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ARITHMETIC FUNCTIONS AND THEIR COPRIMALITY

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Abstract: Let $D \geqslant 3$ be an odd integer and $\ell \geqslant -1$ be a non zero integer such that $\gcd(\ell, D) = 1$. Let $f, g : \mathbb{N} \to \mathbb{N}$ be multiplicative functions such that $f(p) = D$ and $g(p) = p+\ell$ for each prime p. We estimate the number of positive integers $n \leqslant x$ such that $\gcd(f(n), g(n)) = 1$. If D is a prime larger than 3, we also examine the size of the number of positive integers $n \leqslant x$ for which $\gcd(g(n), f(n-1)) = 1$.

Keywords: Arithmetic functions, number of divisors, sum of divisors, shifted primes.

1. Introduction

Given an arithmetical function f and a large number x, examining the number of positive integers $n \leqslant x$ for which $\gcd(n, f(n)) = 1$, has been the focus of several papers. For instance, Paul Erdős [4] established that

$$\#\{n \leqslant x : \gcd(n, \varphi(n)) = 1\} = (1 + o(1))\frac{e^{-\gamma}x}{\log\log\log x} \qquad (x \to \infty),$$

where φ is the Euler function and γ is the Euler constant. A similar result can be obtained if one replaces $\varphi(n)$ by $\sigma(n)$, the sum of the divisors of n. Similarly, letting $\Omega(n)$ stand for the number of prime factors of n counting their multiplicity, Alladi [1] proved that the probability that n and $\Omega(n)$ are relatively prime is equal to $6/\pi^2$ by examining the size of $\{n \leqslant x : \gcd(n, \Omega(n)) = 1\}$. Let $K(x)$ stand for the number of positive integers $n \leqslant x$ such that $\gcd(n\tau(n), \sigma(n)) = 1$, where $\tau(n)$ stands for the number of divisors of n. Some fifty years ago, Kanold [5] showed that there exist positive constants $c_1 < c_2$ and a positive number x_0 such that

$$c_1 < K(x)/\sqrt{x/\log x} < c_2 \qquad (x \geqslant x_0).$$

In 2007, the authors [2] proved that there exists a positive constant c_3 such that

$$K(x) = c_3(1 + o(1))\sqrt{\frac{x}{\log x}} \qquad (x \to \infty).$$

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The analogue problem for counting the number of positive integers n for which

$$\gcd(n\tau(n), \varphi(n)) = 1 \tag{1.1}$$

is trivial. Clearly (1.1) holds for $n = 1, 2$. But these are the only solutions. Indeed, assume that (1.1) holds for some $n \geqslant 3$. Then n is squarefree and it must therefore have an odd prime divisor p, in which case $2|\varphi(n)$ and $2|\tau(n)$, implying that $\gcd(n\tau(n), \varphi(n)) > 1$, thereby proving our claim.

More recently, we obtained (see [3]) asymptotic estimates for the counting functions

$$R(x) := \#\{n \leqslant x : \gcd(\varphi(n), \tau(n)) = \gcd(\sigma(n), \tau(n)) = 1\}$$

and

$$N(x) := \#\{n \leqslant x : \ell(n) = 1\},$$

where $\ell(n) := \gcd(\tau(n), \tau(n+1))$. In fact, we proved that, as $x \to \infty$,

$$R(x) = (c_4 + o(1))\sqrt{\frac{x}{\log x}} \qquad \text{and} \qquad N(x) = (c_5 + o(1))\sqrt{x},$$

where c_4 and c_5 are positive constants.

Let $D \geqslant 3$ be an odd integer and let $\ell \geqslant -1$ be a non zero integer such that $\gcd(\ell, D) = 1$. Let $f, g : \mathbb{N} \to \mathbb{N}$ be multiplicative functions such that $f(p) = D$ and $g(p) = p + \ell$ for each prime p. In this paper, we estimate the number $E(x)$ of positive integers $n \leqslant x$ such that

$$\gcd(f(n), g(n)) = 1. \tag{1.2}$$

Our general result will apply in particular to the case $g(n) = \varphi(n)$ (or $\sigma(n)$) and $f(n) = \tau_k(n)$ with k odd, $k \geqslant 3$, where $\tau_k(n)$ stands for the number of ways one can write n as the product of k positive integers taking into account the order in which they are written. Another valid choice is $f(n) = k^{\omega(n)}$ with k odd, $k \geqslant 3$, where $\omega(n)$ stands for the number of distinct prime factors of n with $\omega(1) = 0$.

Moreover, in the case where $D > 3$ is a prime, we shall also examine the size of the number $S(x)$ of positive integers $n \leqslant x$ for which

$$Z(n) := \gcd(g(n), f(n-1)) = 1.$$

From here on, $\gcd(a, b)$ will be written simply as (a, b). In what follows, we shall denote the logarithmic integral of x by $\mathrm{li}(x)$, that is $\mathrm{li}(x) := \displaystyle\int_2^x \frac{dt}{\log t}$, while Γ stands for the Gamma function. We say that a positive integer n is squarefull if $p^2|n$ for all prime divisors p of n; we will denote by $\mathcal{F}$ the set of squarefull numbers. Moreover, the letters c and C will stand for positive constants, while the letters p and q will always stand for prime numbers. Finally, given any set of positive integers $\mathcal{B}$, the expression $\mathcal{N}(\mathcal{B})$ stands for the multiplicative semi-group generated by $\mathcal{B}$.

Finally, given D and ℓ as above, we let $t_1, t_2, \dots, t_T$ be all those reduced residue classes $\bmod D$ for which $(t_j + \ell, D) = 1$ for $j = 1, 2, \dots, T$.

2. Main results

Theorem 2.1. *There exists a positive constant* c_6 *such that*

$$E(x) = (c_6 + o(1))x \log^{\tau-1} x \qquad (x \to \infty), \tag{2.1}$$

where $\tau = T/\varphi(D)$.

Theorem 2.2. *There exists a positive constant* c_7 *such that*

$$S(x) = (c_7 + o(1))x \log^{\tau-1} x \qquad (x \to \infty), \tag{2.2}$$

where, in this case, $\tau - 1 = -1/(D-1)$.

3. Preliminary results

To prove our results we shall need the following results.

Theorem A (Wirsing). *Let* f *be a non negative multiplicative function for which there exist two positive constants* a_1 *and* $a_2 < 2$ *such that* $f(p^\alpha) \leqslant a_1 a_2^\alpha$ *for each integer* $\alpha \geqslant 2$. *Assume also that there exists a positive constant* C *such that*

$$\sum_{p \leqslant x} f(p) = (C + o(1))\frac{x}{\log x} \qquad (x \to \infty).$$

Then

$$\sum_{n \leqslant x} f(n) = \left(\frac{e^{-\gamma C}}{\Gamma(C)} + o(1)\right)\frac{x}{\log x}\prod_{p \leqslant x}\left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots\right) \qquad (x \to \infty).$$

Theorem B (Levin and Feinleib). *Let* f *be a complex valued multiplicative function satisfying the three conditions*

$$\sum_{p \leqslant x} f(p) = (C + o(1))\frac{x}{\log x} \qquad (x \to \infty),$$

$$\sum_{p \leqslant x} |f(p)| = O\left(\frac{x}{\log x}\right),$$

$$f(p^r) = O((2p)^{c_0 r}),$$

where C *and* c_0 *are positive constants with the additional restriction* $c_0 < 1/2$. *Then,*

$$\sum_{n \leqslant x} f(n) = \frac{e^{-C\tau}}{\Gamma(C)}\frac{x}{\log x}\prod_{p \leqslant x}\left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots\right)$$

$$+ o\left(\frac{x}{\log x}\prod_{p \leqslant x}\left(1 + \frac{|f(p)|}{p} + \frac{|f(p^2)|}{p^2} + \dots\right)\right) \qquad (x \to \infty).$$

Proofs. The results of Theorems A and B can be found in Chapter 4 of the book of Postnikov [6]. ■

4. The proof of Theorem 2.1

Let $\wp_{\ell,D}$ be the set of primes p for which $p \equiv t_j \pmod D$ for $j = 1, \ldots, T$. Furthermore, let $H = \{p : p|D\}$ and set

$$\wp_{\ell,D,H} = \wp_{\ell,D} \cup H.$$

It is well known that

$$\#\{n \leqslant x : n \in \mathcal{F}\} = O(\sqrt{x}) \qquad (x \to \infty). \tag{4.1}$$

Hence, given a non squarefull integer $n \leqslant x$, let us write it as $n = Km$, where $K \in \mathcal{F}$ and $m > 1$ is squarefree with $(K, m) = 1$, so that condition (1.2) can be written as

$$(f(K)f(m), g(K)g(m)) = 1. \tag{4.2}$$

So, for each $K \in \mathcal{F}$, let us set

$$E_K(x) := \#\{n = Km \leqslant x : m > 1 \text{ and (4.2) holds}\},$$

so that, in light of (4.1),

$$E(x) = \sum_{K \in \mathcal{F}} E_K(x) + O(\sqrt{x}). \tag{4.3}$$

By using the Brun-Selberg Sieve, we obtain that for $1 \leqslant K \leqslant \sqrt{x}$,

$$E_K(x) \leqslant E_1\left(\frac{x}{K}\right) \ll \frac{x}{K} \prod_{\substack{p \leqslant \sqrt{x} \\ p \notin \wp_{\ell,D}}} \left(1 - \frac{1}{p}\right) \ll \frac{x}{K} (\log x)^{\tau - 1}, \tag{4.4}$$

while we trivially have that $E_K(x) \ll x/K$ if $\sqrt{x} < K \leqslant x$. Since $\displaystyle\sum_{K \in \mathcal{F}} \frac{1}{K}$ is convergent, it follows from (4.3) and (4.4) that

$$E(x) = \sum_{\substack{K \in \mathcal{F} \\ K < Y_x}} E_K(x) + o(x(\log x)^{\tau - 1}) \qquad (x \to \infty), \tag{4.5}$$

where Y_x is an arbitrary function tending to infinity as $x \to \infty$, which we can also assume to satisfy $\max_{n \leqslant Y_x} f(n) \leqslant \log\log\log x$, say.

Observe that a necessary condition for (4.2) to hold is that

$$(g(K), f(K)D) = 1. \tag{4.6}$$

Now let $\mathcal{K}$ be the set of those $K \in \mathcal{F}$ for which (4.6) holds. Note that the set $\mathcal{K}$ is non empty, since $1 \in \mathcal{K}$. Moreover, define

$$\mathcal{K}_0 = \{K \in \mathcal{K} : p|f(K) \Rightarrow p|D\}.$$

We shall prove that, for every fixed $K \in \mathcal{F}$, as $x \to \infty$,

$$E_K(x) = o(E_1(x)) \qquad \text{if } K \in \mathcal{F} \setminus \mathcal{K}_0, \tag{4.7}$$

$$E_K(x) = (1+o(1)) \prod_{\substack{p|K \\ p \in \wp_{\ell,D,H}}} \left(1+\frac{1}{p}\right)^{-1} \frac{E_1(x)}{K} \qquad \text{if } K \in \mathcal{K}_0, \tag{4.8}$$

$$E_1(x) = (c+o(1))x \log^{\tau-1} x, \tag{4.9}$$

where c is a positive constant. Combining these three estimates with (4.4) and (4.5), Theorem 2.1 will follow immediately.

For a given $K \in \mathcal{K}_0$, letting

$$E(y,K) := \#\{m \in [2,y] : \ m \text{ squarefree and } (m,K)=1\},$$

it is clear that the number of positive integers $n = Km \leqslant x$, with $m > 1$, for which (4.2) holds is equal to $E(x/K, K)$.

Consider the multiplicative function h_K defined on prime powers by $h_K(p^\alpha) = 0$ if $\alpha \geqslant 2$ or if $p|K$, and by

$$h_K(p) = \begin{cases} 1 & \text{if } p \nmid K \text{ and } p \in \wp_{\ell,D,H}, \\ 0 & \text{otherwise.} \end{cases}$$

With this definition of h_K, we have that

$$E(y,K) = \sum_{n \leqslant y} h_K(n). \tag{4.10}$$

To estimate this last sum, we shall consider the Dirichlet series

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{h_K(n)}{n^s} &= \prod_p \left(1 + \frac{h_K(p)}{p} + \frac{h_K(p^2)}{p^2} + \dots\right) \\ &= \prod_{\substack{p|K \\ p \in \wp_{\ell,D,H}}} \left(1+\frac{1}{p^s}\right)^{-1} \prod_{p \in \wp_{\ell,D,H}} \left(1+\frac{1}{p^s}\right). \end{aligned}$$

In light of the fact that

$$\sum_{p \leqslant x} h_K(p) = (\tau + o(1)) \frac{x}{\log x} \qquad (x \to \infty),$$

we may use Theorem A and obtain that, as $x \to \infty$,

$$E_K(x) = (1+o(1)) \prod_{\substack{p|K \\ p \in \wp_{\ell,D,H}}} \left(1+\frac{1}{p}\right)^{-1} \frac{x}{\log x} \exp\{\tau \log\log x + C_D + o(1)\},$$

where C_D is a suitable constant depending only on D. Estimates (4.8) and (4.9) are thus established. It remains to prove (4.7). So, let $K \in \mathcal{F} \setminus \mathcal{K}_0$ be fixed. Then, there exists a prime divisor q of $f(K)$ such that $(q, D) = 1$. If (4.2) holds, then the fact that $p|m$ implies that $(p+\ell, qD) = 1$. Hence, from the Brun-Selberg Sieve, it follows that

$$\begin{aligned} E_K(x) &\ll \frac{x}{K} \prod_{\substack{p \leqslant x \\ (p+\ell, qD)>1}} \left(1 - \frac{1}{p}\right) \\ &\ll \frac{x}{K} \exp\left\{ - \sum_{\substack{p \leqslant x \\ (p+\ell, D)>1}} \frac{1}{p} - \sum_{\substack{p \leqslant x \\ (p+\ell, D)=1 \\ q|p+\ell}} \frac{1}{p} \right\} \\ &\ll \frac{x}{K} \exp\left\{ -\left(1 - \frac{T}{\varphi(D)}\right) \log\log x - \frac{T}{\varphi(D)} \frac{1}{q-1} \log\log x \right\} \\ &\ll \frac{x}{K} \log^{\tau-1} x \cdot \exp\left(- \frac{T}{\varphi(D)(q-1)} \log\log x \right), \end{aligned}$$

thereby implying that (4.7) holds and thus completing the proof of Theorem 2.1.

5. The proof of Theorem 2.2

First observe that the number of those integers $n \leqslant x$ for which n or $n-1$ is a squarefull number is $O(\sqrt{x})$.

Let us write $n = Km$ and $n - 1 = R\nu$, where K and R are squarefull, while m and ν are squarefree, with $(K, m) = 1$ and $(R, \nu) = 1$. Then, for each pair of coprime squarefull numbers K and R, define

$$S_{K,R}(x) = \#\{n \leqslant x : n = Km,\ n-1 = R\nu,\ m > 1, \nu > 1,\ Z(n) = 1\}.$$

With these notations and the above observation, it is clear that

$$S(x) = \sum_{\substack{K,R \in \mathcal{F} \\ (K,R)=1}} S_{K,R}(x) + O(\sqrt{x}). \tag{5.1}$$

Since in this case, $H = D$, it follows that if $n = Km$, $n-1 = R\nu$, $m > 1$, $\nu > 1$ and $Z(n) = 1$, then $\nu \in \wp_{\ell,D,D}$. Consequently, by using the Brun-Selberg Sieve, we obtain that, for each squarefull number R,

$$\sum_{K \in \mathcal{F}} S_{K,R}(x) \ll \begin{cases} \frac{x}{R} \log^{\tau-1} x & \text{if } R \leqslant \sqrt{x}, \\ \frac{x}{R} & \text{if } \sqrt{x} < R \leqslant x. \end{cases} \tag{5.2}$$

Fixing $K \in \mathcal{F}$, we shall estimate the number of integers $n \leqslant x$ such that $K|n$ and for which $n - 1 = R\nu$ with $R \in \mathcal{F}$ and $\nu \in \mathcal{N}(\wp_{\ell,D,D})$.

Similarly as in (5.2), we have

$$\sum_{R\in\mathcal{F}} S_{K,R}(x) \ll \begin{cases} \frac{x}{K}\log^{\tau-1} x & \text{if } K \leqslant \sqrt{x}, \\ \frac{x}{K} & \text{if } \sqrt{x} < K \leqslant x. \end{cases} \tag{5.3}$$

It follows from (5.2) and (5.3) that for an arbitrary function $Y_x \to \infty$,

$$\sum_{\max(K,R)>Y_x} S_{K,R}(x) = o\left(x\log^{\tau-1} x\right) \qquad (x\to\infty). \tag{5.4}$$

So, let us assume that $\max(K,R) \leqslant Y_x$ and define

$$\mathcal{R}_0 = \{R \in \mathcal{F} : q|f(R) \Rightarrow q = D\} \qquad \text{and} \qquad \mathcal{R}_1 = \mathcal{F}\setminus\mathcal{R}_0.$$

Fix $R\in\mathcal{R}_1$ and let $q|f(R)$ with $q \neq D$. Then, $n = Km$ implies that $R\nu + 1 \equiv 0 \pmod K$, while $Z(n) = 1$ implies that $(g(m), Dq) = 1$. Thus, by using the Brun-Selberg Sieve, we have that

$$\sum_{K\in\mathcal{F}} S_{K,R}(x) = o\left(\frac{x}{R}\log^{\tau-1} x\right) \qquad (x\to\infty),$$

so that

$$\sum_{R\in\mathcal{R}_1}\sum_{K\in\mathcal{F}} S_{K,R}(x) = o\left(x\log^{\tau-1} x\right) \qquad (x\to\infty). \tag{5.5}$$

We will say that K, R is an *admissible pair* if $(g(K), Df(R)) = 1$. Observe that it is clear that $S_{K,R}(x) = 0$ if K, R is not an admissible pair, and also that in the case $R \in \mathcal{R}_0$, $(g(K), Df(R)) = 1$ is equivalent to $(g(K), D) = 1$. Finally, observe that $K = 1$, $R = 1$ is an admissible pair.

From (5.4), (5.5) and (5.1), it therefore follows that

$$S(x) = \sum_{\substack{R\in\mathcal{R}_0 \\ K,R \text{ admissible pair} \\ \max(K,R)\leqslant Y_x}} S_{K,R}(x) + o(x\log^{\tau-1} x). \tag{5.6}$$

Let F be the multiplicative function defined by

$$F(p) = \begin{cases} 1 & \text{if } p+\ell \not\equiv 0 \pmod D, \\ 0 & \text{otherwise} \end{cases}$$

and

$$F(p^\alpha) = 0 \text{ if } \alpha \geqslant 2,$$

and define the function $m(x) = \prod_{p\leqslant x}\left(1 + \frac{F(p)}{p}\right)$.

It is clear that if $0 < \varepsilon_x \to 0$ as $x\to\infty$, then $\max_{x^{1-\varepsilon_x}\leqslant y\leqslant x}\left|\frac{m(y)}{m(x)} - 1\right| \to 0$ as $x\to\infty$.

Given an integer $B \geqslant 2$, let χ_B be a character mod B and assume that $\chi_B(n_j) = 1$ for the H distinct residue classes $n_j \pmod{B}$. It is clear that if $H > \varphi(B)/2$, then $\chi_B = \chi_B^{(0)}$ is the principal character mod B.

We now define the functions u and V as follows.

Set $u(n) = \chi_K^{(0)}(n)F(n)$, $n = Km$, $n-1 = R\nu$, so that $u(m) = 1$ if and only if $(m, K) = 1$, m is squarefree and $p|m$ implies that $p + \ell \not\equiv 0 \pmod{D}$. Let V be the multiplicative function defined by

$$V(p) = \begin{cases} 0 & \text{if } (p, R) = 1, \\ 1 & \text{if } p|R, \end{cases}$$

$$V(p^2) = \begin{cases} -1 & \text{if } (p, R) = 1, \\ 0 & \text{if } p|R \end{cases}$$

and $V(p^\alpha) = 0$ if $\alpha \geqslant 3$.

Observe that if $V(\delta) \neq 0$, then we may write $\delta = \delta_1\delta_2^2$ with $\delta_1|R$ and $(\delta_2, R) = 1$, so that $V(\delta) = \mu(\delta_1)\mu(\delta_2)$, where μ stands for the Moebius function. Therefore,

$$V(\delta) = \begin{cases} \mu(\delta_1)\mu(\delta_2) & \text{if } (\delta_1, R) = 1 \text{ and } \delta_2|R, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from this definition that

$$\sum_{\delta|\nu} V(\delta) = \begin{cases} 1 & \text{if } (\nu, R) = 1,\ \nu \text{ squarefree}, \\ 0 & \text{otherwise.} \end{cases}$$

Now, let m_0, ν_0 be the smallest non negative squarefree integers such that

$$Km_0 - R\nu_0 = 1, \tag{5.7}$$

so that all integer solutions of $Km - R\nu = 1$ are given by $m = m_0 + tR$ and $\nu = \nu_0 + tK$ for $t \in \mathbb{Z}$.

With the above definitions, we have

$$S_{K,R}(x) = \sum_{\delta \leqslant x/R} V(\delta) \sum_{\substack{t \leqslant (x-m_0)/KR \\ \nu_0 + tK \equiv 0 \pmod{\delta}}} u(m_0 + tR). \tag{5.8}$$

If $(\delta, K) > 1$, then $(\delta, K) = (\delta_1^2, K)$ and δ_1^2 and K are both squarefull. It follows that if $p|(\delta_1^2, K)$, then $p^2|\delta_1^2$ and $p^2|K$, so that $p^2|\nu_0$ and consequently $p^2|\nu_0 + tK$ for each $t \in \mathbb{Z}$, implying that there each number $\nu_0 + tK$ is squarefull. Hence it follows that in this case, $S_{K,R}(x) = 0$. Therefore, we can from now on assume that $(\delta, K) = 1$ (which holds if and only if $(\delta_1, K) = 1$).

Since $(\delta, K) = 1$, it follows that the congruence $\nu_0 + tK \equiv 0 \pmod{\delta}$ has one solution mod δ, represented by $\nu_0 + t_0K \equiv 0 \pmod{\delta}$, say. This implies that all solutions of the congruence $\nu_0 + tK \equiv 0 \pmod{\delta}$ are given by $t = t_0 + k\delta$, $k \in \mathbb{Z}$.

In light of these observations, (5.8) can be written as

$$\begin{aligned} S_{K,R}(x) &= \sum_{\substack{\delta\leqslant x/R\\(\delta,K)=1}} V(\delta) \sum_{k\leqslant (x-(m_0+t_0R))/\delta R} u(m_0+t_0R+\delta Rk)\\ &= \sum_{\substack{\delta\leqslant x/R\\(\delta,K)=1}} V(\delta)M(\delta)\\ &= \sum_{\substack{\delta\leqslant U_x\\(\delta,K)=1}} V(\delta)M(\delta) + \sum_{\substack{\delta> U_x\\(\delta,K)=1}} V(\delta)M(\delta) = \Sigma_1+\Sigma_2, \end{aligned} \tag{5.9}$$

say, where $U(x)$ is a function chosen so that $U(x)\to\infty$ as $x\to\infty$ and $U(x) = O(\log\log\log x)$.

By the Brun-Selberg Sieve, we obtain that

$$\begin{aligned} \Sigma_2 &\leqslant \sum_{\substack{(\delta,K)=1\\U_x<\delta<\sqrt{x}}} |V(\delta)|\frac{x}{\delta R} \prod_{\substack{p\leqslant x\\p\equiv \ell \pmod D}} \left(1-\frac{1}{p}\right) + \sum_{\sqrt{x}\leqslant\delta\leqslant x/R} \frac{x}{\delta R}|V(\delta)|\\ &\leqslant c\frac{x}{R}(\log x)^{-1/(D-1)} \sum_{\delta>U_x}\frac{|V(\delta)|}{\delta} + \frac{x}{R}\sum_{\sqrt{x}\leqslant\delta\leqslant x/R}\frac{|V(\delta)|}{\delta}. \end{aligned} \tag{5.10}$$

Now, on the one hand,

$$\sum_{\delta>U_x}\frac{|V(\delta)|}{\delta} \leqslant \sum_{\delta_2\mid R}\frac{|\mu(\delta_2)|}{\delta_2}\sum_{\delta_1>U_x/R}\frac{1}{\delta_1^2} \leqslant \prod_{p\mid R}\left(1+\frac{1}{p}\right)\cdot c\frac{\sqrt{R}}{\sqrt{U_x}}, \tag{5.11}$$

while on the other hand,

$$\sum_{\sqrt{x}\leqslant\delta\leqslant x/R}\frac{|V(\delta)|}{\delta} < \sum_{\delta\geqslant\sqrt{x}}\frac{|V(\delta)|}{\delta} \leqslant \prod_{p\mid R}\left(1+\frac{1}{p}\right)\cdot\left(\frac{R}{\sqrt{x}}\right)^{1/2}. \tag{5.12}$$

Gathering (5.11) and (5.12) in (5.10), we obtain

$$\Sigma_2 = o(x(\log x)^{-1/(D-1)}) \qquad (x\to\infty). \tag{5.13}$$

We now consider an estimate for M_δ when $\delta\leqslant U_x$. Recall that

$$M_\delta = \sum_{m_0+t_0R+\delta Rk\leqslant x/K} u(m_0+t_0R+\delta Rk).$$

Let $A = m_0+t_0R$ and $B=\delta R$. One can see that $(A,B)=1$. Indeed, first observe that $(A,R)=1$, since in light of (5.7), we have $(m_0,R)=1$. Now, it follows from (5.7) that $K(m_0+t_0R)-R(\nu_0+t_0K)=1$. But $\delta\mid\nu_0+t_0K$ implies that $(m_0+t_0R,\delta)=1$. Therefore, it follows from these observations that $(A,B)=1$.

Thus, with the above notations, M_δ can be written as

$$\begin{aligned} M_\delta &= \sum_{A+Bk\leqslant x/K} u(A+Bk) = \frac{1}{\varphi(B)} \sum_{\chi \pmod B} \overline{\chi}(A)\cdot \sum_{n\leqslant x/K} \chi(n)u(n) \\ &= M_\delta^{(1)} + M_\delta^{(2)}, \end{aligned}$$

say. These last two expressions can be written as

$$M_\delta^{(1)} = \frac{1}{\varphi(B)} \sum_{n\leqslant x/K} \chi_B^{(0)}(n)u(n), \tag{5.14}$$

$$M_\delta^{(2)} = \frac{1}{\varphi(B)} \sum_{\chi\neq\chi_0} \overline{\chi}(A)\cdot \sum_{n\leqslant x/K} \chi(n)u(n). \tag{5.15}$$

Let $\chi_B \neq \chi_B^{(0)}$. Then $u(p)\chi_B(p) \neq u(p)$ holds for at least one prime $p = p^*$. But then $u(p) = 1$ for every prime $p \equiv p^* \pmod B$ and therefore

$$\frac{1}{\mathrm{li}(x)} \sum_{p\leqslant x} u(p)\chi_B(p) \to \tau = \tau_{\chi_B} \qquad (x\to\infty),$$

with $\mathrm{Re}(\tau_{\chi_B}) < \tau_{\chi_0}$. Hence, it follows from Theorem B that

$$M_\delta^{(2)} = o\left(\frac{x}{\log x} m(x)\right) \qquad (x\to\infty). \tag{5.16}$$

On the other hand,

$$\begin{aligned} \sum_{n\leqslant x/K} u(n)\chi_B^{(0)}(n) = \frac{e^{-\gamma\tau}}{\Gamma(\tau)} \frac{x}{K\log(x/K)} \prod_{p\leqslant x/K}\left(1+\frac{u(p)\chi_B^{(0)}(p)}{p}\right) \\ + o\left(\frac{x}{\log x} m(x)\right), \end{aligned} \tag{5.17}$$

while, for $\delta \leqslant U(x)$, we have $\log(x/K) = (1+o(1))\log x$ as $x\to\infty$.

Now,

$$\prod_{p\leqslant x/K}\left(1+\frac{u(p)\chi_B^{(0)}(p)}{p}\right) = (1+o(1)) \prod_{p\leqslant x}\left(1+\frac{F(p)}{p}\right) \prod_{p\mid KR\delta}\left(1+\frac{F(p)}{p}\right)^{-1}. \tag{5.18}$$

Thus, in light of estimates (5.13) through (5.18), (5.9) becomes

$$S_{K,R}(x) = \frac{e^{-\gamma\tau}}{\Gamma(\tau)} H(K,R)m(x) + o\left(\frac{x}{\log x} m(x)\right), \tag{5.19}$$

where

$$H(K,R) = \sum_{(\delta,K)=1} \frac{V(\delta)}{K\varphi(R\delta)} \prod_{p|RK\delta} \left(1+\frac{F(p)}{p}\right)^{-1}$$
$$= \prod_{p|RK} \left(1+\frac{F(p)}{p}\right)^{-1} \sum_{\delta_2|R} \frac{\mu(\delta_2)}{\delta_2} \frac{1}{K\varphi(R)} \tag{5.20}$$
$$\times \sum_{(\delta_2,RK)=1} \frac{\mu(\delta_2)}{\varphi(\delta_2^2)} \prod_{p|\delta_2} \left(1+\frac{F(p)}{p}\right)^{-1}.$$

Since $\sum_{\delta_2|R} \frac{\mu(\delta_2)}{\delta_2} = \frac{\varphi(R)}{R}$ and

$$\sum_{(\delta_2,RK)=1} \frac{\mu(\delta_2)}{\varphi(\delta_2^2)} \cdot \prod_{p|\delta_2} \left(1+\frac{F(p)}{p}\right)^{-1} = \prod_{p\nmid RK} \left(1-\frac{1}{p(p-1)} \cdot \frac{1}{1+\frac{F(p)}{p}}\right),$$

it follows that (5.20) can be written as

$$H(K,R) = \frac{1}{KR} \prod_{p|RK} \left(1+\frac{F(p)}{p}\right)^{-1} \cdot \prod_{p\nmid RK} \left(1-\frac{1}{p(p-1)} \cdot \frac{1}{1+\frac{F(p)}{p}}\right). \tag{5.21}$$

Note that here we used the fact that

$$\sum_{\delta\leqslant U_x} \frac{V(\delta)}{K\varphi(R\delta)} \prod_{p|RK\delta} \left(1+\frac{F(p)}{p}\right)^{-1} \to H(K,R) \qquad \text{as } U_x \to \infty$$

Since it is clear from (5.21) that

$$0 < \sum_{K,R\in\mathcal{F}}{}^{*} H(K,R) < +\infty,$$

where the star in the sum is there to indicate that we have rightfully ignored those pairs K, R for which either $R \in \mathcal{R}_1$ or K, R is a non admissible pair. The statement of Theorem 2.2 then follows from relation (5.19).

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