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STARK UNITS IN $\mathbb{Z}_p$-EXTENSIONS

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Abstract: We study the connection between the projective limit $\overline{St}_\infty$ of Stark units in $\mathbb{Z}_p$-extensions and a certain Iwasawa module already appeared in the particular cases of circular units or elliptic units. This connection is used to investigate the structure of $\overline{St}_\infty$.

Keywords: Stark units, $\mathbb{Z}_p$-extension.

1. Introduction

Let K/k be a finite abelian extension of number fields. Let p be a prime number and let K_∞ be a $\mathbb{Z}_p$-extension of K abelian over k. If n is a non-negative integer then we let K_n be the unique extension of K contained in K_∞ of degree p^n over K. Let $G_{K_\infty} = \mathrm{Gal}(K_\infty/k)$ be the Galois group of K_∞ over k and let $\Lambda_{K_\infty} = \mathbb{Z}_p[[G_{K_\infty}]]$ be the Iwasawa algebra projective limit of $\mathbb{Z}_p[\mathrm{Gal}(K_n/k)]$, $n \in \mathbb{N}$. In this paper we introduce and study the connection between two objects denoted by $\overline{St}_\infty$ and U_{K_∞} respectively. On the one hand, $\overline{St}_\infty$ is defined in §3 as the projective limit of the p-part $\mathbb{Z}_p \otimes St_n$, where St_n is a certain group defined by using Stark units in K_n. If k is an imaginary quadratic field then the Stark units are in fact elliptic units. In Remark 3.2 below, we also describe the part of $\overline{St}_\infty$ considered for instance in [10] or in [3]. If k is the field of rational numbers and $p \neq 2$ then we obtain the projective limit of cyclotomic units studied in [4], [2], [6], ...etc. If k has at least two infinite places then we obtain the projective limit of the groups defined by Rubin in [11]. On the other hand, U_{K_∞} is a Λ_{K_∞}-submodule of the $\mathbb{Q}_p$-algebra $\mathbb{Q}_p \otimes \Lambda_{K_\infty}$. We may consider it as the torsion free part of the universal distribution determined by the Stark units. We recall that the module U_{K_∞} was introduced in the first time by Sinnott in [13] in the case $k = \mathbb{Q}$. Sinnott used this module to study the index of circular units in the cyclotomic $\mathbb{Z}_p$-extension of K. The second author defined the module U_{K_∞} in [8] when k is an imaginary quadratic field, to study the index of elliptic units

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in $\mathbb{Z}_p$-extensions. In §2 we give two sufficient conditions for U_{K_∞} to be a free Λ-module, where Λ is the Iwasawa algebra $\mathbb{Z}_p[[\Gamma]]$ and $\Gamma := \mathrm{Gal}(K_\infty/K)$. The first condition is denoted by $(\mathrm{HB})_{K/k}$ and derived from the methods employed by Belliard in [1], [2] and in [6], to investigate the Galois structure of the circular units. The second condition is denoted by $(\mathrm{HD})_{K_\infty/k}$ and can be seen as the generalization of the situation considered in [9]. We recall that the aim of [9] is to compute the μ-invariant of the Λ-module $(U_{K_\infty} + \Lambda_{K_\infty})/\Lambda_{K_\infty}$ in some special cases. This invariant is closely related to the behavior of the index of elliptic units in $\mathbb{Z}_p$-extensions. In view of the exact sequence (3.11) below one may ask if the μ-invariant of $(U_{K_\infty} + \Lambda_{K_\infty})/\Lambda_{K_\infty}$ is connected with the μ-invariant of the projective limit of the p-part of the units modulo elliptic units. In §3 we give the exact connection between $\overline{St}_\infty$ and U_{K_∞}. This involves two ideals of Λ_{K_∞}, which are the augmentation ideal and the annihilator of p-power roots of unity in K_∞. Let us notice that in the semi-simple case, that is the case $p \nmid [K : k]$, we have $U_{K_\infty} = \Lambda_{K_\infty}$. If k is an imaginary quadratic field and $p \nmid [K : k]$ then the part (i) of [10, Theorem 7.7] is equivalent to the isomorphism (4.1). The result of Rubin was one of our sources of inspiration. As a final remark, we think that it would be interesting to apply our approach to Rubin-Stark units defined by Rubin in [12].

1.1. Notation

All our number fields are considered as subfields of $\mathbb{C}$, the field of complex numbers. If L is an abelian extension of k then we denote by μ_L the group of roots of unity in L. Also we denote by $\mathcal{O}_L$ the ring of integers of L and by $\mathcal{O}_L^\times$ its group of units. If $\mathfrak{v}$ is a prime ideal of $\mathcal{O}_k$ or an infinite place of k then we denote respectively by $T_\mathfrak{v}(L)$ and by $D_\mathfrak{v}(L)$ the inertia group and the decomposition group of $\mathfrak{v}$ in L/k. Let $\mathfrak{a}$ be a fractional ideal of k. If $\mathfrak{a} \subset \mathcal{O}_k$ then we let $\hat{\mathfrak{a}}$ be the product of the prime ideals of $\mathcal{O}_k$ that divide $\mathfrak{a}$. In general by a ideal of $\mathcal{O}_k$ we shall always mean a nonzero ideal. Usually we shall use G_L to denote the group $\mathrm{Gal}(L/k)$. The cardinal of a finite set X will be denoted $\#X$ or $|X|$ as well. If R is a commutative ring then we denote by $R^\times$ the group of invertible elements in R.

2. The module U_{K_∞}

Let us denote the finite part of the conductor of K_n/k by $\mathfrak{f}_n$. It has the form $\mathfrak{f}_n = \mathfrak{h}\mathfrak{g}_n$, where $\mathfrak{h}$ is prime to $\mathfrak{g}_n$ for all n and does not depend on n, moreover $\mathfrak{g}_n$ is divisible only by those prime ideals of $\mathcal{O}_k$ which ramify in K_∞/K. We may write

$$\mathfrak{h} = \mathfrak{q}_1^{e_1} \cdots \mathfrak{q}_t^{e_t},$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ are the prime divisors of $\mathfrak{h}$. To introduce U_{K_∞}, we need more notation. If D is a finite subgroup of G_{K_∞}, then we set

$$s(D) = \sum_{\sigma \in D} \sigma \in \Lambda_{K_\infty}.$$

Let $\Upsilon = \{\mathfrak{q}_1, \dots, \mathfrak{q}_t\}$ be the set of prime divisors of $\mathfrak{h}$. If $\mathfrak{q} \in \Upsilon$ then the inertia group $T_{\mathfrak{q}}(K_\infty)$ is finite. Let $n \in \mathbb{N} \cup \{\infty\}$ and let $F_{\mathfrak{q}}$ be a Frobenius automorphism at $\mathfrak{q}$ in K_n/k. Then we define

$$(\mathfrak{q}, K_n) = F_{\mathfrak{q}}^{-1} \frac{s(T_{\mathfrak{q}}(K_n))}{|T_{\mathfrak{q}}(K_n)|}.$$

For all divisor $\mathfrak{r} \neq (1)$ of $\hat{\mathfrak{h}}$ we denote by $T_{\mathfrak{r}}(K_n)$ the subgroup of G_{K_n} generated by the inertia groups $T_{\mathfrak{q}}(K_n)$ with $\mathfrak{q}|\mathfrak{r}$. If $\mathfrak{r} = (1)$ then we set $T_{(1)}(K_n) = \{1\}$. Since the fixed field of $T_{\mathfrak{r}}(K_n)$ is the maximal subextension of K_n/k unramified at the primes dividing $\mathfrak{r}$, we deduce that for all nonnegative integers $m \geqslant n$ the restriction of automorphisms gives a surjective map $T_{\mathfrak{r}}(K_m) \longrightarrow T_{\mathfrak{r}}(K_n)$. This map is an isomorphism if $\mathfrak{r} = \mathfrak{q}_i$, for some $i \in \{1, \dots t\}$, since $\mathfrak{q}_i$ does not ramify in K_∞/K. Therefore, $T_{\mathfrak{q}_i}(K_\infty) \simeq T_{\mathfrak{q}_i}(K_n)$ and, as a consequence of this, $T_{\mathfrak{r}}(K_\infty)$ is finite. This implies that the restriction of automorphisms $T_{\mathfrak{r}}(K_\infty) \longrightarrow T_{\mathfrak{r}}(K_n)$, is injective because its kernel is contained in $\mathrm{Gal}(K_\infty/K_n)$ which has no finite subgroup. Thus, we have an isomorphism

$$T_{\mathfrak{r}}(K_\infty) \simeq T_{\mathfrak{r}}(K_n).$$

Definition 2.1. *Let $\mathfrak{s}$ be a divisor of $\hat{\mathfrak{h}}$. If $\mathfrak{s} \neq (1)$, then we denote by $U_{\mathfrak{s}}$ the Λ_{K_∞}-submodule of $\mathbb{Q}_p \otimes \Lambda_{K_\infty}$ generated by all the elements*

$$\alpha(\mathfrak{r}, \mathfrak{s}) = s(T_{\mathfrak{r}}(K_\infty)) \prod_{\mathfrak{q} | \frac{\mathfrak{s}}{\mathfrak{r}}} (1 - (\mathfrak{q}, K_\infty)), \qquad \mathfrak{r}|\mathfrak{s}.$$

Moreover we set $U_{(1)} = \Lambda_{K_\infty}$ and $U_{K_\infty} = U_{\hat{\mathfrak{h}}}$.

Remark 2.1. The modules $U_{\mathfrak{s}}$ were introduced in [13] when k is equal to the field of rational numbers $\mathbb{Q}$. Sinnott used these modules to study the index of cyclotomic units in the cyclotomic $\mathbb{Z}_p$-extension of K. The second author defined the modules $U_{\mathfrak{s}}$ in [8] when k is an imaginary quadratic field, to study the index of elliptic units in $\mathbb{Z}_p$-extensions. Let $n \in \mathbb{N}$ be a nonnegative integer. Let $U_{\mathfrak{s},p}^{(n)}$ be the $\mathbb{Z}_p[G_{K_n}]$-submodule of $\mathbb{Q}_p[G_{K_n}]$ generated by the elements

$$\alpha_n(\mathfrak{r}, \mathfrak{s}) = s(T_{\mathfrak{r}}(K_n)) \prod_{\mathfrak{q} | \frac{\mathfrak{s}}{\mathfrak{r}}} (1 - (\mathfrak{q}, K_n)), \qquad \mathfrak{r}|\mathfrak{s}.$$

Then we have $U_{\mathfrak{s}} = \varprojlim U_{\mathfrak{s},p}^{(n)}$, where the projective limit is relative to the restriction map.

It is clear that $U_{\mathfrak{s}}$ is finitely generated and torsion free as a Λ_{K_∞}-module. Let us remark the double inclusion

$$|T_{\mathfrak{q}}(K_\infty)| U_{\mathfrak{s}\mathfrak{q}} \subset U_{\mathfrak{s}} \qquad \text{and} \qquad |T_{\mathfrak{q}}(K_\infty)| U_{\mathfrak{s}} \subset U_{\mathfrak{s}\mathfrak{q}},$$

for all divisors $\mathfrak{s}$ and $\mathfrak{q}$ of $\hat{\mathfrak{h}}$ such that $\mathfrak{q}$ is prime and does not divide $\mathfrak{s}$. Indeed, easy calculations show that for all divisor $\mathfrak{r}$ of $\mathfrak{s}$,

(1) $|T_{\mathfrak{q}}(K_\infty)|\ \alpha(\mathfrak{r}, \mathfrak{q}\mathfrak{s}) = |T_{\mathfrak{q}}(K_\infty)|(1-(\mathfrak{q}, K_\infty))\alpha(\mathfrak{r}, \mathfrak{s}) \in U_{\mathfrak{s}}$, and
(2) $|T_{\mathfrak{q}}(K_\infty)|\ \alpha(\mathfrak{r}\mathfrak{q}, \mathfrak{q}\mathfrak{s}) = |T_{\mathfrak{q}}(K_\infty)|(\sum_{y\in Y} y)\alpha(\mathfrak{r}, \mathfrak{s}) \in U_{\mathfrak{s}}$, where Y is a any complete system of representatives of the quotient group $T_{\mathfrak{r}\mathfrak{q}}/T_{\mathfrak{r}}$.

For the reverse inclusion, we have

$$|T_{\mathfrak{q}}(K_\infty)|\ \alpha(\mathfrak{r}, \mathfrak{s}) = |T_{\mathfrak{q}}(K_\infty)|\ \alpha(\mathfrak{r}, \mathfrak{q}\mathfrak{s}) + |T_{\mathfrak{q}}(K_\infty) \cap T_{\mathfrak{r}}(K_\infty)| F_{\mathfrak{q}}^{-1} \alpha(\mathfrak{r}\mathfrak{q}, \mathfrak{q}\mathfrak{s}).$$

As usual, let us denote by $\Gamma = \mathrm{Gal}(K_\infty/K)$ the Galois group of K_∞/K and by

$$\Lambda = \mathbb{Z}_p[[\Gamma]]$$

the Iwasawa algebra projective limit of $\mathbb{Z}_p[\mathrm{Gal}(K_n/K)]$, $n \in \mathbb{N}$. Finally, let $d_{\mathfrak{s}}$ be the product of the orders $|T_{\mathfrak{q}}(K_\infty)|$ for the prime ideals $\mathfrak{q}|\mathfrak{s}$. The above inclusions show that multiplication by $d_{\mathfrak{s}}$ gives an injective homomorphism of Λ-modules

$$U_{\mathfrak{s}} \hookrightarrow \Lambda_{K_\infty}$$

with cokernel annihilated by $d_{\mathfrak{s}}^2$. In particular, since Λ is an integral domain, we have the following

Proposition 2.1. *For all* $\mathfrak{s}|\widehat{\mathfrak{h}}$, $U_{\mathfrak{s}}$ *is a torsion free* Λ*-module with rank* $[K:k]$.

Remark 2.2. From the discussion preceding the above proposition we see that

$$p \nmid d_{\mathfrak{s}} \Longrightarrow U_{\mathfrak{s}} = \Lambda_{K_\infty}.$$

We shall prove in the sequel that U_{K_∞} is a free Λ-module when a certain condition is satisfied. This condition is essentially the hypothesis (HB) introduced by Belliard in [1, page 34]. Belliard used (HB) to compute, in the case $k = \mathbb{Q}$ and p a prime odd number, unramified in $K/\mathbb{Q}$, the Tate cohomology groups $\hat{H}^i(G_n^m, \mathcal{C}_m)$, for all nonnegative integers $m \geqslant n \geqslant 0$, where $\mathrm{G}_n^m = \mathrm{Gal}(K_m/K_n)$ and $\mathcal{C}_m$ is Sinnott's group of circular units of K_m. Later he showed in [6, appendix] that (HB) implies that the projective limit of $\mathbb{Z}_p \otimes \mathcal{C}_n$ with respect to the norm map is free over Λ. The hypothesis (HB) of Belliard may be generalized as follows. For each divisor $\mathfrak{r}$ of $\hat{\mathfrak{h}}$ we denote by $K_\infty(\mathfrak{r})$ the subfield of K_∞ fixed by $T_{\mathfrak{r}'}(K_\infty)$, where $\mathfrak{r}'$ is such that $\mathfrak{r}\mathfrak{r}' = \hat{\mathfrak{h}}$,

$$K_\infty(\mathfrak{r}) = K_\infty^{T_{\mathfrak{r}'}(K_\infty)}.$$

This is a $\mathbb{Z}_p$-extension of $K \cap K_\infty(\mathfrak{r})$. Let us set $L_\infty := K_\infty((1))$, and let $N_{\mathfrak{r}}$ be the ideal of the abelian group ring $\mathbb{Z}_p[\mathrm{Gal}(K_\infty(\mathfrak{r})/L_\infty)]$ generated by the traces $s(T_{\mathfrak{q}}(K_\infty(\mathfrak{r})))$, where $\mathfrak{q}$ runs over the prime divisors of $\mathfrak{r}$. If $\mathfrak{r} = (1)$ then we set $N_{\mathfrak{r}} = 0$. We denote by $(\mathrm{HB})_{K/k}$ the following hypothesis

$$(\mathbf{HB})_{K/k}: \quad \mathbb{Z}_p[\mathrm{Gal}(K_\infty(\mathfrak{r})/L_\infty)]/N_{\mathfrak{r}} \text{ is } \mathbb{Z}_p\text{−free for all } \mathfrak{r}|\hat{\mathfrak{h}}.$$

The extension $K_\infty(\mathfrak{r})$ is the compositum of L_∞ and $K \cap K_\infty(\mathfrak{r})$. Hence the Galois group $\mathrm{Gal}(K_\infty(\mathfrak{r})/K \cap L_\infty)$ is equal to the direct product

$$\mathrm{Gal}(K_\infty(\mathfrak{r})/K \cap L_\infty) = \mathrm{Gal}(K_\infty(\mathfrak{r})/L_\infty) \times \mathrm{Gal}(K_\infty(\mathfrak{r})/K \cap K_\infty(\mathfrak{r})).$$

Let us also observe that we have a natural isomorphism

$$\mathrm{Gal}(K_\infty/K) \simeq \mathrm{Gal}(K_\infty(\mathfrak{r})/K \cap K_\infty(\mathfrak{r})).$$

Let X be a subset of $\mathrm{Gal}(K_\infty/k)$ containing the identity map, and such that the restriction of automorphisms gives a bijection $X \longrightarrow \mathrm{Gal}(K \cap L_\infty/k)$. Further, let $\Sigma_\mathfrak{r}$ be the $\mathbb{Z}_p$-module freely generated by the symbols $\sigma[\gamma]$, with $\sigma \in \mathrm{Gal}(K_\infty(\mathfrak{r})/L_\infty)$ and $\gamma \in X$. In other words,

$$\Sigma_\mathfrak{r} = \bigoplus_{\gamma \in X} \mathbb{Z}_p[\mathrm{Gal}(K_\infty(\mathfrak{r})/L_\infty)][\gamma]$$

We make $\Sigma_\mathfrak{r}$ into a $\mathbb{Z}_p[\mathrm{Gal}(K_\infty/L_\infty)]$-module by taking for all $g \in \mathrm{Gal}(K_\infty/L_\infty)$, all $\sigma \in \mathrm{Gal}(K_\infty(\mathfrak{r})/L_\infty)$ and all $\gamma \in X$

$$g.(\sigma[\gamma]) := (\overline{g}\sigma)[\gamma],$$

where $\overline{g}$ is the restriction of g to $\mathrm{Gal}(K_\infty(\mathfrak{r})/L_\infty)$. Let us also define the following submodule of $\Sigma_\mathfrak{r}$

$$S_\mathfrak{r} := \bigoplus_{\gamma \in X} N_\mathfrak{r}[\gamma]$$

In particular, if $\mathfrak{r} = (1)$ then $S_{(1)} = 0$. Hence, we may rewrite the hypothesis $(\mathrm{HB})_{K/k}$ as follows

$$(\mathbf{HB})_{K/k}: \quad \Sigma_\mathfrak{r}/S_\mathfrak{r} \text{ is } \mathbb{Z}_p\text{-free for all } \mathfrak{r}|\hat{\mathfrak{h}}. \tag{2.1}$$

Now consider the Λ-homomorphism

$$\rho_\mathfrak{r} : \Sigma_\mathfrak{r} \otimes_{\mathbb{Z}_p} \Lambda \longrightarrow U_{K_\infty}$$

such that

$$\rho_\mathfrak{r}(\sigma[\gamma] \otimes \lambda) = \widetilde{\sigma}\gamma\lambda\alpha(\mathfrak{r}', \hat{\mathfrak{h}}),$$

where $\widetilde{\sigma}$ is any extension of σ to K_∞. By the very definition of $\alpha(\mathfrak{r}', \hat{\mathfrak{h}})$, the image $\rho_\mathfrak{r}(\sigma[\gamma] \otimes \lambda)$ does not depend on the choice of $\widetilde{\sigma}$. Indeed, let $\widetilde{\sigma}\tau$, $\tau \in T_{\mathfrak{r}'}(K_\infty)$, be any extension of σ to K_∞. Then, $\tau s(T_{\mathfrak{r}'}(K_\infty)) = s(T_{\mathfrak{r}'}(K_\infty))$ and $\widetilde{\sigma}\tau\gamma\lambda\alpha(\mathfrak{r}', \hat{\mathfrak{h}}) = \widetilde{\sigma}\gamma\lambda\tau\alpha(\mathfrak{r}', \hat{\mathfrak{h}}) = \widetilde{\sigma}\gamma\lambda\alpha(\mathfrak{r}', \hat{\mathfrak{h}})$. Let us set

$$\Sigma = \bigoplus_{\mathfrak{r}|\hat{\mathfrak{h}}} \Sigma_\mathfrak{r} \qquad \text{and} \qquad S = \bigoplus_{\mathfrak{r}|\hat{\mathfrak{h}}} S_\mathfrak{r}$$

We have the following result

Proposition 2.2. *Suppose that* $(\mathrm{HB})_{K/k}$ *is satisfied. Then, there exists a surjective* Λ*-homomorphism*

$$\Psi : (\Sigma/S) \otimes_{\mathbb{Z}_p} \Lambda \longrightarrow U_{K_\infty}.$$

Proof. We adapt the proof of [1, Proposition 3.5]. Let $\prec$ be a total order defined on the set of divisors of $\hat{\mathfrak{h}}$ such that

$$\mathfrak{r}|\mathfrak{s} \Longrightarrow \mathfrak{r} \prec \mathfrak{s}.$$

This order will allow us to construct Ψ inductively. If $\mathfrak{s}|\hat{\mathfrak{h}}$ then we denote by $V_{\mathfrak{s}}$ the Λ_{K_∞}-submodule of U_{K_∞} generated by the elements $\alpha(\mathfrak{r}', \hat{\mathfrak{h}})$, with $\mathfrak{r} \prec \mathfrak{s}$. It is obvious that $\rho_{\mathfrak{s}}(\Sigma_{\mathfrak{s}} \otimes_{\mathbb{Z}_p} \Lambda) \subset V_{\mathfrak{s}}$. Define

$$\Sigma'_{\mathfrak{s}} = \bigoplus_{\mathfrak{r} \prec \mathfrak{s}} \Sigma_{\mathfrak{r}} \qquad \text{and} \qquad S'_{\mathfrak{s}} = \bigoplus_{\mathfrak{r} \prec \mathfrak{s}} S_{\mathfrak{r}}.$$

Our aim is to define a surjective Λ-homomorphism $\Psi_{\mathfrak{s}} : \Sigma'_{\mathfrak{s}}/S'_{\mathfrak{s}} \otimes_{\mathbb{Z}_p} \Lambda \longrightarrow V_{\mathfrak{s}}$. But let us first make the following remarks. If $\mathfrak{s}_1$ is the successor of $\mathfrak{s}$, then the natural homomorphism of $\mathbb{Z}_p$-modules

$$\Sigma_{\mathfrak{s}_1} \longrightarrow \Sigma'_{\mathfrak{s}_1}/(S'_{\mathfrak{s}_1} + \Sigma'_{\mathfrak{s}}) \simeq (\Sigma'_{\mathfrak{s}_1}/S'_{\mathfrak{s}_1})/(\Sigma'_{\mathfrak{s}}/S'_{\mathfrak{s}})$$

is surjective and has kernel equal to $S_{\mathfrak{s}_1}$. Therefore, since Λ is $\mathbb{Z}_p$-flat, the Λ-modules $M_{\mathfrak{s}} := \Sigma'_{\mathfrak{s}}/S'_{\mathfrak{s}} \otimes_{\mathbb{Z}_p} \Lambda$ are such that

$$M_{\mathfrak{s}_1}/M_{\mathfrak{s}} \simeq (\Sigma_{\mathfrak{s}_1} \otimes_{\mathbb{Z}_p} \Lambda)/(S_{\mathfrak{s}_1} \otimes_{\mathbb{Z}_p} \Lambda) \simeq (\Sigma_{\mathfrak{s}_1}/S_{\mathfrak{s}_1}) \otimes_{\mathbb{Z}_p} \Lambda \tag{2.2}$$

But $\rho_{\mathfrak{s}_1}(S_{\mathfrak{s}_1} \otimes \Lambda) \subset V_{\mathfrak{s}}$. Indeed, if $\mathfrak{q}$ is a prime ideal of k dividing $\mathfrak{s}_1$ and $\mathfrak{r}$ is defined by $\mathfrak{r}\mathfrak{q} = \mathfrak{s}_1$, then $\mathfrak{r} \prec \mathfrak{s}$, $\mathfrak{r}' = \mathfrak{q}\mathfrak{s}'_1$ and, in addition, the restriction of automorphisms gives the following exact sequence

$$1 \longrightarrow T_{\frac{\mathfrak{r}'}{\mathfrak{q}}}(K_\infty) \longrightarrow T_{\mathfrak{r}'}(K_\infty) \longrightarrow T_{\mathfrak{q}}(K_\infty(\mathfrak{s}_1)) \longrightarrow 1,$$

from which we deduce

$$s(T_{\frac{\mathfrak{r}'}{\mathfrak{q}}}(K_\infty)) \sum \widetilde{\sigma} = s(T_{\mathfrak{r}'}(K_\infty)),$$

where the sum $\sum \widetilde{\sigma}$ is over all $\sigma \in T_{\mathfrak{q}}(K_\infty(\mathfrak{s}_1))$. Therefore, we obtain

$$\rho_{\mathfrak{s}_1}\big(s(T_{\mathfrak{q}}(K_\infty(\mathfrak{s}_1)))[\gamma] \otimes \lambda\big) = \gamma\lambda\alpha(\mathfrak{r}', \hat{\mathfrak{h}})(1 - F_{\mathfrak{q}}^{-1}) \in V_{\mathfrak{s}}. \tag{2.3}$$

Hence, we may use the first isomorphism of (2.2) to deduce from $\rho_{\mathfrak{s}_1}$ a homomorphism of Λ-modules

$$\widetilde{\rho}_{\mathfrak{s}_1} : M_{\mathfrak{s}_1}/M_{\mathfrak{s}} \longrightarrow V_{\mathfrak{s}_1}/V_{\mathfrak{s}}.$$

If $\mathfrak{s} = (1)$ then we let $\Psi_{\mathfrak{s}} = \rho_{\mathfrak{s}}$. Suppose we have constructed $\Psi_{\mathfrak{s}}$ for some $\mathfrak{s}|\hat{\mathfrak{h}}$ and let $\mathfrak{s}_1$ be the successor of $\mathfrak{s}$. The hypothesis $(\mathrm{HB})_{K/k}$ being satisfied we deduce first that $\Sigma_{\mathfrak{s}_1}/S_{\mathfrak{s}_1}$ is $\mathbb{Z}_p$-free, thanks to (2.1). But this implies that $M_{\mathfrak{s}_1}/M_{\mathfrak{s}}$ is Λ-free by the second isomorphism of (2.2). Consequently, $M_{\mathfrak{s}_1} \simeq M_{\mathfrak{s}} \oplus M_{\mathfrak{s}_1}/M_{\mathfrak{s}}$

as Λ-modules. This allows us to define $\Psi_{\mathfrak{s}_1}$ by using $\Psi_{\mathfrak{s}}$ and $\widetilde{\rho}_{\mathfrak{s}_1}$. In addition, we obtain the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{\mathfrak{s}} & \longrightarrow & M_{\mathfrak{s}_1} & \longrightarrow & M_{\mathfrak{s}_1}/M_{\mathfrak{s}} & \longrightarrow & 0 \\ & & \Psi_{\mathfrak{s}}\big\downarrow & & \Psi_{\mathfrak{s}_1}\big\downarrow & & \widetilde{\rho}_{\mathfrak{s}_1}\big\downarrow & & \\ 0 & \longrightarrow & V_{\mathfrak{s}} & \longrightarrow & V_{\mathfrak{s}_1} & \longrightarrow & V_{\mathfrak{s}_1}/V_{\mathfrak{s}} & \longrightarrow & 0 \end{array}$$

Since $\Psi_{\mathfrak{s}}$ is surjective by hypothesis and $\widetilde{\rho}_{\mathfrak{s}_1}$ is surjective by construction, we deduce that $\Psi_{\mathfrak{s}_1}$ is also surjective. This completes the proof of the proposition. ■

Corollary 2.1. *Suppose that (HB)$_{K/k}$ is satisfied. Then the map Ψ of Proposition 2.2 is an isomorphism. In particular U_{K_∞} is Λ-free of rank $[K:k]$.*

Proof. Suppose we know that Σ/S has rank equal to $[K:k]$ as a $\mathbb{Z}_p$-module. Then, the kernel of Ψ is a torsion Λ-module since Ψ is surjective between two Λ-modules which necessarily have the same rank. But Hypothesis (HB)$_{K/k}$ implies that $(\Sigma/S)\otimes_{\mathbb{Z}_p}\Lambda$ is Λ-free. Therefore $\mathrm{Ker}\Psi = 0$. Hence we have to check that Σ/S is a $\mathbb{Z}_p$-module of rank $[K:k]$. In other words, we have to prove that

$$Y := \bigoplus_{\mathfrak{r}|\hat{\mathfrak{h}}} \mathbb{Z}_p[\mathrm{Gal}(K_\infty(\mathfrak{r})/L_\infty)]/N_{\mathfrak{r}}$$

is a $\mathbb{Z}_p$-module of rank equal to $[K : K\cap L_\infty]$. Since Y is a $\mathbb{Z}_p[\mathrm{Gal}(K_\infty/L_\infty)]$-module we shall use character theory. Let χ be an irreducible p-adic character of $\mathrm{Gal}(K_\infty/L_\infty)$, that is a group homomorphism $\mathrm{Gal}(K_\infty/L_\infty)\longrightarrow \bar{\mathbb{Q}}_p^\times$. Let e_χ be the idempotent of $\bar{\mathbb{Q}}_p[\mathrm{Gal}(K_\infty/L_\infty)]$ associated to χ. Let $\mathfrak{r}|\hat{\mathfrak{h}}$, then the action of $\mathrm{Gal}(K_\infty/L_\infty)$ is so that

$$\chi \text{ is not trivial on } T_{\mathfrak{r}'}(K_\infty) \Longrightarrow (\bar{\mathbb{Q}}_p\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\mathrm{Gal}(K_\infty(\mathfrak{r})/L_\infty)])e_\chi = 0.$$

This is the case if $\mathfrak{r}_\chi \nmid \mathfrak{r}$, where $\mathfrak{r}_\chi$ is the product of the prime divisors $\mathfrak{q}$ of $\mathfrak{h}$ for which χ is not trivial on $T_{\mathfrak{q}}(K_\infty)$. On the other hand, if $\mathfrak{r}_\chi|\mathfrak{r}$ and $\mathfrak{r}_\chi\neq\mathfrak{r}$ then $\mathfrak{r}$ has a prime divisor $\mathfrak{q}$ such that χ is trivial on $T_{\mathfrak{q}}(K_\infty)$. Therefore,

$$(\bar{\mathbb{Q}}_p\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\mathrm{Gal}(K_\infty(\mathfrak{r})/L_\infty)])e_\chi = (\bar{\mathbb{Q}}_p\otimes_{\mathbb{Z}_p} N_{\mathfrak{r}})e_\chi.$$

Thus we have

$$\begin{aligned}(\bar{\mathbb{Q}}_p\otimes_{\mathbb{Z}_p} Y)e_\chi &= (\bar{\mathbb{Q}}_p\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\mathrm{Gal}(K_\infty(\mathfrak{r}_\chi)/L_\infty)]/N_{\mathfrak{r}_\chi})e_\chi\\ &= (\bar{\mathbb{Q}}_p\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\mathrm{Gal}(K_\infty(\mathfrak{r}_\chi)/L_\infty)])e_\chi,\end{aligned}$$

which is a $\bar{\mathbb{Q}}_p$-vector space of dimension 1. This proves that $\bar{\mathbb{Q}}_p\otimes_{\mathbb{Z}_p}Y$ is a $\bar{\mathbb{Q}}_p$-vector space of dimension $[K:K\cap L_\infty]$ and Y is a $\mathbb{Z}_p$-module of rank $[K:K\cap L_\infty]$. The proof of the corollary is now complete. ■

Remark 2.3. Exactly as in [1, section IV] one may prove that if one of the following conditions holds then the hypothesis (HB)$_{K/k}$ is satisfied.

(1) The p-Sylow subgroup of $\mathrm{Gal}(K/K \cap L_\infty)$ is cyclic.
(2) The p-Sylow of $\mathrm{Gal}(K/K \cap L_\infty)$ is equal to the direct product of the p-Sylow subgroups of the inertia groups $T_{\mathfrak{q}}(K)$, for the prime divisors $\mathfrak{q}$ of $\mathfrak{h}$.
(3) The ideal $\mathfrak{h}$ is divisible by at most two prime ideals of $\mathcal{O}_k$.

Proposition 2.3. *Let $U^0_{K_\infty}$ be the kernel in U_{K_∞} of the restriction map to k and suppose that (HB)$_{K/k}$ is satisfied. Then, $U^0_{K_\infty}$ is a free Λ-module of rank $[K:k]$.*

Proof. Let us remark that $\alpha(\mathfrak{r}, \hat{\mathfrak{h}}) \in U^0_{K_\infty}$, for all $\mathfrak{r} \neq \hat{\mathfrak{h}}$. Therefore, if we let W be the Λ_{K_∞}-submodule of U_{K_∞} generated by $\alpha(\mathfrak{r}, \hat{\mathfrak{h}})$, $\mathfrak{r} \neq \hat{\mathfrak{h}}$, then

$$U^0_{K_\infty} = s(\mathrm{Gal}(K_\infty/L_\infty))\mathcal{I}_\infty + W,$$

where $\mathcal{I}_\infty$ is the kernel in Λ_{K_∞} of the restriction map to k. This ideal of Λ_{K_∞} is called the augmentation ideal, cf. [7, Definition 5.2.2]. It is also obtained as the projective limit of the augmentation ideals of $\mathbb{Z}_p[G_L]$, where L/k runs over all the finite subextensions of K_∞/k. Thus, $\mathcal{I}_\infty$ is the ideal of Λ_{K_∞} generated by $\sigma - 1$, $\sigma \in \mathrm{Gal}(K_\infty/k)$. We leave it to the reader to prove that $\mathcal{I}_\infty$ is a finitely generated Λ-module. In an other request, the restriction map to k gives us the following short exact sequence of Λ-modules

$$0 \longrightarrow U^0_{K_\infty} \longrightarrow U_{K_\infty} \longrightarrow [K_\infty : L_\infty]\mathbb{Z}_p \longrightarrow 0.$$

Since $\mathbb{Z}_p$, equipped with the trivial action of Γ, is a finitely generated torsion Λ-module, we deduce from above and from Proposition 2.1 that $U^0_{K_\infty}$ is a finitely generated Λ-module of rank $[K:k]$. Let δ be a topological generator of $\Gamma := \mathrm{Gal}(K_\infty/K)$. Then, one may easily prove that $s(\mathrm{Gal}(K_\infty/L_\infty))\mathcal{I}_\infty$ consists of all the elements $s(\mathrm{Gal}(K_\infty/L_\infty))x$, with $x = \sum_{\gamma \in X} \lambda_\gamma \gamma$, where the coefficients $\lambda_\gamma \in \Lambda$ and satisfy $\sum_{\gamma \in X} \lambda_\gamma \in \Lambda(\delta - 1)$. In particular, we are able to modify the definition of the map Ψ of Proposition 2.2 to obtain, by induction, a surjective Λ-homomorphism

$$\Psi^0 : (\Sigma/S) \otimes_{\mathbb{Z}_p} \Lambda \longrightarrow U^0_{K_\infty}.$$

We only have to modify the definition of $\rho_{(1)}$ and $V_{(1)}$ as follows. We let $V_{(1)} := s(\mathrm{Gal}(K_\infty/L_\infty))\mathcal{I}_\infty$ and

$$\rho_{(1)}([\gamma] \otimes \lambda) = \begin{cases} \lambda(\gamma - 1)s(\mathrm{Gal}(K_\infty/L_\infty)), & \text{if } \gamma \neq 1 \\ \lambda(\delta - 1)s(\mathrm{Gal}(K_\infty/L_\infty)), & \text{if } \gamma = 1. \end{cases}$$

With these two points changed, the construction of Ψ^0 is now identical to the construction of Ψ. Further, we argue as in the proof of corollary 2.1 to conclude that $U^0_{K_\infty}$ is a free Λ-module of rank $[K:k]$. ■

The hypothesis (HB)$_{K/k}$ is not necessary for U_{K_∞} to be Λ-free. Indeed, we are able to prove that U_{K_∞} is a free Λ-module if the condition (HD)$_{K_\infty/k}$ stated below is satisfied. It is easy to see that (HB)$_{K/k}$ and (HD)$_{K_\infty/k}$ are completely independent.

(HD)$_{K_\infty/k}$: The prime divisors of $\mathfrak{h}$ split completely in L_∞/k.

It is plain that the hypothesis $(\mathrm{HD})_{K_\infty/k}$ depends on the choice of the $\mathbb{Z}_p$-extension K_∞ of K whereas $(\mathrm{HB})_{K/k}$ does not.

Remark 2.4. Let us notice that the hypothesis $(\mathrm{HD})_{K_\infty/k}$ implies in particular that the elements $\alpha(\mathfrak{r},\hat{\mathfrak{h}})$ are in $\mathbb{Q}_p[\mathrm{Gal}(K_\infty/L_\infty)]$.

Let us denote by Ξ the $\mathbb{Z}_p$-module freely generated by the symbols $\sigma[\mathfrak{r}]$, with $\sigma\in\mathrm{Gal}(K_\infty(\mathfrak{r})/L_\infty)$ and $\mathfrak{r}|\hat{\mathfrak{h}}$. In other words, the $\mathbb{Z}_p$-module Ξ is the direct sum of $\mathbb{Z}_p[\mathrm{Gal}(K_\infty(\mathfrak{r})/L_\infty)][\mathfrak{r}]$ for all $\mathfrak{r}|\hat{\mathfrak{h}}$,

$$\Xi=\bigoplus_{\mathfrak{r}|\hat{\mathfrak{h}}}\mathbb{Z}_p[\mathrm{Gal}(K_\infty(\mathfrak{r})/L_\infty)][\mathfrak{r}].$$

We have a natural homomorphism of $\Lambda[\mathrm{Gal}(K_\infty/L_\infty)]$-modules

$$\Phi:\Xi\otimes\Lambda\longrightarrow U_{K_\infty}$$

such that $\Phi(\sigma[\mathfrak{r}]\otimes\lambda)=\widetilde{\sigma}\lambda\alpha(\mathfrak{r}',\hat{\mathfrak{h}})$, for all $\mathfrak{r}|\hat{\mathfrak{h}}$, all $\sigma\in\mathrm{Gal}(K_\infty(\mathfrak{r})/L_\infty)$ and all $\lambda\in\Lambda$, where $\widetilde{\sigma}$ is any extension of σ to K_∞. Let us assume, until the next proposition, that the hypothesis $(\mathrm{HD})_{K_\infty/k}$ is satisfied. A first consequence is the following decomposition of U_{K_∞},

$$U_{K_\infty}=\bigoplus_{\gamma\in X}\gamma\mathrm{Im}\Phi.$$

In particular, $\mathrm{rank}_\Lambda(\mathrm{Im}\Phi)=[K_\infty:L_\infty]$. Moreover, we may define the submodule $\mathcal{R}$ of Ξ generated by all the sums

$$rel(\mathfrak{q},\mathfrak{r})=s(T_{\mathfrak{q}}(K_\infty(\mathfrak{r}\mathfrak{q})))[\mathfrak{r}\mathfrak{q}]-\big(1_\infty(\mathfrak{r})-(\mathfrak{q},K_\infty(\mathfrak{r})/k)^{-1}\big)[\mathfrak{r}],$$

where $\mathfrak{q}\nmid\mathfrak{r}$, $1_\infty(\mathfrak{r})$ is the identity map of $K_\infty(\mathfrak{r})$ and $(\mathfrak{q},K_\infty(\mathfrak{r})/k)$ is the Frobenius automorphism of $K_\infty(\mathfrak{r})/k$ at $\mathfrak{q}$. As for (2.3) we have

$$\begin{aligned}\Phi\big(s(T_{\mathfrak{q}}(K_\infty(\mathfrak{r}\mathfrak{q})))[\mathfrak{r}\mathfrak{q}]\otimes 1\big)&=(1-(\mathfrak{q},K_\infty))\alpha(\mathfrak{r}',\hat{\mathfrak{h}})\\&=(1-F_{\mathfrak{q}}^{-1})\alpha(\mathfrak{r}',\hat{\mathfrak{h}})\\&=\Phi\big((1_\infty(\mathfrak{r})-(\mathfrak{q},K_\infty(\mathfrak{r})/k)^{-1})[\mathfrak{r}]\otimes 1\big).\end{aligned}$$

This shows that the sums $rel(\mathfrak{q},\mathfrak{r})\otimes 1$ are in $\mathrm{Ker}\Phi$. Let $A:=\Xi/\mathcal{R}$, then, here also one may use character theory to check that $\mathrm{rank}_{\mathbb{Z}_p}(A)\leqslant[K_\infty:L_\infty]$. Indeed, let χ be a p-adic character of $\mathrm{Gal}(K_\infty/L_\infty)$ and let notation be as in the proof of Corollary 2.1. If $\mathfrak{r}|\hat{\mathfrak{h}}$ is such that $\mathfrak{r}_\chi|\mathfrak{r}$ and $\mathfrak{r}_\chi\neq\mathfrak{r}$ then $\mathfrak{r}$ has a prime divisor $\mathfrak{q}$ such that χ is trivial on $T_{\mathfrak{q}}(K_n)$. Let us put $\mathfrak{r}=\mathfrak{s}\mathfrak{q}$. Since $\mathcal{R}$ contains the sum $rel(\mathfrak{q},\mathfrak{s})$ we see that

$$\begin{aligned}&\big(\bar{\mathbb{Q}}_p\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\mathrm{Gal}(K_\infty(\mathfrak{r})/L_\infty)][\mathfrak{r}]\big)e_\chi\\&\qquad\subset\big(\bar{\mathbb{Q}}_p\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\mathrm{Gal}(K_\infty(\mathfrak{s})/L_\infty)][\mathfrak{s}]\big)e_\chi+\big(\bar{\mathbb{Q}}_p\otimes_{\mathbb{Z}_p}\mathcal{R}\big)e_\chi.\end{aligned}$$

Hence, by applying the same reasoning to $\mathfrak{s}$ if necessary and so on, we arrive at the inclusion

$$\big(\bar{\mathbb{Q}}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mathrm{Gal}(K_\infty(\mathfrak{r})/L_\infty)][\mathfrak{r}]\big)e_\chi \subset \big(\bar{\mathbb{Q}}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mathrm{Gal}(K_\infty(\mathfrak{r}_\chi)/L_\infty)][\mathfrak{r}_\chi]\big)e_\chi + \big(\bar{\mathbb{Q}}_p \otimes_{\mathbb{Z}_p} \mathcal{R}\big)e_\chi.$$

This implies that

$$\dim[\big(\bar{\mathbb{Q}}_p \otimes_{\mathbb{Z}_p} (\Xi/\mathcal{R})\big)e_\chi] \leqslant 1,$$

and that $\mathrm{rank}_{\mathbb{Z}_p}(A) \leqslant [K_\infty : L_\infty]$. Let $T(A)$ be the $\mathbb{Z}_p$-torsion submodule of A and let $A' := A/T(A)$. Since $\mathrm{Im}\Phi$ is a torsion free $\mathbb{Z}_p$-module and has Λ-rank equal to $[K_\infty : L_\infty]$, the condition $(\mathrm{HD})_{K_\infty/k}$, when satisfied, allows us to derive from Φ a $\Lambda[\mathrm{Gal}(K_\infty/L_\infty)]$-isomorphism

$$\bar{\Phi} : A' \otimes \Lambda \longrightarrow \mathrm{Im}\Phi.$$

In an other request, the inclusion $\mathrm{Im}\Phi \subset U_{K_\infty}$ gives us the decomposition

$$U^0_{K_\infty} = \Big[\bigoplus_{\gamma \in X, \gamma \neq 1} (\gamma - 1)\mathrm{Im}\Phi\Big] \bigoplus \mathrm{Im}\Phi \cap U^0_{K_\infty}.$$

Moreover, $\mathrm{Im}\Phi \cap U^0_{K_\infty}$ is the $\Lambda[\mathrm{Gal}(K_\infty/L_\infty)]$ generated by the elements $\alpha(\mathfrak{r}, \hat{\mathfrak{h}})$, $\mathfrak{r} \neq \hat{\mathfrak{h}}$, and by $(\delta - 1)s(\mathrm{Gal}(K_\infty/L_\infty))$, where δ is any fixed topological generator of $\Gamma := \mathrm{Gal}(K_\infty/K)$. Let us define the $\Lambda[\mathrm{Gal}(K_\infty/L_\infty)]$-homomorphism $\Phi^0 : \Xi \otimes \Lambda \longrightarrow U_{K_\infty}$, such that

$$\Phi^0(\sigma[\mathfrak{r}] \otimes \lambda) = \begin{cases} \widetilde{\sigma}\lambda\alpha(\mathfrak{r}', \hat{\mathfrak{h}}) & \text{if } \mathfrak{r} \neq (1) \\ (\delta - 1)\widetilde{\sigma}\lambda s(\mathrm{Gal}(K_\infty/L_\infty)) & \text{if } \mathfrak{r} = (1). \end{cases}$$

But since $\mathrm{Im}\Phi^0 = \mathrm{Im}\Phi \cap U^0_{K_\infty}$ and $\mathcal{R} \subset \mathrm{Ker}\Phi^0$ we obtain, exactly as above, an isomorphism of $\Lambda[\mathrm{Gal}(K_\infty/L_\infty)]$-modules $\bar{\Phi}^0 : A' \otimes \Lambda \longrightarrow \mathrm{Im}\Phi \cap U^0_{K_\infty}$. Thus we have proved

Proposition 2.4. *If the prime divisors of $\mathfrak{h}$ split completely in L_∞/k then U_{K_∞} and $U^0_{K_\infty}$ are isomorphic as a $\Lambda[\mathrm{Gal}(K_\infty/L_\infty)]$-modules to the direct product*

$$(A' \otimes \Lambda)^{[K \cap L_\infty : k]}.$$

In particular U_{K_∞} and $U^0_{K_\infty}$ are free Λ-modules.

Remark 2.5. When $k = \mathbb{Q}$, K_∞ is the cyclotomic $\mathbb{Z}_p$-extension of K. In this case, every prime of K is finitely decomposed in K_∞/K. In particular, the hypothesis $(\mathrm{HD})_{K_\infty/k}$ is not satisfied if $k = \mathbb{Q}$. Now, suppose that $p \neq 2$ and that k is an imaginary quadratic field. Let k_∞ be the anticyclotomic $\mathbb{Z}_p$-extension of k. If the prime divisors of $\mathfrak{h}$ don't split in $k/\mathbb{Q}$, then they split completely in k_∞ thanks to [5, Theorem 5.2 Chap. 13 page 311]. Moreover, if we assume that these primes split completely in $K^{T_{\hat{\mathfrak{h}}}(K)} = K \cap L_\infty$ then the condition $(\mathrm{HD})_{K_\infty/k}$ holds in $K_\infty := Kk_\infty$. It is interesting to remark that this is exactly the situation considered in [9].

3. Stark units

Let P be the set of infinite places of k. For each divisor $\mathfrak{r}$ of $\hat{\mathfrak{h}}$ and each nonnegative integer n we denote by $P_n(\mathfrak{r})$ the set of prime ideals that divide $\mathfrak{r}\mathfrak{g}_n$. Also we set

$$K_n(\mathfrak{r}) = K_n^{T_{\mathfrak{r}'}(K_n)}, \qquad \mathfrak{r}\mathfrak{r}' = \hat{\mathfrak{h}}.$$

For this section we suppose the following

(1) There exists in P a place v which splits completely in K/k. Denote by w a place of K_∞ above v.
(2) Stark's conjecture $St(K_n(\mathfrak{r})/k, P_n(\mathfrak{r}) \cup P)$, holds for all $\mathfrak{r}$ and all $n \in \mathbb{N}$ such that $P_n(\mathfrak{r}) \neq \emptyset$.

Recall that $St(K_n(\mathfrak{r})/k, P_n(\mathfrak{r}) \cup P)$ is stated in [14, page 89, Conjecture 2.2]. This conjecture claims the existence of an element $\varepsilon = \varepsilon_n(\mathfrak{r}) \in K_n(\mathfrak{r})$ unique up to a root of unity such that

(i) If we denote by $e = e_n(\mathfrak{r})$ the number of roots of unity in $K_n(\mathfrak{r})$ then the extension $K_n(\mathfrak{r})(\varepsilon^{1/e})/k$ is abelian.
(ii) If $\#(P_n(\mathfrak{r}) \cup P) \geqslant 3$ then $|\varepsilon|_{w'} = 1$ for all $w' \nmid v$. If $P_n(\mathfrak{r}) \cup P = \{v, v'\}$ then $|\varepsilon|_{\sigma w'} = |\varepsilon|_{w'}$ for all $\sigma \in \mathrm{Gal}(K_n(\mathfrak{r})/k)$ and $w'|v'$, moreover, $|\varepsilon|_{w'} = 1$ for any place w' such that $w' \nmid v$ and $w' \nmid v'$.
(iii) We have

$$L'_{\mathfrak{r},n}(0, \chi) = -\frac{1}{e_n(\mathfrak{r})} \sum_{\sigma \in \mathrm{Gal}(K_n(\mathfrak{r})/k)} \chi(\sigma) \log(|\varepsilon^\sigma|_w), \tag{3.1}$$

for all complex characters χ of $\mathrm{Gal}(K_n(\mathfrak{r})/k)$.

Here $s \longmapsto L_{\mathfrak{r},n}(s, \chi)$ is the L-function associated to χ, defined for the complex numbers s such that $\Re(s) > 1$, by the Euler product

$$L_{\mathfrak{r},n}(s, \chi) = \prod_{\mathfrak{l} \nmid \mathfrak{r}\mathfrak{g}_n} \left(1 - \chi(\sigma_\mathfrak{l}) N(\mathfrak{l})^{-s}\right)^{-1},$$

where $\mathfrak{l}$ runs through all the nonzero prime ideals of $\mathcal{O}_k$ not dividing $\mathfrak{r}\mathfrak{g}_n$, and $\sigma_\mathfrak{l}$ is the Frobenius automorphism of $K_n(\mathfrak{r})/k$ at $\mathfrak{l}$.

Definition 3.1. *In the sequel we shall fix a nonnegative integer n_0 such that all the prime ideals of $\mathcal{O}_k$ which are ramified in K_∞/K divide $\mathfrak{g}_{n_0}$.*

We have the following Euler system property:

Lemma 3.1. *Let $\mathfrak{q}$ be a prime ideal dividing $\mathfrak{h}$ and let $\mathfrak{r}$ be a divisor of $\hat{\mathfrak{h}}$ such that $\mathfrak{q} \nmid \mathfrak{r}$. Then for all $n \geqslant n_0$ we have*

$$N_{K_n(\mathfrak{r}\mathfrak{q})/K_n(\mathfrak{r})}(\varepsilon_n(\mathfrak{r}\mathfrak{q}))^{e_n(\mathfrak{r})} = \varepsilon_n(\mathfrak{r})^{(1-(\mathfrak{q}, K_n(\mathfrak{r})/k)^{-1})e_n(\mathfrak{r}\mathfrak{q})}, \tag{3.2}$$

where $(\mathfrak{q}, K_n(\mathfrak{r})/k)$ is the Frobenius automorphism of $K_n(\mathfrak{r})/k$ at $\mathfrak{q}$.

Proof. The formula (3.2) may be derived from the proofs of [14, page 92, Propositions 3.4 and 3.5]. A direct proof may also be found in [11, Theorem 4.1]. Let us indicate how to check it. Since both sides of the equality are $e_n(\mathfrak{r})$-th powers, it suffices to prove that they both have the same absolute value at all places of $K_n(\mathfrak{r})$. But, notice that, since $\#(P_n(\mathfrak{r}) \cup P) \geqslant 3$, these elements of $K_n(\mathfrak{r})$ have absolute value equal to 1 at the places w' such that $w' \nmid v$. Let us denote by $E_n(\mathfrak{r})$ the following subgroup of $K_n(\mathfrak{r})^\times$:

$$E_n(\mathfrak{r}) := \{\varepsilon \in K_n(\mathfrak{r}) / \, |\varepsilon|_{w'} = 1 \qquad \text{for all} \ \ w' \nmid v\}. \tag{3.3}$$

Let $\mathbb{R}[G_{K_n(\mathfrak{r})}]$ be the group ring of $G_{K_n(\mathfrak{r})}$ over the field of real numbers and $l_{\mathfrak{r},n} : K_n(\mathfrak{r})^\times \longrightarrow \mathbb{R}[G_{K_n(\mathfrak{r})}]$ be the logarithm map, defined for $x \in K_n(\mathfrak{r})^\times$ by

$$l_{\mathfrak{r},n}(x) = \sum_{\sigma \in G_{K_n(\mathfrak{r})}} \log(|x^\sigma|_w)\sigma. \tag{3.4}$$

The map $l_{\mathfrak{r},n}$ is a $\mathbb{Z}[G_{K_n(\mathfrak{r})}]$-homomorphism satisfying $E_n(\mathfrak{r}) \cap \mathrm{Ker} l_{\mathfrak{r},n} = \mu_{K_n(\mathfrak{r})}$. Let A (resp. B) be the member to the left (resp. right) of (3.2). We have to check the equality $l_{\mathfrak{r},n}(A) = l_{\mathfrak{r},n}(B)$. Let χ be a complex character of $G_{K_n(\mathfrak{r})}$ and let e_χ be the idempotent of $\mathbb{C}[G_{K_n(\mathfrak{r})}]$ associated to χ. By (3.1) we have

$$l_{\mathfrak{r},n}(B)e_\chi = -e_n(\mathfrak{r})e_n(\mathfrak{r}\mathfrak{q})L'_{\mathfrak{r},n}(0,\chi)(1-\chi(\mathfrak{q}))e_\chi.$$

On the other hand, if $\widetilde{\chi}$ is the character of $G_{K_n(\mathfrak{r}\mathfrak{q})}$ inflated from χ then we deduce, again from (3.1),

$$l_{\mathfrak{r},n}(A)e_\chi = -e_n(\mathfrak{r})e_n(\mathfrak{r}\mathfrak{q})L'_{\mathfrak{r}\mathfrak{q},n}(0,\widetilde{\chi})e_\chi,$$

Since we have the relation $L'_{\mathfrak{r}\mathfrak{q},n}(0,\widetilde{\chi}) = L'_{\mathfrak{r},n}(0,\chi)(1-\chi(\mathfrak{q}))$ we get the equality

$$l_{\mathfrak{r},n}(A)e_\chi = l_{\mathfrak{r},n}(B)e_\chi$$

for all χ and hence $l_{\mathfrak{r},n}(A) = l_{\mathfrak{r},n}(B)$. ■

Lemma 3.2. *Let $\mathfrak{r}$ be a divisor of $\hat{\mathfrak{h}}$. Then for all $m \geqslant n \geqslant n_0$ we have*

$$N_{K_m(\mathfrak{r})/K_n(\mathfrak{r})}(\varepsilon_m(\mathfrak{r}))^{e_n(\mathfrak{r})} = \varepsilon_n(\mathfrak{r})^{e_m(\mathfrak{r})}. \tag{3.5}$$

Proof. This is exactly the situation described in [14, proof of Proposition 3.5, page 92]. On the other hand we may argue as in the proof of the above lemma 3.1. The details are left to the reader. ■

Lemma 3.3. *Let Ω be the set of places of k which are in $P - \{v\}$ or associated to a prime ideal of $\mathcal{O}_k$ ramified in K_∞/K and let $\theta \in \Omega$. Let $\mathfrak{r}|\hat{\mathfrak{h}}$. If $L \subset K_n(\mathfrak{r})$ is the fixed field of $D_\theta(K_n(\mathfrak{r}))$. Then, for all $n \geqslant n_0$ such that $\#(P_n(\mathfrak{r}) \cup P) \geqslant 3$ we have*

$$N_{K_n(\mathfrak{r})/L}(\varepsilon_n(\mathfrak{r}))^{e_n(\mathfrak{r})} = 1. \tag{3.6}$$

Proof. Remark that $C := N_{K_n(\mathfrak{r})/L}(\varepsilon_n(\mathfrak{r})) \in E_n(\mathfrak{r})$. Moreover, if χ is a complex character of $G_{K_n(\mathfrak{r})}$ not trivial on $D_\theta(K_n(\mathfrak{r}))$ then it is clear that $l_{\mathfrak{r},n}(C)e_\chi = 0$. If χ is trivial on $D_\theta(K_n(\mathfrak{r}))$ then we have

$$l_{\mathfrak{r},n}(C)e_\chi = -e_n(\mathfrak{r})[K_n(\mathfrak{r}) : L]L'_{\mathfrak{r},n}(0, \chi)e_\chi.$$

But since $\#(P_n(\mathfrak{r})\cup P) \geqslant 3$ we have $L'_{\mathfrak{r},n}(0, \chi) = 0$, even if χ is the trivial character, by [14, Proposition 3.4, page 24]. Therefore C is a root of unity in $K_n(\mathfrak{r})$. ■

Remark 3.1. At this stage, we notice that if k is neither $\mathbb{Q}$ nor imaginary quadratic then the condition $\#(P_n(\mathfrak{r}) \cup P) \geqslant 3$ will be fulfilled for all $n \geqslant n_0$. As a matter of fact, it follows from the above lemma that if some $v' \in P - \{v\}$ splits in K (and hence in K_∞) then $\varepsilon_n(\mathfrak{r})$ will be a root of unity for all n and all $\mathfrak{r}$. In particular, to get Stark units $\varepsilon_n(\mathfrak{r})$ which are not simply roots of unity we have to suppose that every $v' \in P - \{v\}$ is real and becomes complex in K. Furthermore, if we assume Leopoldt's conjecture then this latter condition limits the possible $\mathbb{Z}_p$-extensions K_∞ of K that are abelian over k: if v is real then K_∞ must be the cyclotomic $\mathbb{Z}_p$-extension, and if v is complex then the compositum of all such extensions is a $\mathbb{Z}_p^2$-extension.

Let $\mathcal{F}_n(\mathfrak{r})$ be the annihilator in $\mathbb{Z}[\mathrm{Gal}(K_n(\mathfrak{r})/k)]$ of $\mu_{K_n(\mathfrak{r})}$ and let $\eta \in \mathcal{F}_n(\mathfrak{r})$. By [14, chap.IV, Lemma 1.1 and Proposition 1.2] there exists in $K_n(\mathfrak{r})$ an element $\varepsilon_n(\mathfrak{r}, \eta)$ satisfying

$$\varepsilon_n(\mathfrak{r}, \eta)^{e_n(\mathfrak{r})} = \varepsilon_n(\mathfrak{r})^\eta.$$

If $\eta' \in \mathbb{Z}[\mathrm{Gal}(K_n/k)]$ annihilates μ_{K_n} and has image equal to η by the restriction map to $\mathbb{Z}[\mathrm{Gal}(K_n(\mathfrak{r})/k)]$ then we set $\varepsilon_n(\mathfrak{r}, \eta') := \varepsilon_n(\mathfrak{r}, \eta)$.

Definition 3.2. *Let n be a nonnegative integer. Then we denote by D_n the subgroup of $K_n^\times$ generated by μ_{K_n} and by $\varepsilon_n(\mathfrak{r}, \eta)$ for all $\mathfrak{r}$ and all η. We denote the intersection $D_n \cap \mathcal{O}_{K_n}^\times$ by St_n and the tensor product $\mathbb{Z}_p \otimes St_n$ by $\overline{St}_n$.*

Let us observe that D_n is naturally a $\mathbb{Z}[\mathrm{Gal}(K_n/k)]$-module and contains the Stark units $\varepsilon_n(\mathfrak{r})$. We are interested in studying how the Λ_{K_∞}-module U_{K_∞} is related to the projective limit

$$\overline{St}_\infty := \varprojlim \overline{St}_n,$$

relative to the norm map.

Remark 3.2.

(1) If $k = \mathbb{Q}$ then K is a totally real abelian number field and K_∞ is the cyclotomic $\mathbb{Z}_p$-extension of K. In this case $e_n(\mathfrak{r}) = 2$ and the Stark units are just cyclotomic units. We refer the reader to [14, pages 79 and 80] for the details. The relations (3.2) and (3.5) are satisfied even without the corresponding number of roots of unity on each side. If p is odd then $\overline{St}_\infty$ is equal to the inverse limit of $\mathbb{Z}_p \otimes C_n$, where C_n is Sinnott's group of circular units of K_n.

(2) Suppose that k is an imaginary quadratic field and let n be a nonnegative integer. Then define St_n^0 to be the subgroup of St_n generated by μ_{K_n} and by $\varepsilon_n(\mathfrak{r},\eta(\sigma-1))$ for all $\mathfrak{r}|\hat{\mathfrak{h}}$, $\eta \in \mathcal{F}_n(\mathfrak{r})$ and all $\sigma \in G_{K_n(\mathfrak{r})}$. Let $\overline{St}_n^0$ be the tensor product $\mathbb{Z}_p \otimes St_n^0$. Then the projective limit of the group of elliptic units considered by Rubin, in [10] for instance, is equal to

$$\overline{St}_\infty^0 := \varprojlim \overline{St}_n^0,$$

(3) If k has at least two infinite places then St_n is a subgroup of the group $C_{\mathrm{Stark},n} = C_{\mathrm{Stark}}$ of K_n defined by Rubin in [11, §4], but the projective limit of $\mathbb{Z}_p \otimes C_{\mathrm{Stark},n}$ is also equal to $\overline{St}_\infty$.

Definition 3.3. *We denote by* ρ_n *the following idempotent of* $\mathbb{Q}[G_{K_n}]$

$$\rho_n = \prod_{\mathfrak{u}\in\Omega}(1 - \frac{s(D_{\mathfrak{u}}(K_n))}{|D_{\mathfrak{u}}(K_n)|}),$$

where Ω *is defined in Lemma 3.3.*

Theorem 3.1. *Let* $R_n = \mathbb{Z}[G_{K_n}]$ *and let* U_{K_n} *be the* R_n*-submodule of* $\mathbb{Q}[G_{K_n}]$ *generated by the elements* $\alpha_n(\mathfrak{r},\hat{\mathfrak{h}})$ *defined in Remark 2.1. Let* $\mathcal{F}_n \subset R_n$ *be the annihilator of* μ_{K_n}*. Let* $\mathcal{F}_nU_{K_n}$ *be the* R_n*-submodule of* U_{K_n} *generated by the elements* $\eta\alpha_n(\mathfrak{r},\hat{\mathfrak{h}})$, $\eta \in \mathcal{F}_n$ *and* $\mathfrak{r}|\hat{\mathfrak{h}}$. *If* $n \geqslant n_0$, *then there exists a well-defined surjective* R_n*-homomorphism* $\mathcal{Z}_n : D_n \longrightarrow \rho_n\mathcal{F}_nU_{K_n}$, *such that*

$$\mathcal{Z}_n(\varepsilon_n(\mathfrak{r},\eta)) = \rho_n\eta\alpha_n(\mathfrak{r}',\hat{\mathfrak{h}}).$$

Moreover, $\mathrm{Ker}\mathcal{Z}_n = \mu_{K_n}$. *If* $\#\Omega = 1$ *then the image of* St_n *by* $\mathcal{Z}_n$ *is* $\mathcal{Z}_n(St_n) = \rho_n(\mathcal{F}_nU_{K_n})^0$, *where* $(\mathcal{F}_nU_{K_n})^0$ *is the kernel in* $\mathcal{F}_nU_{K_n}$ *of the restriction map to* k.

Proof. Let us first describe both $\rho_n\mathcal{F}_nU_{K_n}$ and D_n/μ_{K_n} by generators and relations. The relations are the distribution relations deduced from the two lemmas 3.1 and 3.3. The generators are defined as follows. For every divisor $\mathfrak{r}$ of $\hat{\mathfrak{h}}$ we denote by $\mathcal{F}_n(\mathfrak{r}) \subset \mathbb{Z}[\mathrm{Gal}(K_n(\mathfrak{r})/k)]$ the annihilator of $\mu_{K_n(\mathfrak{r})}$. Then, we consider the $\mathbb{Z}$-module Σ_n, freely generated by the symbols $\eta[\mathfrak{r}]$, with $\eta \in \mathcal{F}_n(\mathfrak{r})$ and $\mathfrak{r}|\hat{\mathfrak{h}}$. In other words,

$$\Sigma_n = \bigoplus_{\mathfrak{r}|\hat{\mathfrak{h}}} \mathcal{F}_n(\mathfrak{r})[\mathfrak{r}],$$

Let us remark that Σ_n is naturally an R_n-module. Let S_n be the R_n-submodule of Σ_n generated by the following sums

(1) $rel_n(\mathfrak{q},\mathfrak{r},\eta) = s(T_{\mathfrak{q}}(K_n(\mathfrak{r}\mathfrak{q})))\eta[\mathfrak{r}\mathfrak{q}] - \big(1_n(\mathfrak{r}) - (\mathfrak{q}, K_n(\mathfrak{r})/k)^{-1}\big)\mathrm{res}(\eta)[\mathfrak{r}]$, where $\mathfrak{q} \nmid \mathfrak{r}$, $1_n(\mathfrak{r})$ denotes the identity map of $K_n(\mathfrak{r})$, $\eta \in \mathcal{F}_n[\mathfrak{r}\mathfrak{q}]$ and $\mathrm{res}(\eta)$ is the image of η by the restriction map to $\mathbb{Z}[\mathrm{Gal}(K_n(\mathfrak{r})/k)]$.

(2) $s(D_{\mathfrak{v}}(K_n(\mathfrak{r})))\eta[\mathfrak{r}]$, $\mathfrak{v} \in \Omega$, $\eta \in \mathcal{F}_n(\mathfrak{r})$ and $\mathfrak{r}|\hat{\mathfrak{h}}$.

We define two R_n-homomorphisms

$$\Theta_n : \Sigma_n \longrightarrow D_n/\mu_{K_n} \qquad \text{and} \qquad \Theta'_n : \Sigma_n \longrightarrow \rho_n \mathcal{F}_n U_{K_n}$$

such that for given $\mathfrak{r}|\hat{\mathfrak{h}}$ and $\eta \in \mathcal{F}_n(\mathfrak{r})$ we have

$$\Theta_n(\eta[\mathfrak{r}]) = \varepsilon_n(\mathfrak{r}, \eta) \text{ modulo } \mu_{K_n} \qquad \text{and} \qquad \Theta'_n(\eta[\mathfrak{r}]) = \rho_n \eta' \alpha_n(\mathfrak{r}', \hat{\mathfrak{h}}).$$

where η' is any element of $\mathcal{F}_n$ whose image by the restriction map $\mathcal{F}_n \longrightarrow \mathcal{F}_n(\mathfrak{r})$ is equal to η. This map is onto as follows from the description of the annihilator of roots of unity given in [14, Lemme 1.1 page 82]. Recall that $\mathfrak{r}'|\hat{\mathfrak{h}}$ is such that $\mathfrak{r}\mathfrak{r}' = \hat{\mathfrak{h}}$. The maps Θ_n and Θ'_n are obviously surjective. Furthermore, we deduce from the lemmas 3.1 and 3.3, the inclusion $S_n \subset \mathrm{Ker}(\Theta_n)$. On the other hand a simple computation shows that $S_n \subset \mathrm{Ker}(\Theta'_n)$. The last step now is to prove that Σ_n/S_n, D_n/μ_{K_n} and $\rho_n \mathcal{F}_n U_{K_n}$ have the same rank as $\mathbb{Z}$-modules. Since Θ_n and Θ'_n are onto we have

$$\mathrm{rank}_{\mathbb{Z}}(\Sigma_n/S_n) \geqslant \mathrm{rank}_{\mathbb{Z}}(\rho_n \mathcal{F}_n U_{K_n}) \qquad \text{and} \qquad \mathrm{rank}_{\mathbb{Z}}(\Sigma_n/S_n) \geqslant \mathrm{rank}_{\mathbb{Z}}(D_n).$$

Let us show the other inequalities. We use character theory. Let χ be an irreducible complex character of $G_{K_n} = \mathrm{Gal}(K_n/k)$. Let e_χ be the idempotent of $\mathbb{C}[G_{K_n}]$ associated to χ. We observe first that for any ideal $\mathfrak{r}|\hat{\mathfrak{h}}$ we have

$$\chi \text{ is not trivial on } T_{\mathfrak{r}'}(K_n) = \mathrm{Gal}(K_n/K_n(\mathfrak{r})) \implies \big(\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{F}_n(\mathfrak{r})[\mathfrak{r}]\big)e_\chi = 0. \quad (3.7)$$

Let $\mathfrak{r}_\chi$ be the product of the prime divisors $\mathfrak{q}$ of $\mathfrak{h}$ for which χ is not trivial on $T_{\mathfrak{q}}(K_n)$ ($\mathfrak{r}_\chi = 1$ if $\chi = 1$). If $\mathfrak{r}_\chi \nmid \mathfrak{r}$ then χ is not trivial on $\mathrm{Gal}(K_n/K_n(\mathfrak{r}))$ and then $\big(\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{F}_n(\mathfrak{r})[\mathfrak{r}]\big)e_\chi = 0$, thanks to (3.7). Moreover, if $\mathfrak{r}_\chi|\mathfrak{r}$ and $\mathfrak{r}_\chi \neq \mathfrak{r}$ then $\mathfrak{r}$ has a prime divisor $\mathfrak{q}$ such that χ is trivial on $T_{\mathfrak{q}}(K_n)$. Let us put $\mathfrak{r} = \mathfrak{s}\mathfrak{q}$ and let $\eta \in \mathcal{F}_n(\mathfrak{r})$. Since S_n contains the sum $rel_n(\mathfrak{q}, \mathfrak{s}, \eta)$ we see that

$$\big(\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{F}_n(\mathfrak{r})[\mathfrak{r}]\big)e_\chi \subset \big(\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{F}_n(\mathfrak{s})[\mathfrak{s}]\big)e_\chi + \big(\mathbb{C} \otimes_{\mathbb{Z}} S_n\big)e_\chi.$$

Hence, by applying the same argument to $\mathfrak{s}$ if necessary and so on, we obtain the inclusion

$$\big(\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{F}_n(\mathfrak{r})[\mathfrak{r}]\big)e_\chi \subset \big(\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{F}_n(\mathfrak{r}_\chi)[\mathfrak{r}_\chi]\big)e_\chi + \big(\mathbb{C} \otimes_{\mathbb{Z}} S_n\big)e_\chi.$$

This implies that

$$\dim[\big(\mathbb{C} \otimes_{\mathbb{Z}} (\Sigma_n/S_n)\big)e_\chi] \leqslant 1. \quad (3.8)$$

Furthermore, the equality

$$e_n \alpha(\mathfrak{r}'_\chi, \hat{\mathfrak{h}})e_\chi = e_n |T_{\mathfrak{r}'_\chi}(K_n)| e_\chi,$$

where $e_n = e_n(\hat{\mathfrak{h}})$ is the number of roots of unity in K_n shows that $\mathcal{F}_n U_{K_n} e_\chi \neq 0$. Thus, if $\rho_n \mathcal{F}_n U_{K_n} e_\chi = 0$ then there exists $\mathfrak{u} \in \Omega$ such that χ is trivial on $D_{\mathfrak{u}}(K_n)$. In particular,

$$\big(\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{F}_n(\mathfrak{r})[\mathfrak{r}]\big)e_\chi = \big(\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{F}_n(\mathfrak{r})s(D_{\mathfrak{u}}(K_n(\mathfrak{r})))[\mathfrak{r}]\big)e_\chi, \quad (3.9)$$

for all $\mathfrak{r}|\hat{\mathfrak{h}}$. Thus,

$$\rho_n\mathcal{F}_nU_{K_n}e_\chi = 0 \Longrightarrow \big(\mathbb{C}\otimes_{\mathbb{Z}}(\Sigma_n/S_n)\big)e_\chi = 0 \tag{3.10}$$

Since the $\mathbb{Z}$-rank of $\rho_n\mathcal{F}_nU_{K_n}$ is the number of irreducibles complex characters χ of $\mathrm{Gal}(K_n/k)$ such that $\rho_n\mathcal{F}_nU_{K_n}e_\chi \neq 0$, (3.8) and (3.10) clearly show that

$$\mathrm{rank}_{\mathbb{Z}}(\Sigma_n/S_n) \leqslant \mathrm{rank}_{\mathbb{Z}}(\rho_n\mathcal{F}_nU_{K_n}).$$

Let us now compute the $\mathbb{Z}$-rank of D_n. We may replace D_n by its image by the logarithm map $l_n := l_{\hat{\mathfrak{h}},n}$ defined by the formula (3.4) since $\mathrm{Ker}l_n \cap D_n = \mu_{K_n}$. Let χ be an irreducible complex character of G_{K_n} such that $l_n(D_n)e_\chi = 0$. Let $\mathfrak{r}_\chi$ be as above, and let $\eta \in \mathcal{F}_n(\mathfrak{r}_\chi)$. Then we have

$$l_n\Big(\varepsilon_n(\mathfrak{r}_\chi,\eta)\Big)e_\chi = -\eta[K_n : K_n(\mathfrak{r}_\chi)]L'_{\mathfrak{r}_\chi,n}(0,\chi)e_\chi,$$

thanks to (3.1). Here χ is considered as a character of $\mathrm{Gal}(K_n(\mathfrak{r}_\chi)/k)$. Finally we see that $L'_{\mathfrak{r}_\chi,n}(0,\chi) = 0$, which means that there exists some $\mathfrak{u} \in \Omega$ such that χ is trivial on $D_{\mathfrak{u}}(K_n)$, thanks to [14, Chap.I, Proposition 3.4, page 24]. The equality (3.9) then implies $\big(\mathbb{C}\otimes_{\mathbb{Z}}(\Sigma_n/S_n)\big)e_\chi = 0$; and again by (3.8) we conclude that

$$\mathrm{rank}_{\mathbb{Z}}(\Sigma_n/S_n) \leqslant \mathrm{rank}_{\mathbb{Z}}(l_n(D_n)) = \mathrm{rank}_{\mathbb{Z}}(D_n).$$

Finally we have proved that D_n, $\rho_n\mathcal{F}_nU_{K_n}$ and Σ_n/S_n have the same $\mathbb{Z}$-rank. In particular the torsion group of Σ_n/S_n is $\mathrm{Ker}\Theta_n/S_n = \mathrm{Ker}\Theta'_n/S_n$. This proves that the formula of Theorem 3.1 defines a surjective R_n-homomorphism from D_n into $\rho_n\mathcal{F}_nU_{K_n}$, with kernel equal to μ_{K_n}. Let us suppose that $\#\Omega = 1$. Then, $k = \mathbb{Q}$ or k is an imaginary quadratic field and $\mathfrak{g}_n = \mathfrak{p}^{i_n}$, where $\mathfrak{p}$ is a fixed prime dividing p. In this case, for $n \geqslant n_0$, $\mathfrak{r} \neq (1)$ and any $\eta \in \mathcal{F}_n$, the element $\varepsilon_n(\mathfrak{r},\eta) \in St_n$. Moreover, by the properties (ii) and (iii) of Stark units, $\varepsilon_n((1),\eta) \in St_n$, if and only if, $\eta \in \mathcal{F}_n^0$, the kernel in $\mathcal{F}_n$ of the restriction map to k. But, the R_n-submodule of $\mathcal{F}_nU_{K_n}$ generated by $\eta\alpha_n(\mathfrak{r}',\hat{\mathfrak{h}})$, with $\mathfrak{r}|\hat{\mathfrak{h}}$ and $\eta \in \mathcal{F}_n$ are such that $\mathfrak{r} \neq (1)$ or $\eta \in (\mathcal{F}_n)^0$ is equal to $(\mathcal{F}_nU_{K_n})^0$. This proves the last assertion of the theorem. ■

If $n \in \mathbb{N}$ then we denote by $\mu_{K_n,p} = \mathbb{Z}_p \otimes \mu_{K_n}$ the p-part of μ_{K_n} and by $\mathcal{F}_{n,p}$ the annihilator in $\mathbb{Z}_p[G_{K_n}]$ of $\mu_{K_n,p}$. The projective limit of $\mathcal{F}_{n,p}$ will be denoted $\mathcal{F}_\infty$. This is the annihilator in Λ_{K_∞} of $\mu_{K_\infty,p} = \mathbb{Z}_p \otimes \mu_{K_\infty}$. Also in the sequel we shall need the following idempotent of $\mathbb{Q}_p \otimes \Lambda_{K_\infty}$,

$$\nu = \prod_{\substack{\mathfrak{u}\in P\\ \mathfrak{u}\neq \upsilon}} (1 - \frac{s(D_{\mathfrak{u}}(K_\infty))}{|D_{\mathfrak{u}}(K_\infty)|}).$$

If $P = \{\upsilon\}$ then we set $\nu = 1$.

Theorem 3.2. *Let $(\mathcal{F}_\infty U_{K_\infty})^0$ be the kernel in $\mathcal{F}_\infty U_{K_\infty}$ of the restriction map to k, and let*

$$M_\infty := \begin{cases} \mathcal{F}_\infty U_{K_\infty} & \text{if } \#\Omega \geqslant 2,\\ (\mathcal{F}_\infty U_{K_\infty})^0 & \text{if } \#\Omega = 1.\end{cases}$$

Then there exists an exact sequence of Λ_{K_∞}*-modules*

$$0 \longrightarrow \varprojlim_n \mu_{K_n,p} \longrightarrow \overline{St}_\infty \longrightarrow \nu M_\infty \longrightarrow 0.$$

Proof. Let $U_{K_{n,p}}$ be the $\mathbb{Z}_p[G_{K_n}]$-submodule of $\mathbb{Q}_p[G_{K_n}]$ generated by U_{K_n}. Then, the tensor product with $\mathbb{Z}_p$ of the exact sequence given by the above theorem 3.1 gives, for $n \geqslant n_0$, the following projective system of exact sequences

$$0 \longrightarrow \mu_{K_n,p} \longrightarrow \overline{St}_n \longrightarrow \rho_n M_n \longrightarrow 0,$$

where $M_n := \mathcal{F}_{n,p} U_{K_{n,p}}$ if $\#\Omega \geqslant 2$ and $M_n := (\mathcal{F}_{n,p} U_{K_{n,p}})^0$, the kernel in $\mathcal{F}_{n,p} U_{K_{n,p}}$ of the restriction map to k. The coherence of these exact sequences is a consequence of Lemma 3.2. Therefore, the $\mathbb{Z}_p$-modules $\overline{St}_n$ being compact, we obtain the exact sequence

$$0 \longrightarrow \varprojlim_n \mu_{K_n,p} \longrightarrow \overline{St}_\infty \longrightarrow \varprojlim_n \rho_n M_n \longrightarrow 0.$$

But $\varprojlim_n \rho_n M_n = \rho M_\infty$, where $\rho := (\rho_n)_n$. Furthermore, $\rho = \nu(\rho'_n)_n$, where ρ'_n is the product of $1 - \frac{s(D_{\mathfrak{p}}(K_n))}{|D_{\mathfrak{p}}(K_n)|}$ for all the prime ideals of $\mathcal{O}_k$ which ramify in K_∞/K. Let $\rho' := (\rho'_n)_n$ then the map

$$\begin{aligned} h : \nu M_\infty &\longrightarrow \rho M_\infty \\ \nu x &\longmapsto \rho' \nu x. \end{aligned}$$

is clearly a surjective Λ_{K_∞}-homomorphism. Here νM_∞ and ρM_∞ are considered as submodules of the Λ_{K_∞}-algebra $\mathbb{Q}_p[[G_{K_\infty}]]$, projective limit of $\mathbb{Q}_p[G_{K_n}]$. Let us prove that h is injective. Let $x = (x_n)_n \in \nu M_\infty$ be such that $\rho' x = 0$. Let us take $n_1 \geqslant n_0$ sufficiently large so that the prime ideals which ramify in K_∞/K are totally ramified in K_∞/K_{n_1}. Then, for all $n \geqslant n_1$, we may write $\rho'_n = 1 - s(\mathrm{Gal}(K_n/K_{n_1}))\alpha_n$, where $\alpha_n \in \mathbb{Q}_p[G_{K_n}]$. Thus $\rho'_n x_n = 0$ implies that $x_n = s(\mathrm{Gal}(K_n/K_{n_1}))\alpha_n x_n$. In particular, x_n is invariant under the action of $\mathrm{Gal}(K_n/K_{n_1})$. But, there exists $0 \neq d \in \mathbb{N}$ such that $dx_n \in \mathbb{Z}_p[G_{K_n}]$ for all n. Our first conclusion is that $dx_n = s(\mathrm{Gal}(K_n/K_{n_1}))y_n$, for some $y_n \in \mathbb{Z}_p[G_{K_n}]$. Since x_n is the image of x_m by the restriction map to K_n we obtain, for all $m \geqslant n \geqslant n_1$ the relation $dx_n = p^{m-n} s(\mathrm{Gal}(K_n/K_{n_1}))\mathrm{res}_{K_m/K_n}(y_m)$, where res_{K_m/K_n} is the restriction map from K_m to K_n. Since we may take m sufficiently large the above equality shows that $dx_n \in p^i \mathbb{Z}_p[G_{K_n}]$, for all $i \geqslant 0$, which clearly means that $x_n = 0$. This completes the proof of the theorem. ■

Remark 3.3. In case $k = \mathbb{Q}$ we have $\#\Omega = 1$ and $\varprojlim_n \mu_{K_n,p} = 0$. Therefore the exact sequence of Theorem 3.2 is actually an isomorphism

$$\overline{St}_\infty \simeq (\mathcal{F}_\infty U_{K_\infty})^0.$$

Although the cyclotomic units are intensively studied by so many authors, this isomorphism does not seem to have been previously noticed. On the other hand, we deduce from Theorem 3.2 that when k is an imaginary quadratic field we have an exact sequence of Λ_{K_∞}-modules

$$0 \longrightarrow \varprojlim_n \mu_{K_n,p} \longrightarrow \overline{St}^0_\infty \longrightarrow \mathcal{F}_\infty \mathcal{I}_\infty U_{K_\infty} \longrightarrow 0, \tag{3.11}$$

where $\mathcal{I}_\infty$ be the augmentation ideal of Λ_{K_∞}, that is the kernel in Λ_{K_∞} of the restriction map to k (see also the proof of Proposition 2.3). This exact sequence is already proved in the semi-simple case by Rubin in [10, Theorem 7.7]

4. On the Λ-structure of $\overline{St}_\infty$

We observe first that $\varprojlim \mu_{K_n,p} = 0$ unless $\mu_{K_\infty,p}$ is infinite. On the other hand, in view of the Remark 3.1, we suppose until the end of this paper, that $P = \{v\}$ or the places in $P - \{v\}$ are all real and become complex in K. Then, by the above Theorem 3.2, the Λ-torsion and the Λ-rank of $\overline{St}_\infty$, are respectively, $\varprojlim \mu_{K_n,p}$ and

$$\mathrm{rank}_\Lambda(\overline{St}_\infty) = \begin{cases} \mathrm{rank}_\Lambda(U_{K_\infty}) = [K:k] & \text{if } P = \{v\} \\ \#\{\chi \in \hat{G_K} \text{ such that } \nu e_\chi \neq 0\} & \text{if } \#P \geqslant 2 \end{cases}$$

where $\hat{G_K}$ is the group of complex characters of $G_K = \mathrm{Gal}(K/k)$.

The results of the sections 2 and 3 may be used to determine the structure of $\overline{St}_\infty$ as we shall prove in two special cases.

Proposition 4.1. *Suppose that $\mu_{K_\infty,p} = 1$ and that at least one of the hypothesis (HB)$_{K/k}$ or (HD)$_{K_\infty/k}$ is satisfied. Then $\overline{St}_\infty$ is Λ-free.*

Proof. If $\mu_{K_\infty,p} = 1$ then $\mathcal{F}_\infty = \Lambda_{K_\infty}$ and $\varprojlim \mu_{K_n,p} = 0$. Hence we have

$$\overline{St}_\infty \simeq \begin{cases} \nu U_{K_\infty}, & \text{if } \#\Omega \geqslant 2 \\ U^0_{K_\infty}, & \text{if } \#\Omega = 1. \end{cases}$$

thanks to Theorem 3.2. Therefore, in case $\#\Omega \geqslant 2$ the Proposition is a straightforward consequence of Corollary 2.1 and Proposition 2.4. If k is imaginary quadratic then we use Proposition 2.3 and Proposition 2.4. If $k = \mathbb{Q}$ we only need Proposition 2.3 to conclude, since (HD)$_{K_\infty/\mathbb{Q}}$ is never satisfied. ∎

Proposition 4.2. *If $p \nmid [K:k]$ then $\overline{St}_\infty$ is isomorphic as a Λ_{K_∞}-module to*

$$\overline{St}_\infty \simeq \begin{cases} \Lambda_{K_\infty} & \text{if } k = \mathbb{Q} \\ \varprojlim \mu_{K_n,p} \oplus (\mathcal{F}_\infty \cap \mathcal{I}_\infty) & \text{if } k \text{ is imaginary quadratic and } \#\Omega = 1 \\ \varprojlim \mu_{K_n,p} \oplus \nu\mathcal{F}_\infty & \text{otherwise,} \end{cases} \tag{4.1}$$

where $\mathcal{I}_\infty$ be the augmentation ideal of Λ_{K_∞}, that is the kernel in Λ_{K_∞} of the restriction map to k (see the proof of Proposition 2.3 for other descriptions of $\mathcal{I}_\infty$).

Proof. The key of this proposition is to prove that the exact sequence of Theorem 3.2 splits in the semi-simple case. But let us make some preliminary remarks. In case $p \nmid [K:k]$ we have $U_{K_\infty} = \Lambda_{K_\infty}$ by Remark 2.2. Thus, the module M_∞ of Theorem 3.2 satisfies

$$M_\infty = \begin{cases} \mathcal{F}_\infty \cap \mathcal{I}_\infty & \text{if } \#\Omega = 1 \\ \mathcal{F}_\infty & \text{if } \#\Omega \geqslant 2. \end{cases}$$

Moreover, if $k = \mathbb{Q}$ then $M_\infty = \mathcal{I}_\infty$. Indeed, If $p = 2$ then $\mu_{K_\infty,p} = \{-1, 1\}$ and $\mathcal{F}_\infty = \mathcal{I}_\infty + 2\Lambda_{K_\infty}$. If $p \neq 2$ then $\mu_{K_\infty,p} = 1$ and $\mathcal{F}_\infty = \Lambda_{K_\infty}$. But, in the semi-simple case $\mathcal{I}_\infty$ is Λ_{K_∞}-free and, if $\mu_{K_\infty,p}$ is infinite, the modules $\mathcal{F}_\infty$ and $\mathcal{F}_\infty \cap \mathcal{I}_\infty$ are Λ_{K_∞}-free. To prove these properties, we use the decomposition $\mathrm{Gal}(K_\infty/k) = W \times \Gamma$, where W is the $\mathbb{Z}$-torsion subgroup of $\mathrm{Gal}(K_\infty/k)$. The restriction of automorphisms induces an isomorphism from W to $\mathrm{Gal}(K/k)$. In particular, $\Lambda_{K_\infty} = \Lambda[W]$, the group algebra with coefficients in Λ. Hence, $\mathcal{I}_\infty$ is generated by $\Lambda \cap \mathcal{I}_\infty$ and by $w - 1$, $w \in W$. Using the fact that $\Lambda \cap \mathcal{I}_\infty$ is the projective limit of the augmentation ideals of $\mathbb{Z}_p[\mathrm{Gal}(K_n/K)]$, one may easily prove that $\Lambda \cap \mathcal{I}_\infty = (\delta - 1)\Lambda$, where δ is any topological generator of Γ. Let χ_0 be the trivial character of W. Then, $(1 - e_{\chi_0}) \in \mathcal{I}_\infty$; moreover, the following equalities

$$\begin{aligned} \delta - 1 &= [(\delta-1)(1-e_{\chi_0}) + e_{\chi_0}][1 - e_{\chi_0} + (\delta-1)e_{\chi_0}] \\ w - 1 &= [(w-1)(1-e_{\chi_0})][1 - e_{\chi_0} + (\delta-1)e_{\chi_0}], \quad \forall w \in W, \end{aligned}$$

show that $\mathcal{I}_\infty = (1 - e_{\chi_0} + (\delta-1)e_{\chi_0})\Lambda_{K_\infty}$. Since $(1 - e_{\chi_0} + (\delta-1)e_{\chi_0})$ is not torsion we deduce that this element is indeed a basis of $\mathcal{I}_\infty$. Now, suppose that $\mu_{K_\infty,p}$ is infinite. Then, the action of $\mathrm{Gal}(K_\infty/k)$ on $\mu_{K_\infty,p}$ and the isomorphim $\mathrm{Hom}(\mu_{K_\infty,p}, \mu_{K_\infty,p}) \simeq \mathbb{Z}_p$ give an exact sequence of Λ_{K_∞}-modules

$$0 \longrightarrow \mathcal{F}_\infty \longrightarrow \Lambda_{K_\infty} \xrightarrow{\Psi} \mathbb{Z}_p \longrightarrow 0.$$

such that $\Psi(a) = a$ for all $a \in \mathbb{Z}_p$. In particular, $\mathcal{F}_\infty$ is generated by $\Lambda \cap \mathcal{F}_\infty$ and by $w - \Psi(w)$, $w \in W$. Let δ be as above and let $x \in \Lambda$. Since the restriction δ_n of δ to K_n generates $\mathrm{Gal}(K_n/K)$ we may write x in the form $x = (\delta - \Psi(\delta))y + z$, where $y \in \Lambda$ and $z \in \mathbb{Z}_p$. If $x \in \mathcal{F}_\infty$ then $0 = \Psi(x) = \Psi(z) = z$. Thus, $\Lambda \cap \mathcal{F}_\infty = (\delta - \Psi(\delta))\Lambda$. Further, the restriction of Ψ to W is an irreducible $\mathbb{Z}_p$-character of W. We denote by e_Ψ the idempotent associated to this character. Then, exactly as we did for $\mathcal{I}_\infty$, we easily check that $\mathcal{F}_\infty$ is free and $\mathcal{F}_\infty = (1 - e_\Psi + (\delta - \Psi(\delta))e_\Psi)\Lambda_{K_\infty}$. In particular, we see that the module

$$\mathcal{F}_\infty \cap \mathcal{I}_\infty = \mathcal{F}_\infty \mathcal{I}_\infty$$

is Λ_{K_∞}-free. By we have proved, the exact sequence of Theorem 3.2 split over Λ_{K_∞} if $k = \mathbb{Q}$ or k is an imaginary quadratic field. This implies (4.1) in these two cases. Now, let us suppose $\#\Omega \geqslant 2$ and $\nu \neq 1$. If $\mu_{K_\infty,p}$ is finite then (4.1) is equivalent to Theorem 3.2. On the other hand, if $\mu_{K_\infty,p}$ is infinite then $(\varprojlim \mu_{K_n,p})e_\chi = 0$ for all irreducible $\mathbb{Z}_p$-character χ of W such that $\chi \neq \Psi$. Further, if $\mathfrak{u} \in P - \{\upsilon\}$ then $D_{\mathfrak{u}}(K_\infty)e_\Psi = 0$ and hence $\nu e_\Psi = e_\Psi$. In particular, $e_\Psi \nu \mathcal{F}_\infty = (\delta - \Psi(\delta))e_\Psi \Lambda_{K_\infty}$, which proves that $e_\Psi \nu \mathcal{F}_\infty$ is free over $e_\Psi \Lambda_{K_\infty} = e_\Psi \Lambda$. Therefore, we have $\overline{St}_\infty e_\chi \simeq (\varprojlim \mu_{K_n,p})e_\chi \oplus \nu \mathcal{F}_\infty e_\chi$, for all irreducible $\mathbb{Z}_p$-character χ of W. Now we proved (4.1) in all cases. ■

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