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A UNIFORM VERSION OF STIRLING'S FORMULA

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Abstract: A uniform version of Stirling's formula, suitable for future applications, is obtained.
Keywords: Euler's Γ function, Stirling's formula.

With applications in mind, in this paper we prove the following uniform version of Stirling's formula for the Euler Γ function. Let $B_n(z)$ and B_n be the n-th Bernoulli polynomial and the n-th Bernoulli number, respectively.

Theorem. *Let $N \geqslant 1$ be an integer, $B \geqslant 1$ and let $z, s \in \mathbb{C}$ satisfy*

$$\Re(z+s) \geqslant 0, \qquad |s| \leqslant \frac{3}{5}|z|, \qquad N \leqslant 2B|z|.$$

Then

$$\begin{aligned}\log\Gamma(z+s) = &\left(z+s-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi \\ &+ \sum_{j=1}^{N} \frac{(-1)^{j+1}B_{j+1}(s)}{j(j+1)}\frac{1}{z^j} \\ &+ O\left(\frac{1}{|z|^{N+1}}\left((N+\frac{|s|^2}{N^2})|s|^N + B^N N!\right)\right).\end{aligned}$$

Remarks.

1. Taking $s = 0$ we get a uniform version of the classical Stirling formula for $\log\Gamma(z)$ in the range $\Re z \geqslant 0$ and $|z| \geqslant \frac{N}{2B}$, with remainder term $O(B^N N!/|z|^{N+1})$.
2. Using $\Gamma(z)\Gamma(1-z) = \pi/\sin z$ we can replace $\Re z \geqslant 0$ by $|\arg z| \leqslant \pi - \delta$ with

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2010 Mathematics Subject Classification: primary: 33B15 a positive δ. In this case the remainder term has the form

$$O\left(\frac{B^N N!}{|z|^{N+1}}+\frac{1-\operatorname{sgn}\Re z}{2}e^{-2\pi|z|\sin\delta}\right).$$

We start the proof of the Theorem with the following formula, see (5) on p. 21 of Bateman's project [1]. Let z, s be as in the Theorem and fix δ, λ such that

$$0<\delta<\pi/4, \qquad 0<\lambda<2\pi, \qquad \lambda\cos\delta \geqslant 5. \tag{1}$$

Moreover, choose $|\beta| \leqslant \pi/2-\delta$ such that $|\arg(z+s)+\beta| \leqslant \delta$. Then

$$\log\Gamma(z+s)=\left(z+s-\frac{1}{2}\right)\log(z+s)-z-s+\frac{1}{2}\log 2\pi+\int_0^{\infty e^{i\beta}}\lambda(w)e^{-w(z+s)}\mathrm{d}w,$$

where

$$\lambda(w)=\frac{1}{w}\left(\frac{1}{e^w-1}-\frac{1}{w}+\frac{1}{2}\right).$$

Hence

$$\log\Gamma(z+s)=\left(z+s-\frac{1}{2}\right)\log z-z+\frac{1}{2}\log 2\pi+A(z,s)+B(z,s)+C(z,s) \tag{2}$$

with

$$A(z,s)=\left(z+s-\frac{1}{2}\right)\log(1+\frac{s}{z})-s$$

$$B(z,s)=\int_0^{\lambda e^{i\beta}}\lambda(w)e^{-w(z+s)}\mathrm{d}w$$

$$C(z,s)=\int_{\lambda e^{i\beta}}^{\infty e^{i\beta}}\lambda(w)e^{-w(z+s)}\mathrm{d}w.$$

We need several lemmas.

Lemma 1. *Under the assumptions of the Theorem we have*

$$A(z,s)=\sum_{j=1}^{N}\frac{W_j(s)}{z^j}+O\left(\frac{1}{N}\left(1+\frac{|s|}{N}\right)\frac{|s|^{N+1}}{|z|^{N+1}}\right),$$

where

$$W_j(s)=(-1)^j\frac{\frac{j+1}{2}-s}{j(j+1)}s^j.$$

Proof. Clearly

$$A(z,s)=\left(z+s-\frac{1}{2}\right)\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\frac{s^j}{z^j}-s=\sum_{j=1}^{\infty}\frac{W_j(s)}{z^j}.$$

Writing $\theta_0 = |s|/|z|$ we have $0 < \theta_0 \leqslant \theta \leqslant 3/5$, hence

$$\sum_{j=N+1}^{\infty} \frac{|W_j(s)|}{|z|^j} \ll \sum_{j=N+1}^{\infty} \frac{\theta_0^j}{j} + |s| \sum_{j=N+1}^{\infty} \frac{\theta_0^j}{j(j+1)} \ll \frac{\theta_0^{N+1}}{N} + \frac{|s|}{N^2}\theta_0^{N+1}$$
$$\ll \frac{|s|}{N^2}\theta_0^{N+1}\left(\frac{N}{|s|}+1\right) \ll \frac{1}{N}\left(1+\frac{|s|}{N}\right)\frac{|s|^{N+1}}{|z|^{N+1}},$$

and the lemma follows. ■

Lemma 2. *For $B \geqslant 1$ and $AB \geqslant k \geqslant 0$ we have*

$$\int_A^{\infty} x^k e^{-x} \mathrm{d}x \leqslant (k+1)(AB)^k e^{-A}.$$

Proof. We argue by induction. For $k=0$ both sides are equal to e^{-A}. For $k \geqslant 0$ we integrate by parts and use the inductive hypothesis (recalling the condition on A) thus obtaining

$$\int_A^{\infty} x^{k+1} e^{-x} \mathrm{d}x = A^{k+1}e^{-A} + (k+1)\int_A^{\infty} x^k e^{-x}\mathrm{d}x$$
$$\leqslant A^{k+1}e^{-A} + (k+1)^2(AB)^k e^{-A} \leqslant (k+2)(AB)^{k+1}e^{-A},$$

and the lemma follows. ■

Lemma 3. *For $|w| < 2\pi$ we have*

$$\lambda(w) = \sum_{k=0}^{\infty} \alpha_k w^k, \qquad \alpha_k = \frac{B_{k+2}}{(k+2)!}.$$

Proof. This follows from the power series expansion of $\frac{1}{e^w-1}$ and from the values of B_0 and B_1, see (1), (3) and (4) on p. 35–36 of Bateman's project [1]. ■

Lemma 4. *For $k \geqslant 0$ we have*

$$|\alpha_k| \leqslant \frac{1}{12(2\pi)^k}.$$

Proof. By Lemma 3 we have that $\alpha_k = 0$ for odd k's, see (17) on p. 38 of Bateman's project [1]. If $k = 2m$ then by (22) on p. 38 of Bateman's project [1]

$$\alpha_{2m} = (-1)^{m+1}\frac{2\zeta(2m+2)}{(2\pi)^{2m+2}}$$

where $\zeta(s)$ is the Riemann zeta function. Hence

$$(2\pi)^k|\alpha_k| \leqslant \frac{\zeta(2)}{2\pi^2} = \frac{1}{12},$$

and the result follows. ■

Lemma 5. *Under the assumptions of the Theorem we have*

$$B(z,s)=\sum_{k=0}^{N-1}\frac{\alpha_k k!}{(z+s)^{k+1}}+O\left(\frac{N!}{2^N|z|^{N+1}}\right).$$

Proof. Since $0<\lambda<2\pi$, by Lemmas 3 and 4 for $|w|\leqslant\lambda$ we have

$$\lambda(w)=\sum_{k=0}^{N-1}\alpha_k w^k+O\left(\sum_{k=N}^{\infty}\frac{|w|^k}{(2\pi)^k}\right)=\sum_{k=0}^{N-1}\alpha_k w^k+O\left(\frac{|w|^N}{(2\pi)^N}\right).$$

Hence, writing $\zeta=z+s$ and recalling (1) and the definition of the Γ function, using Cauchy's theorem we obtain

$$\begin{aligned}B(z,s)&=\sum_{k=0}^{N-1}\alpha_k\int_0^{\lambda e^{i\beta}}w^k e^{-\zeta w}\mathrm{d}w+O\left(\int_0^{\lambda e^{i\beta}}\left(\frac{|w|}{2\pi}\right)^N e^{-|\zeta||w|\cos\delta}\mathrm{d}w\right)\\&=\sum_{k=0}^{N-1}\alpha_k\frac{\Gamma(k+1)}{\zeta^{k+1}}+O\left(\left|\int_{\lambda e^{i\beta}}^{\infty e^{i\beta}}w^k e^{-\zeta w}\mathrm{d}w\right|\right)\\&\quad+O\left(\int_0^{\lambda}\left(\frac{x}{2\pi}\right)^N e^{-x|\zeta|\cos\delta}\mathrm{d}x\right)\\&=\sum_{k=0}^{N-1}\alpha_k\left(\frac{k!}{\zeta^{k+1}}+r_k(\zeta)\right)+R_N(\zeta),\end{aligned}$$

say. Concerning $r_k(\zeta)$ we have

$$r_k(\zeta)\ll\frac{1}{(|\zeta|\cos\delta)^{k+1}}\int_{\lambda|\zeta|\cos\delta}^{\infty}x^k e^{-x}\mathrm{d}x,$$

and recalling (1) and the assumptions of the Theorem we get

$$B\lambda|\zeta|\cos\delta\geqslant B\frac{2}{5}\lambda|z|\cos\delta\geqslant 2B|z|\geqslant N\geqslant k.$$

Hence we can apply Lemma 2 thus obtaining

$$r_k(\zeta)\ll\frac{(k+1)(\lambda B|\zeta|\cos\delta)^k e^{-\lambda|\zeta|\cos\delta}}{(|\zeta|\cos\delta)^{k+1}}=\frac{(k+1)(\lambda B)^k}{|\zeta|\cos\delta}e^{-\lambda|\zeta|\cos\delta}.$$

Therefore, recalling the power series expansion of e^x, the assumptions of the Theorem and (1)

$$\begin{aligned}\sum_{k=0}^{N-1}\alpha_k r_k(\zeta)&\ll\lambda\sum_{k=0}^{N-1}\left(\frac{\lambda B}{2\pi}\right)^k(k+1)\frac{e^{-\lambda|\zeta|\cos\delta}}{\lambda|\zeta|\cos\delta}\ll\frac{B^N e^{-\lambda|\zeta|\cos\delta}}{\lambda|\zeta|\cos\delta}\\&\ll\frac{B^N N!}{(\lambda|\zeta|\cos\delta)^{N+1}}\leqslant\frac{B^N N!}{(2|z|)^{N+1}}\leqslant\frac{B^N N!}{2^N|z|^{N+1}}.\end{aligned}\tag{3}$$

Concerning $R_N(\zeta)$ we have

$$R_N(\zeta) \ll \frac{1}{(2\pi)^N} \frac{1}{(|\zeta| \cos\delta)^{N+1}} \int_0^{\lambda|\zeta|\cos\delta} x^N e^{-x} \mathrm{d}x$$
$$\ll \frac{1}{(\lambda|\zeta|\cos\delta)^{N+1}} \int_0^{\infty} x^N e^{-x} \mathrm{d}x \ll \frac{N!}{(\lambda|\zeta|\cos\delta)^{N+1}}.$$

Hence arguing as above we get

$$R_N(\zeta) \ll \frac{N!}{2^N |z|^{N+1}},$$

and the lemma follows. ∎

Lemma 6. *For $|w| < 1$ and integers $m \geqslant 1$, $k \geqslant 0$ we have*

$$S_{m,k}(w) := \sum_{l=m}^{\infty} \binom{k+l}{l} w^l = \sum_{l=0}^{k} \binom{k+m}{l} \frac{w^{k+m-l}}{(1-w)^{k-l+1}}.$$

Proof. We have

$$S_{m,k}(w) = \sum_{l=m}^{\infty} \frac{(k+l)\cdots(l+1)}{k!} w^l = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}w^k} \left(\sum_{l=m}^{\infty} w^{k+l} \right) = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}w^k} \left(\frac{w^{k+m}}{1-w} \right)$$
$$= \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} \frac{\mathrm{d}^l}{\mathrm{d}w^l} w^{k+m} \frac{\mathrm{d}^{k-l}}{\mathrm{d}w^{k-l}} \left(\frac{1}{1-w} \right)$$
$$= \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} \frac{(k+m)!}{(k+m-l)!} w^{k+m-l} (k-l)! \frac{1}{(1-w)^{k-l+1}}$$
$$= \frac{1}{k!} \sum_{l=0}^{k} \binom{k+m}{l} \frac{w^{k+m-l}}{(1-w)^{k-l+1}}$$

and the result follows. ∎

Lemma 7. *For $1 \leqslant N \leqslant \frac{3}{2}x - 1$ and $x > 0$ we have*

$$\Phi_N(x) := \sum_{m=1}^{N} \frac{x^m}{m!} \leqslant \frac{(2x)^N}{N!}.$$

Proof. By induction on N. For $N = 1$ the inequality is trivial. For $N > 1$ we have

$$\Phi_{N+1}(x) = \Phi_N(x) + \frac{x^{N+1}}{(N+1)!} \leqslant \frac{(2x)^N}{N!} + \frac{x^{N+1}}{(N+1)!}$$
$$= \frac{(2x)^{N+1}}{(N+1)!} \left(\frac{N+1}{2x} + \frac{1}{2^{N+1}} \right) \leqslant \frac{(2x)^{N+1}}{(N+1)!}$$

since $\frac{N+1}{2x} \leqslant 3/4$ and $\frac{1}{2^{N+1}} \leqslant 1/4$. ∎

Lemma 8. *Under the assumptions of the Theorem we have*

$$B(z,s)=\sum_{k=0}^{N-1}\frac{U_k(s)}{z^{k+1}}+O\left(\frac{N|s|^N+B^N N!}{|z|^{N+1}}\right)$$

for certain polynomials $U_k\in\mathbb{C}[s]$.

Proof. From the expansion for $|w|<1$

$$\frac{1}{(1+w)^{k+1}}=\sum_{l=0}^{\infty}\binom{-k-1}{l}w^l$$

and the identity

$$\binom{-k-1}{l}=\binom{k+l}{l}(-1)^l,$$

by Lemma 5 we obtain (using the notation of Lemma 6)

$$\begin{aligned}B(z,s)&=\sum_{k=0}^{N-1}\frac{\alpha_k k!}{z^{k+1}}\left(\sum_{l=0}^{N-k-1}\binom{-k-1}{l}\left(\frac{s}{z}\right)^l\right)\\&\quad+\sum_{k=0}^{N-1}\frac{\alpha_k k!}{z^{k+1}}S_{N-k,k}\left(-\frac{s}{z}\right)+O\left(\frac{B^N N!}{2^N|z|^{N+1}}\right)\\&=\sum_{k=0}^{N-1}\frac{U_k(s)}{z^{k+1}}+R_N(z,s)+O\left(\frac{B^N N!}{2^N|z|^{N+1}}\right),\end{aligned}$$

say, with $U_k\in\mathbb{C}[s]$. Observing that $\binom{N}{l}=\frac{N\cdots(N-l+1)}{l!}\leqslant\frac{N^l}{l!}$, using the notation of Lemma 7 and writing $\theta_0=|s|/|z|$, by Lemma 6 with $w=\theta_0$ we get

$$\begin{aligned}R_N(z,s)&\ll\frac{1}{|z|}\sum_{k=0}^{N-1}\frac{|\alpha_k|k!}{|z|^k}\sum_{l=N-k}^{\infty}\binom{k+l}{l}\theta_0^l\\&\ll\frac{1}{|z|}\sum_{k=0}^{N-1}\frac{k!}{(2\pi|z|)^k}\sum_{l=0}^{k}\binom{N}{l}\frac{\theta_0^{N-l}}{(1-\theta_0)^{k-l+1}}\\&\ll\frac{\theta_0^N}{|z|}\sum_{k=0}^{N-1}\frac{k!}{(2\pi|z|(1-\theta_0))^k}\sum_{l=0}^{k}\frac{1}{l!}\left(\frac{N(1-\theta_0)}{\theta_0}\right)^l\\&=\frac{\theta_0^N}{|z|}\sum_{k=0}^{N-1}\frac{k!}{(2\pi|z|(1-\theta_0))^k}\left(\Phi_k\left(\frac{N(1-\theta_0)}{\theta_0}\right)+1\right).\end{aligned}$$

But $k+1\leqslant\frac{3}{2}\frac{N(1-\theta_0)}{\theta_0}$ since $\theta_0\leqslant 3/5$ and $k\leqslant N-1$, hence by Lemma 7 we have

$$\Phi_k\left(\frac{N(1-\theta_0)}{\theta_0}\right)+1\ll\frac{1}{k!}\left(\frac{2N(1-\theta_0)}{\theta_0}\right)^k$$

and therefore we obtain

$$R_N(z,s) \ll \frac{\theta_0^N}{|z|}\sum_{k=0}^{N-1}\frac{k!}{(2\pi|z|(1-\theta_0))^k}\frac{1}{k!}\left(\frac{2N(1-\theta_0)}{\theta_0}\right)^k \ll \frac{\theta_0^N}{|z|}\sum_{k=0}^{N-1}\left(\frac{N}{\pi|s|}\right)^k.$$

Now we consider two cases. If $N/(\pi|s|) \leqslant 1+1/N$ then

$$R_N(z,s) \ll \frac{\theta_0^N}{|z|}N = N\frac{|s|^N}{|z|^{N+1}},$$

while if $N/(\pi|s|) > 1+1/N$ then

$$R_N(z,s) \ll \frac{\theta_0^N}{|z|}\frac{\left(\frac{N}{\pi|s|}\right)^N - 1}{\left(\frac{N}{\pi|s|}\right) - 1} \leqslant \frac{\theta_0^N}{|z|}N\left(\frac{N}{\pi|s|}\right)^N = \frac{N}{|z|^{N+1}}\frac{N^N}{\pi^N} \ll \frac{N!}{|z|^{N+1}},$$

and the lemma follows. ■

Lemma 9. *Under the assumptions of the Theorem we have*

$$C(z,s) \ll \frac{N!}{2^N|z|^{N+1}}.$$

Proof. Since $\lambda(w)$ is bounded on the path of integration, writing $\zeta = z+s$ we have

$$C(z,s) \ll \int_\lambda^\infty e^{-|\zeta|x\cos\delta}\mathrm{d}x = \frac{1}{|\zeta|\cos\delta}e^{-|\zeta|\lambda\cos\delta} \ll \frac{1}{|\zeta|\lambda\cos\delta}e^{-|\zeta|\lambda\cos\delta},$$

and the lemma follows arguing as in (3). ■

Now we are ready for the proof of the Theorem. From (2) and Lemmas 1, 8 and 9 we have

$$\log\Gamma(z+s) = \left(z+s-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + \sum_{j=1}^N \frac{P_j(s)}{z^j} + E_N(z,s) \qquad (4)$$

where $P_j(s) = W_j(s) + U_{j-1}(s)$ are polynomials and, considering separately the cases $|s| \leqslant N$ and $|s| > N$, $E_N(z,s)$ satisfies

$$E_N(z,s) \ll \frac{1}{|z|^{N+1}}\left(\left(N+\frac{|s|^2}{N^2}\right)|s|^N + B^N N!\right). \qquad (5)$$

Comparing (4) and (5) with the classical Stirling formula, see (12) on p.48 of Bateman's project [1], by the uniqueness of the asymptotic expansion we have

$$P_j(s) = \frac{(-1)^{j+1}B_{j+1}(s)}{j(j+1)}$$

and the Theorem is proved.

References

[1] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, vol. I, McGraw-Hill, 1953.

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