = n, n_i \in \mathrm{Z}^+\}\right|$, there holds

$$\sum_{n \leqslant x} d(1,1,2,2;n) = \text{main terms} + \Delta_1(x), \Delta_1(x) = O\left(x^{\theta+\varepsilon}\right),$$

where $\theta < \frac{1}{2}$ is some positive constant, and ε is any sufficiently small positive constant. Since the estimate of $\Delta_1(x)$ can be reduced to the estimate of exponential sums which can be estimated by van der Corput's method, work of [K] causes great interest of number theorists for getting new permissible values of θ. In [K] an admissible value $\theta = \frac{5}{12}$ was derived. Subsequently this was improved in [M1]($\theta = \frac{83}{201}$), [MS]($\theta = \frac{45}{109}$), [L1]($\theta = \frac{2}{5}$), [Y]($\theta = \frac{3}{8}$), [L2]($\theta = \frac{7}{19}$), [ZC]($\theta = \frac{4}{11} = 0.3636...$), [LW1]($\theta = \frac{21}{58} = 0.3620...$) and [W1]($\theta = \frac{47}{130} = 0.3615...$). In this paper we shall deduce the following new result.

Theorem 1.1. $\theta = \frac{13}{36} = 0.3611...$ *is permissible.*

To deduce Theorem 1.1, we shall use the new bound for triple exponential sums with monomials of the recent paper [RS](which is best possible in some sense under that setting), and we shall follow the approach of [ZC] to simplify the arguments of [W1]. Note that in [ZC] the authors used the expansion of the remainder term of the Dirichlet divisor problem obtained via the complex analysis (involving the analytical theory of the Riemann zeta-function), and thus the treatments are somewhat coincise. On the other hand, the formula of [V] for generalized many dimensional divisor problems has been suspected by us in another paper (from which the results of [L2] and [SW] on the distribution of 4-full integers also turn out to be invalid). If we use our recent result of [L4] for triple exponential sums (improving Theorem 1.1 of [RS]), we can improve the error term $O(x^{\theta+\varepsilon})$ to $O(x^{\theta}\exp(c\sqrt{\log x}))(c > 0)$(note that the factor "$x^{\varepsilon}$" can not be reduced by the method of [RS]).

Using quite the similar arguments, we can improve the result $\frac{4}{11}$ of [B] on the squarefree divisor problem and the result $\frac{221}{608}$ of [W4].

Theorem 1.2. *Under the Riemann hypothesis(RH),* $\Delta_2(x) = O\left(x^{\frac{13}{36}} + \varepsilon\right)$ *holds.*

Theorem 1.3. *Under RH,* $\left|\left\{m^2 + n^2 \leqslant x, (m,n) = 1\right\}\right| = \frac{6}{\pi^2}x + O\left(x^{\frac{13}{36}} + \varepsilon\right).$

2. Proof of Theorem 1.1

For the sake of simplicity, we just quote but do not list again the existing important results for estimating exponential sums of the literatures.

In view of the Basic Lemma, the Proposition 2, the proof of Case 2 of Proposition 1, and the arguments in treating Case 1 of Proposition 1 of [ZC], to deduce our result it suffices to deduce that

$$\begin{aligned} T(M,N) :&= \sum_{m\sim M} a_m \sum_{n\sim N} b_n e\left(\tfrac{2\sqrt{nx}}{m}\right) \\ &\ll M^{\frac{1}{2}} N^{\frac{3}{4}} x^{\theta' + \frac{1}{2}\varepsilon}, \qquad \theta' = \theta - \tfrac{1}{4},\ \theta = \tfrac{13}{36}, \end{aligned} \tag{1}$$

where $M \leqslant x^{1-2\theta}$, $N \leqslant x^{1-2\theta}$, $\delta = \varepsilon^2$, $a_m = d(m)m^{-\frac{1}{2}}M^{\frac{1}{2}}x^{-\delta} \leqslant 1$, $b_n = d(n)n^{-\frac{3}{4}}N^{\frac{3}{4}}x^{-\delta} \leqslant 1$, and $e(\xi) = \exp(2\pi i\xi)$. Using (2.1) of [W2] with the exponent pair $(k,\lambda) = (\frac{1}{2}, \frac{1}{2})$, we get

$$\begin{aligned} x^{-\delta}T(M,N) \ll M^{\frac{1}{2}}N^{\frac{3}{4}}\Big(&\sqrt[20]{x^2M^{-1}N^2} + \sqrt[36]{x^3N^4} + \sqrt[20]{x^2N^3M^{-2}} \\ &+ \sqrt[44]{x^2M^2N^9} + \sqrt[36]{M^6N^7} + M^{\frac{1}{2}}N^{-\frac{1}{4}} \\ &+ \sqrt[44]{x^{-1}M^8N^{12}} + \sqrt[4]{x^{-1}M^4}\Big). \end{aligned} \tag{2}$$

Similarly, using (2.1) of [W2] with the exponent pair $(\frac{1}{2}, \frac{1}{2})$, but with the roles of M and N changed, we obtain

$$\begin{aligned} x^{-\delta}T(M,N) \ll M^{\frac{1}{2}}N^{\frac{3}{4}}\Big(&\sqrt[20]{x^2M} + \sqrt[36]{x^3M^4} + \sqrt[20]{x^2M^2N^{-1}} \\ &+ \sqrt[44]{x^2M^{14}N^{-3}} + \sqrt[36]{M^{16}N^{-3}} + \sqrt[44]{x^{-1}M^{24}N^{-5}} \\ &+ N^{\frac{1}{4}} + Mx^{-\frac{1}{4}}\Big). \end{aligned} \tag{3}$$

Using (2.1) of [W2] with the exponent pair $(k,\lambda) = (\frac{2}{7}, \frac{4}{7})$, we obtain

$$\begin{aligned} x^{-\delta}T(M,N) \ll M^{\frac{1}{2}}N^{\frac{3}{4}}\Big(&\sqrt[116]{x^{11}M^9} + \sqrt[20]{x^2M^2N^{-1}} + \sqrt[108]{x^9M^{13}N^{-8}} \\ &+ \sqrt[44]{x^2M^{14}N^{-3}} + M^{\frac{1}{2}}N^{-\frac{1}{4}} + N^{\frac{1}{4}} \\ &+ \sqrt[44]{x^{-1}M^{26}N^{-6}} + Mx^{-\frac{1}{4}}\Big). \end{aligned} \tag{4}$$

Using Corollary 1 of [LW2] we obtain

$$\begin{aligned} x^{-\delta}T(M,N) \ll M^{\frac{1}{2}}N^{\frac{3}{4}}\Big(&\sqrt[12]{x^2M^{-4}N} + \sqrt[16]{x^2M^{-1}} + \sqrt[8]{xN^{-1}} + \sqrt[12]{xM} \\ &+ M^{\frac{1}{2}}N^{-\frac{1}{4}} + N^{\frac{1}{4}} + \sqrt[4]{x^{-1}M^4}\Big). \end{aligned} \tag{5}$$

Our purpose is to deduce from (2), (3), (4) and (5) the following estimate

$$T'(M,N) := M^{-\frac{1}{2}}N^{-\frac{3}{4}}x^{-\delta}T(M,N) \ll M^{\frac{1}{2}}N^{-\frac{1}{4}} + x^{\theta'}. \tag{6}$$

First we note that the following terms of (2)–(5)

$$\sqrt[36]{M^6N^7}, \qquad \sqrt[44]{x^{-1}M^8N^{12}}, \qquad Mx^{-\frac{1}{4}}, \qquad N^{\frac{1}{4}}, \qquad \sqrt[12]{xM}$$

can be estimated trivially as $O(x^{\theta'})$, by using only $M \leqslant x^{1-2\theta}$ and $N \leqslant x^{1-2\theta}$. Thus, from (3) and (4) we get

$$T'(M,N) \ll \sqrt[8]{xN^{-1}} + M^{\frac{1}{2}}N^{-\frac{1}{4}} + \sqrt[16]{x^2M^{-1}} + \sum_{1\leqslant i\leqslant 5} R_i + x^{\theta'},$$

where

$$\begin{aligned}
R_1 &= \min\left(\sqrt[12]{x^2M^{-4}N},\ \sqrt[116]{x^{11}M^9}\right) \leqslant \sqrt[572]{x^{62}N^9},\\
R_2 &= \min\left(\sqrt[12]{x^2M^{-4}N},\ \sqrt[20]{x^2M^2N^{-1}}\right) \leqslant \sqrt[52]{x^6N^{-1}},\\
R_3 &= \min\left(\sqrt[12]{x^2M^{-4}N},\ \sqrt[108]{x^9M^{13}N^{-8}}\right) \leqslant \sqrt[588]{x^{62}N^{-19}} \leqslant x^{\theta'},\\
R_4 &= \min\left(\sqrt[12]{x^2M^{-4}N},\ \sqrt[44]{x^2M^{14}N^{-3}}\right) \leqslant \sqrt[172]{x^{18}N},\\
R_5 &= \min\left(\sqrt[12]{x^2M^{-4}N},\ \sqrt[44]{x^{-1}M^{26}N^{-6}}\right) \leqslant \sqrt[244]{x^{25}N}.
\end{aligned}$$

(we have used $N \gg 1$). Using $N \leqslant x^{1-2\theta}$ we find that $R_1 + R_4 + R_5 = O(x^{\theta'})$. Thus we get

$$T'(M,N) \ll M^{\frac{1}{2}}N^{-\frac{1}{4}} + \sqrt[8]{xN^{-1}} + \sqrt[16]{x^2M^{-1}} + \sqrt[52]{x^6N^{-1}} + x^{\theta'}. \tag{7}$$

From (3) and (7) we have

$$T'(M,N) \ll \sqrt[8]{xN^{-1}} + M^{\frac{1}{2}}N^{-\frac{1}{4}} + \sqrt[52]{x^6N^{-1}} + \sum_{6\leqslant i\leqslant 11} R_i + x^{\theta'}, \tag{8}$$

where

$$\begin{aligned}
R_6 &= \min\left(\sqrt[16]{x^2M^{-1}},\ \sqrt[20]{x^2M}\right) \leqslant x^{\frac{1}{9}} = x^{\theta'},\\
R_7 &= \min\left(\sqrt[16]{x^2M^{-1}},\ \sqrt[36]{x^3M^4}\right) \leqslant x^{0.11},\\
R_8 &= \min\left(\sqrt[16]{x^2M^{-1}},\ \sqrt[20]{x^2M^2N^{-1}}\right) \leqslant \sqrt[52]{x^6N^{-1}},\\
R_9 &= \min\left(\sqrt[16]{x^2M^{-1}},\ \sqrt[44]{x^2M^{14}N^{-3}}\right) \leqslant \sqrt[268]{x^{30}N^{-3}},\\
R_{10} &= \min\left(\sqrt[16]{x^2M^{-1}},\ \sqrt[36]{M^{16}N^{-13}}\right) \leqslant \sqrt[292]{x^{321}N^{-13}} \leqslant x^{\theta'},\\
R_{11} &= \min\left(\sqrt[16]{x^2M^{-1}},\ \sqrt[44]{x^{-1}M^{24}N^{-5}}\right) \leqslant \sqrt[428]{x^{47}N^{-5}} \leqslant x^{\theta'}.
\end{aligned}$$

(we have used $N \gg 1$). Thus from (8) we find that, if $N \geqslant x^{6-52\theta'}$ then (6) holds.Thus, to show (6) we can assume that $N < x^{6-52\theta'}$. Now from (2) we get (by using$M \leqslant x^{1-2\theta}$ also)

$$T'(M,N) \ll \sqrt[20]{x^2M^{-1}N^2} + \sqrt[20]{x^2M^{-2}N^3} + M^{\frac{1}{2}}N^{-\frac{1}{4}} + x^{\theta'},$$

and thus (6) follows if $M \geqslant N$. Thus we can suppose that $M < N < x^{6-52\theta'}$, and we find that (3) implies (6). Thus (6) always holds.

We give other bounds for $T'(M,N)$, so as to we can deduce (1) by combining (6).

Obviously, for two suitable numbers M_1 andM_2 with $\frac{1}{4}M \leqslant M_1M_2 \leqslant 2M$ and $1 \ll M_1 \ll M_2$ (by splitting the range of summation) we have (here $L = \log x$)

$$T'(M,N) \ll L^2N^{-\frac{3}{4}}M_1^{-\frac{1}{2}} \sum_{n\sim N}\sum_{m_1\sim M_1}\left|\sum_{m_2\in I(m_1)} m_2^{-\frac{1}{2}}e(-f(n,m_1,m_2))\right|, \qquad (9)$$

where $f(n,m_1,m_2) = 2(nx)^{\frac{1}{2}}m_1^{-1}m_2^{-1}$, and $I(m_1) = (M_2, \min(2M_2, 2Mm_1^{-1})]$. If $M_2 \geqslant (Nx)^{\frac{1}{2}}M^{-1}$, using first a partial summation and then the exponent pair $(\frac{1}{2},\frac{1}{2})$, we get

$$\sum_{m_2\in I(m_1)} m_2^{-\frac{1}{2}}e(f(n,m_1,m_2)) \ll M_2^{-\frac{1}{2}}\left((Nx)^{\frac{1}{4}}M^{-\frac{1}{2}} + M_2M(Nx)^{-\frac{1}{2}}\right),$$

and consequently

$$T'(M,N) \ll M^2M_2^{-1}x^{-\frac{1}{4}} + M^{\frac{3}{2}}x^{-\frac{1}{2}} \ll x^{\theta'}.$$

Assuming in the following that $M_2 < (Nx)^{\frac{1}{2}}M^{-1}$. Using Theorem 1 of [RS] to the triple sum of (9) we have $x^{-\delta}T'(M,N) \ll M^{-\frac{1}{2}}N^{-\frac{3}{4}}(\sqrt[8]{x^{-1}M^{10}N^7} + \sqrt[4]{M_2M^3N^3} + NM^{\frac{1}{2}}M_1^{\frac{1}{2}} + \sqrt[8]{xM^2M_1^2N^2})$.

$$\ll \sqrt[8]{x^{-1}M^6N} + \sqrt[4]{M_2M} + \sqrt[4]{MN} + \sqrt[8]{xM^{-1}N}, \qquad (10)$$

where we have used $M_1 \ll M^{\frac{1}{2}}$. By (6) and (10) we have

$$x^{-\delta}T'(M,N) \ll \sqrt[4]{MM_2} + \sum_{12\leqslant i\leqslant 14} R_i + x^{\theta'}, \qquad (11)$$

where

$$R_{12} = \min(\sqrt[4]{M^2N^{-1}}, \sqrt[8]{x^{-1}M^6N}) \leqslant \sqrt[12]{x^{-1}M^8} \leqslant x^{\theta'},$$
$$R_{13} = \min(\sqrt[4]{M^2N^{-1}}, \sqrt[4]{MN}) \leqslant M^{\frac{3}{8}} \leqslant x^{\theta'},$$
$$R_{14} = \min(\sqrt[4]{M^2N^{-1}}, \sqrt[8]{xM^{-1}N}) \leqslant \sqrt[12]{xM} \leqslant x^{\theta'}.$$

(by $M \leqslant x^{1-2\theta}$).

Using Theorem 3 of [L3] we get

$$\sum_{m_2\in I(m_1)} m_2^{-\frac{1}{2}}e(-f(n,m_1,m_2)) \ll \left|\sum_{u\in J} u^{-\frac{1}{2}}e(g(u,n,m_1))\right| + (M_2F^{-1})^{\frac{1}{2}} + LM_2^{-\frac{1}{2}}, \tag{12}$$

here J is the image of the interval $I(m_1)$ by the mapping $2(nx)^{\frac{1}{2}}m_1^{-1}\xi^{-2}(\xi \in I(m_1))$, and $F = (Nx)^{\frac{1}{2}}M^{-1}$. It is easy to see that $u \in J$ implies $u \simeq U, U = FM_2^{-1}$. Let $C(u) = (Uu^{-1})^{\frac{1}{2}}$. By (2.1) of [W1] with the exponent pair $(\frac{1}{2}, \frac{1}{2})$ we get

$$x^{-\delta}\sum_{m_1\sim M_1}\sum_{n\sim N}\left|\sum_{u\in J} C(u)e(g(u,n,m_1))\right| \ll \sqrt[6]{(M_1N)^5FU^4} + (M_1N)^{\frac{1}{2}}U + M_1NU^{\frac{1}{2}} + F^{-\frac{1}{2}}M_1NU.$$

Consequently, from (9) and (12) we get

$$x^{-\delta}T'(M,N) \ll \sqrt[4]{xM^{-4}M_1^2} + \sqrt[12]{x^2M^{-6}N^3M_1^6} + \sqrt[4]{MN} + x^{\theta'}. \tag{13}$$

From (11) and (13) we get(noting that $M_1 \ll M^{\frac{1}{2}}$)

$$x^{-\delta}T'(M,N) \ll \sum_{15\leqslant i\leqslant 16} R_i + \sqrt[4]{MN} + x^{\theta'}, \tag{14}$$

here

$$R_{15} = \min\left(\sqrt[4]{xM^{-4}M_1^2}, \sqrt[4]{MM_2}\right) \leqslant x^{\frac{1}{12}},$$

$$R_{16} = \min\left(\sqrt[12]{x^2M^{-6}N^3M_1^6}, \sqrt[4]{MM_2}\right) \leqslant \sqrt[36]{x^2M^6N^3}.$$

From (6) and (14) we finally achieve that(see (11) for R_{13})

$$x^{-\delta}T'(M,N) \ll R_{17} + R_{13} + x^{\theta'},$$

where $R_{13} \ll x^{\theta'}$, and

$$R_{17} = \min\left(\sqrt[36]{x^2M^6N^3}, \sqrt[4]{M^2N^{-1}}\right) \leqslant \sqrt[24]{xM^6} \ll x^{\theta'}.$$

Thus (1) always holds, and the proof of Theorem 1.1 is finished.

3. Proof of Theorem 1.2

Again let $\theta = \frac{13}{36}$. Then similarly with p. 134 of [B] it suffices to deduce that

$$S(M,N) := \sum_{m\sim M}\sum_{n\sim N} A_mB_ne\left(\frac{2n^{\frac{1}{2}}x^{\frac{1}{2}}}{m}\right) \ll x^{\theta'+\frac{\varepsilon}{2}}, \tag{15}$$

where $1 \leqslant M, N \leqslant x^{1-2\theta}$, $\theta' = \theta - \frac{1}{4}$ and $A_m = \mu(m)m^{-\frac{1}{2}}, B_n = d(n)n^{-\frac{3}{4}}(\mu(m)$ and $d(n)$ are the Mobius function and the Dirichlet divisor functions respectively). Using quite the same method we get as (6) the following($\delta = \varepsilon^2$)

$$x^{-\delta} S(M,N) \ll M^{\frac{1}{2}} N^{-\frac{1}{4}} + x^{\theta'}. \tag{16}$$

Using the decomposition of Vaughan's type for$\mu(.)$(see [J] for instance), for any function f we have

$$\sum_{m\sim M} \mu(m) f(m) = \sum_{U<r\leqslant \frac{2M}{V}} \mu(r) \sum_{rh\sim M} b_h f(hr) - \sum_{r\leqslant UV} c(r) \sum_{rh\sim M} f(rh), \tag{17}$$

where U and V are two parameters with $1 \leqslant U, V \leqslant M$, the additional coefficients are real numbers satisfying $b_s, c(s) \ll d(s)$. We use (17) with the choice $U = MN^{-\frac{1}{2}} x^{-2\theta'}$, and $V = M^2 x^{-4\theta'}$. Note that $UV \ll M/V$, thus to get (15) it suffices to show both

$$S_1 := M^{-\frac{1}{2}} N^{-\frac{3}{4}} \left(\sum_{n\sim N} \sum_{rh\sim M, r\sim R, h\sim H} a_n b_r c_h e(g(r,h,n)) \right) \ll x^{\theta' + \frac{\varepsilon}{3}}, \tag{18}$$

and

$$S_2 := M^{-\frac{1}{2}} N^{-\frac{3}{4}} \left(\sum_{n\sim N} \sum_{rh\sim M, r\sim R, h\sim H} a_n b_r e(g(r,h,n)) \right) \ll x^{\theta' + \frac{\varepsilon}{3}}, \tag{19}$$

where $|a_n|, |b_r|, |c_h| \leqslant 1$, $g(r,h,n) = 2x^{\frac{1}{2}} n^{\frac{1}{2}} (rh)^{-1}$, in (18) we have $U \ll R \ll M/V$, and in (19) we have $R \ll U$. Let $F = (xN)^{\frac{1}{2}} M^{-1}$. To deduce (18), we first use (2.1) of Lemma 2.1 of [W3], and this gives

$$\begin{aligned} x^{-\delta} S_1 &\ll \sqrt[12]{F^2 M^2 N R} + \sqrt[4]{N^{-1} H^2} + \sqrt[4]{N R^2} + \sqrt[4]{F^{-2} N M^2} \\ &\ll \sqrt[12]{x N^2 R} + \sqrt[4]{N R^2} + x^{\theta'} := \sum_{1\leqslant i\leqslant 2} P_i + x^{\theta'}. \end{aligned} \tag{20}$$

(using $R \gg U$and $M \ll x^{1-2\theta}$). By Theorem 1 of [RS] we get

$$\begin{aligned} x^{-\delta} S_1 &\ll \sqrt[4]{FH} + \sqrt[4]{RM} + \sqrt[4]{NH^2} + F^{-\frac{1}{2}} MN \\ &\ll \sqrt[8]{x N M^{-2} H^2} + \sqrt[4]{N H^2} + x^{\theta'} := \sum_{1\leqslant i\leqslant 2} G_i + x^{\theta'}. \end{aligned} \tag{21}$$

(using $R \ll M/V$ and $M \ll x^{1-2\theta}$). Let $Q = \sqrt[4]{M^2 N^{-1}}$. Note that an estimate similar with (16) holds if we write $rh = m$ in S_1. Thus by (16), (20) and (21) we have

$$x^{-\delta} S_1 \ll \sum_{1\leqslant i\leqslant 2} \sum_{1\leqslant j\leqslant 2} T_{ij} + x^{\theta'}, T_{ij} = \min(P_i, G_j, Q).$$

We can deduce by the usual method that

$$\begin{aligned}
T_{11} &\ll \min\left(\sqrt[32]{x^3N^5}, Q\right) \ll \sqrt[52]{x^3M^{10}} \ll x^{\theta'},\\
T_{12} &\ll \min\left(\sqrt[28]{x^2M^2N^5}, Q\right) \ll \sqrt[24]{xM^6} \ll x^{\theta'},\\
T_{21} &\ll \min\left(\sqrt[12]{xN^2}, Q\right) \ll \sqrt[20]{xM^4} \ll x^{\theta'},\\
T_{22} &\ll \min\left(\sqrt[4]{MN}, Q\right) \ll M^{\frac{3}{8}} \ll x^{\theta'}.
\end{aligned}$$

Thus (18) always holds. We then note that (19) is very easy to derive when we first use the B-process of van der Corput's method to the variable h(see the proof of Lemma 2.2 of [W3], for instance), and then use Theorem 1 of [RS]. We omit the routine details.

The proof of Theorem 1.2 is thus finished.

4. Proof of Theorem 1.3

From p.70 of [W4] and the treatment for proving Theorem 1.2, we easily see that Theorem 1.3 follows exactly by using the same procedures.

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Address: H.-Q. Liu: Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China.

E-mail: mnpqr@163.com

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