Functiones et Approximatio
45.1 (2011), 23–41

ABOUT AN ANALOGUE OF IHARA'S LEMMA FOR SHIMURA CURVES

Miriam Ciavarella, Lea Terracini

Abstract: The object of this work is to present the status of art of an open problem: to provide an analogue for the cohomology of Shimura curves of the Ihara's lemma [Iha73] which holds for modular curves. We will formulate our conjecture and locate it in the more general setting of the congruence subgroup problem. We will exploit the relationship between cohomology of Shimura curves and certain spaces of modular forms to establish some consequences of the conjecture about congruence modules of modular forms and about the problem of raising the level.

Keywords: cohomology of Shimura curves, congruence subgroups, Hecke algebras.

Introduction

Let N be a positive integer and q be a prime number not dividing N. We denote by $\mathcal{M}(N)$ the Hecke modules arising from the cohomology with integer coefficients of the modular curve of level N. Ihara's lemma [Iha73] establishes that the sum of the two natural degeneracy maps $\alpha : \mathcal{M}(N)^2 \to \mathcal{M}(Nq)$ has torsion-free cokernel. This result has been widely applied to detect congruences between modular forms [Rib83b] and for establishing the freeness of a local component of $\mathcal{M}(N)$ over the Hecke algebra in the non-minimal case [Wil95], [TW95], [DD97], [DR97].

Ihara's Lemma has been generalized by Diamond and Taylor [DT94] to certain Hecke components of the ℓ-adic cohomology of Shimura curves with good reduction at ℓ. Their proof relies on a crystalline cohomology argument. The reader can refer to Section 2 for a more detailed exposition of known results.

In this work (Section 1) we state a conjecture concerning a problem of Ihara for the ℓ-adic cohomology of Shimura curves in some case of bad reduction. In Section 2 we give a review of the literature about Ihara's problems in some particular situations. In Section 3 we relate our conjecture to more general conjectures about the subgroup congruence property. In the last section, we use the conjecture to generalize to non-minimal levels some Galois representations theoretic results about local components of Hecke algebras and cohomology of Shimura curves proved in the minimal case in [Ter03, Cia09].

2010 Mathematics Subject Classification: primary: 11F80, 11F33, 11G18, 20G30

1. The conjecture

Let B be an indefinite quaternion algebra over $\mathbf{Q}$ of discriminant $\Delta \neq 1$ and let ν be its reduced norm. Let R be a maximal order in B. If p is a prime not dividing Δ, (included $p=\infty$) we fix an isomorphism of $\mathbf{Q}_p$-algebras $i_p : B_p \to M_2(\mathbf{Q}_p)$ such that $i_p(R_p) = M_2(\mathbf{Z}_p)$ if $p \neq \infty$, where $B_p = B \otimes_{\mathbf{Q}} \mathbf{Q}_p$ and $R_p = R \otimes_{\mathbf{Z}} \mathbf{Z}_p$.

We will denote by $B_{\mathbf{A}}$ the adelization of B, by $B_{\mathbf{A}}^{\times}$ the topological group of invertible elements in $B_{\mathbf{A}}$ and $B_{\mathbf{A}}^{\times,\infty}$ the subgroup of finite idèles.

We fix a prime $\ell > 2$ and we suppose $\Delta = \Delta'\ell$. Let N be an integer prime to Δ; if $p \not|\Delta$, we define:

$$K_p^0(N) = i_p^{-1}\left\{\gamma \in GL_2(\mathbf{Z}_p) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N\right\}$$
$$K_p^1(N) = i_p^{-1}\left\{\gamma \in GL_2(\mathbf{Z}_p) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod N\right\}.$$

We define
$$V_0(N) = \prod_{p \not| N} R_p^{\times} \times \prod_{p|N} K_p^0(N).$$
$$V_1(N) = \prod_{p \not| N\ell} R_p^{\times} \times \prod_{p|N} K_p^1(N) \times (1+\pi_\ell R_\ell)$$
where π_ℓ is a uniformizer of R_ℓ.

We introduce a Dirichlet character $\psi : (\mathbf{Z}/\Delta\ell N\mathbf{Z})^{\times} \to \mathbf{C}^{\times}$; as a general hypothesis we will assume that the order of ψ is prime to ℓ. By an abuse of notation, we will denote by ψ also the adelization of the Dirichlet character ψ and we will denote by ψ_p the composition of ψ with the inclusion $\mathbf{Q}_p^{\times} \to \mathbf{A}^{\times}$. We fix a regular character $\chi : \mathbf{Z}_{\ell^2}^{\times} \to \overline{\mathbf{Q}}^{\times}$ such that $\chi|_{\mathbf{Z}_\ell^{\times}} = \psi_\ell|_{\mathbf{Z}_\ell^{\times}}$. We observe that χ is not uniquely determinated by ψ; we extend χ to $\mathbf{Q}_{\ell^2}^{\times}$ by putting $\chi(\ell) = -\psi_\ell(\ell)$. If we fix an embedding of $\overline{\mathbf{Q}}$ in $\overline{\mathbf{Q}}_\ell$, then we can regard the values of χ in this field.

By local classfield theory, χ may be regarded as a character of I_ℓ. Then there is a ℓ-type τ associated to χ (for a reference see [CDT99]), $\tau = \chi \oplus \chi^\sigma$ where σ is the non trivial element of $\mathrm{Gal}(\mathbf{Q}_{\ell^2}/\mathbf{Q}_\ell)$. We put $\widehat{\psi} = \prod_{p|N} \psi_p \times \chi$; then $\widehat{\psi}$ is a character of $V_0(N)$ and its kernel is $V_1(N)$ where

For $i = 0, 1$, we consider the adelic Shimura curve associated to the group $V_i(N)$:
$$\mathbf{X}_i(N) = B_{\mathbf{Q}}^{\times} \setminus B_{\mathbf{A}}^{\times}/K_\infty^{+} \times V_i(N)$$
where $K_\infty^{+} = \mathbf{R}^{\times} SO_2(\mathbf{R})$.

Let $\mathcal{O}$ be the ring of integers of a finite extension K of $\mathbf{Q}_\ell$ containing $\mathbf{Q}_{\ell^2}$ and the image of $\widehat{\psi}$; let k be its residue field.

Since $K_p^0(N)/K_p^1(N) \simeq (\mathbf{Z}/p\mathbf{Z})^{\times}$ and $R_\ell^{\times}/(1+\pi_\ell R_\ell) \simeq \mathbf{F}_{\ell^2}^{\times}$ we can regard $\widehat{\psi} = \prod_{p|N} \psi_p \times \chi$ as a character of the group $V_0(N)/V_1(N) \cong (\mathbf{Z}/N\mathbf{Z})^{\times} \times \mathbf{F}_{\ell^2}^{\times}$ by isomorphims i_p. Then $V_0(N)/V_1(N)$ naturally acts on $H^1(\mathbf{X}_1(N), \mathcal{O})$ via the projection $\mathbf{X}_1(N) \to \mathbf{X}_0(N)$. As a general hypothesis, we suppose that the order of the group $V_0(N)/V_1(N)$ is invertible in $\mathcal{O}$, thus if we denote by $H^1(\mathbf{X}_1(N), \mathcal{O})^{\widehat{\psi}}$ the sub-Hecke-module of $H^1(\mathbf{X}_1(N), \mathcal{O})$ on which $V_0(N)/V_1(N)$ acts by the character $\widehat{\psi}$, then $H^1(\mathbf{X}_1(N), \mathcal{O})^{\widehat{\psi}}$ is a direct summand of $H^1(\mathbf{X}_1(N), \mathcal{O})$.

Let S be a square-free number prime to $N\Delta$; by abuse of notation we will denote by S also the finite set of rational primes p_j such that $\prod_j p_j = S$. For $i = 0, 1$ we define the group

$$V_i(N,S) = \left\{ u \in V_i(N) \mid u_q \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod q \text{ for } q|S \right\};$$

we observe that $V_0(N,S) = V_0(NS)$ and if $S = \emptyset$ then $V_i(N,S) = V_i(N)$. For any prime number q not dividing ΔN let η_q be the idèle in $B_{\mathbf{A}}^{\times}$ defined by $\eta_{q,v} = 1$ if $v \neq q$ and $\eta_{q,q} = i_q^{-1}\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$. Then both $V_i(N,q)$ and $\eta_q V_i(N,q)\eta_q^{-1}$ are subgroups of $V_i(N)$. The two injections (inclusion and conjugation by η_q) give rise to degeneracy maps on the Shimura curves $\mathbf{X}_i(N,q) \to \mathbf{X}_i(N)$ where, for $i = 0, 1$ and S as above, $\mathbf{X}_i(N,S)$ is the Shimura curve associated to the group $V_i(N,S)$. The degeneracy maps commute with the action of $V_0(N)/V_1(N)$ so that there is a map of cohomology groups

$$\alpha : H^1(\mathbf{X}_1(N), \mathcal{O})^{\widehat{\psi}} \oplus H^1(\mathbf{X}_1(N), \mathcal{O})^{\widehat{\psi}} \to H^1(\mathbf{X}_1(N,q), \mathcal{O})^{\widehat{\psi}} \tag{1}$$

defined by $\alpha = \mathrm{Res} \oplus |_{\eta_q}$ where Res is the usual restriction map and $|_{\eta_q}$ is the map on cohomology induced by conjugation by η_q.

Essentially the open problem is to verify that the map α has torsion-free cokernel. We will refer to this kind of problems as "Problem of Ihara"; the motivation for this name will be clear in the Section 2.

One can naturally define an Hecke algebra acting over each cohomology group defined above (see [Hid88], [Ter03], [Cia09]). A simplified version of the problem of Ihara would be to establish the torsion-free property of the cokernel of the map α restricted to non-Eisenstein components of the cohomology. Thus we formulate the following weaker conjecture:

Conjecture 1.1. *Let q be a prime number such that $q \not| N\Delta\ell$. We fix a maximal non Eisenstein ideal $\mathfrak{m}$ of the Hecke algebra $\mathbf{T}^{\widehat{\psi}}(N)$ acting on the group $H^1(\mathbf{X}_1(N), \mathcal{O})^{\widehat{\psi}}$. Let $\mathbf{T}^{\widehat{\psi}}(N,q)$ be the Hecke algebra acting on $H^1(\mathbf{X}_1(N,q), \mathcal{O})^{\widehat{\psi}}$, let $\mathfrak{m}_q$ be the inverse image of $\mathfrak{m}$ under the natural map $\mathbf{T}^{\widehat{\psi}}(N,q) \to \mathbf{T}^{\widehat{\psi}}(N)$. The map*

$$\alpha_{\mathfrak{m}} : H^1(\mathbf{X}_1(N), \mathcal{O})^{\widehat{\psi}}_{\mathfrak{m}} \times H^1(\mathbf{X}_1(N), \mathcal{O})^{\widehat{\psi}}_{\mathfrak{m}} \to H^1(\mathbf{X}_1(N,q), \mathcal{O})^{\widehat{\psi}}_{\mathfrak{m}_q}$$

has a torsion free cokernel.

It easily follows from the Hochschild-Serre spectral sequence that

$$H^1(\mathbf{X}_1(N,S),\mathcal{O})^{\widehat{\psi}} \simeq H^1(\mathbf{X}_0(NS),\mathcal{O}(\widehat{\psi}))$$

where $\mathcal{O}(\widehat{\psi})$ is the sheaf $B^\times \setminus B^\times_{\mathbf{A}} \times \mathcal{O}/K^+_\infty \times V_0(NS)$, $B^\times$ acts on $B^\times_{\mathbf{A}} \times \mathcal{O}$ on the left by $\alpha \cdot (g,m) = (\alpha g, m)$ and $K^+_\infty \times V_0(NS)$ acts on the right by $(g,m)\cdot v = (g,m)\cdot(v_\infty, v^\infty) = (gv, \widehat{\psi}(v^\infty)m)$ where v_∞ and v^∞ are respectively the infinite and finite part of v.

We define $\Phi_0(N) = (GL_2^+(\mathbf{R}) \times V_0(N)) \cap B^\times_{\mathbf{Q}}$, and for $i = 0,1$, we define $\Phi_i(N,S) = (GL_2^+(\mathbf{R}) \times V_i(N,S)) \cap B^\times_{\mathbf{Q}}$. We observe that $\Phi_0(N,S) = \Phi_0(NS)$. By the isomorphism i_∞, the group $\Phi_i(N,S)$ can be considered as a discrete subgroup having finite covolume in $SL_2(\mathbf{R})$ [Vig80]. Since we suppose $\Delta \neq 1$, then B is a division algebra, $SL_2(\mathbf{R})/\Phi_i(N,S)$ is compact and $\Phi_i(N,S) \setminus SL_2(\mathbf{R})/SO_2(\mathbf{R})$ is a compact Riemann surface.

By simplicity, we shall assume that the group $\Phi_0(NS)$ does not contain elliptic elements. If not, there is a standard technique to add a prime s in the level such that $\Phi_0(NS)$ has not elliptic elements and the local Hecke component of the cohomology at level NSs is the same that at level NS.

By translating to the cohomology of groups we obtain (see [Hid93], Appendix) $H^1(\mathbf{X}_1(N,S),\mathcal{O})^{\widehat{\psi}} \simeq H^1(\Phi_0(NS),\mathcal{O}(\widetilde{\psi}))$, where $\widetilde{\psi}$ is the restriction of $\widehat{\psi}$ to $\Phi_0(NS)/\Phi_1(N,S)$ and $\mathcal{O}(\widetilde{\psi})$ is $\mathcal{O}$ with the action of $\Phi_0(NS)$ given by $a \mapsto \widetilde{\psi}^{-1}(\gamma)a$. By this analysis, we can translate the map α in cohomology of groups. By strong approximation, write $\eta_q = \delta_q g_\infty u$, with $\delta_q \in B^\times$, $g_\infty \in GL_2^+(\mathbf{R})$ and $u \in V_0(Nq)$. Consider first the map

$$\begin{array}{rcl} |_{\eta_q} : H^1(\Phi_0(N),\mathcal{O}(\widetilde{\psi})) & \to & H^1(\Phi_0(N,q),\mathcal{O}(\widetilde{\psi})) \\ x & \longmapsto & x|_{\eta_q} \end{array}$$

working as follows: let ξ be a cocycle representing the cohomology class x in $H^1(\Phi_0(N),\mathcal{O}(\widetilde{\psi}))$; then $x|_{\eta_q}$ is represented by the cocycle $\xi'(\gamma) = \widehat{\psi}^{-1}(u)\xi(\delta_q\gamma\delta_q^{-1})$. It is easy to see that this definition does not depend on the choice of the approximation. Define now:

$$\begin{array}{rcl} \alpha : H^1(\Phi_0(N),\mathcal{O}(\widetilde{\psi})) \times H^1(\Phi_0(N),\mathcal{O}(\widetilde{\psi})) & \to & H^1(\Phi_0(N,q),\mathcal{O}(\widetilde{\psi})) \\ (x,y) & \longmapsto & \mathrm{Res}(x) + y|_{\eta_q} \end{array}$$

where Res is the restriction map $\mathrm{Res}_{\Phi_0(N)/\Phi_0(N,q)}$ from $\Phi_0(N)$ to $\Phi_0(N,q)$.

We observe that by Nakayama's lemma, Conjecture 1.1 is equivalent to prove that the map

$$\alpha_{\mathfrak{m}} \otimes k : H^1(\Phi_0(N),k(\widetilde{\psi}))_{\mathfrak{m}} \times H^1(\Phi_0(N),k(\widetilde{\psi}))_{\mathfrak{m}} \to H^1(\Phi_0(N,q),k(\widetilde{\psi}))_{\mathfrak{m}_q}$$

is injective.

2. The literature about the problem of Ihara

Besides the intrinsic interest of Ihara's problem, it turned out to be an essential tool to solve problems of raising the level of modular forms. For this reason, it has been generalized in many ways, and the literature about this problem is very rich. In this section we will present the most relevant results about this subject.

2.1. The modular case

Let us consider a quaternion algebra B with discriminant $\Delta = 1$. Under this assumption it is known that $B \cong M_2(\mathbf{Q})$. Let N be a positive integer number and let q be a prime number $q \nmid N$. We consider the modular curve $X(N)$ (resp. $X_1(N)$) associated to the congruence subgroup $\Gamma(N)$ (resp. $\Gamma_1(N)$) and let $X(N,q)$ (resp. $X_1(N,q)$) be the modular curve associated to $\Gamma(N) \cap \Gamma_0(q)$ (resp. $\Gamma_1(N) \cap \Gamma_0(q)$) [Shi71]. In 1983, Ribet [Rib83b], observed that for all prime numbers ℓ, the problem of showing that the map

$$\alpha \otimes_{\mathcal{O}} k : H^1(X(N), \mathbf{Z}/\ell\mathbf{Z}) \oplus H^1(X(N), \mathbf{Z}/\ell\mathbf{Z}) \to H^1(X(N,q), \mathbf{Z}/\ell\mathbf{Z})$$

is injective is just a special case of Lemma 3.2 of Ihara [Iha73]. For this reason we are referring to this kind of results as the "problem of Ihara". In [Rib83b] Ribet gives a more direct proof of the injectivity of $\alpha \otimes_{\mathcal{O}} k$ working with the parabolic cohomology groups . The key of his proof is the knowledge of the amalgamamated product

$$\Gamma = \Gamma(N) *_{\Gamma(N,q)} \left(\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma(N) \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \right)$$

and the injectivity of $\alpha \otimes_{\mathcal{O}} k$ can be deduced (via the exact sequence of Lyndon [Ser80]) by the vanishing of the parabolic cohomology group of Γ with coefficients in $\mathbf{F}_\ell$.

In [CDT99] Conrad-Diamond and Taylor prove the problem of Ihara related to a map $\alpha_{\mathfrak{m}}$ obtained by considering the localization of the weight n cohomology to a non-Eisenstein maximal ideal of the Hecke algebra acting on it. They require that the prime q does not divide $N\ell$. Also in this case, the key of the proof is the knowledge of the amalgamated product Γ and its congruence subgroup property [Ser70].

In [Dia91], Diamond generalizes the work of Ribet to an arbitrary weight n cohomology, under the hypothesis that $q \nmid N(n-2)!$. Also in this case the argument is the vanishing of the parabolic cohomology group of the amalgama Γ.

2.2. The quaternionic case

We now consider an indefinite quaternion algebra B over $\mathbf{Q}$ of discriminant $\Delta > 1$. The group-theoretic approaches working for elliptic modular curves seem to be inadequate for Shimura curves. In particular, the congruence subgroup property for q-arithmetic subgroups of $B^\times$ is conjectured, but unfortunately not known.

In [DT94], Diamond and Taylor prove a problem of Ihara for Shimura curves in a special non-Eisenstein case, using a Galois theoretic argument. They make the following hypotheses:

1. the prime ℓ does not divide the discriminant Δ of B;
2. they work with Shimura curves arising from an open compact subgroup $U \subset B_{\mathbf{A}}^{\times}$ sufficiently small: it must be contained in the group

$$V(N) := \prod_{p \nmid N} R_p^{\times} \times \prod_{p \mid N} K_p^1(N)$$

for N prime to Δ.

We observe that hypotheses 1. and 2., assure that the Shimura curve associated to such a group U, have good reduction at ℓ, so they give a new argument for results about the problem of Ihara, which uses the arithmetic geometry of the curves.

We observe that the reason for which the approach of Diamond-Taylor fails in our case, is that under our hypotheses, the Shimura curves $\mathbf{X}_i(N,S)$ do not have good reduction at ℓ (see for example [Car86]).

3. A reformulation in terms of the subgroup congruence property

If G is a subgroup of $B_{\mathbf{A}}^{\times}$, we denote by $G^{(1)}$ the subgroup of reduced norm 1 elements of G.

For $p \neq \ell$ we denote by T_p the Hecke operator at p acting on $H^1(\Phi_0(N), \mathcal{O}(\tilde{\psi}))$ and $H^1(\Phi_0(N), k(\tilde{\psi}))$ and by U_p the Hecke operator at p acting on $H^1(\Phi_0(N), \mathcal{O}(\tilde{\psi}))$ and $H^1(\Phi_0(Np), k(\tilde{\psi}))$, see for example [Hid88]. We observe that for p not dividing $N\Delta$

$$T_p(x) = \mathrm{Cor}_{\Phi_0(N)/\Phi_0(Np)}(x|_{\eta_p}) \tag{2}$$

The following properties of Hecke operators are easily proved by straightforward calculations:

Proposition 3.1. *Let p, q be primes not dividing $N\Delta\ell$. Then for every $x \in H^1(\Phi_0(N), \mathcal{O}(\widetilde{\psi}))$ the following equalities hold:*

$$T_p(x|_{\eta_q}) = T_p(x)|_{\eta_q} \qquad \text{if } p \neq q,$$

$$T_p(\mathrm{Res}_{\Phi_0(N)/\Phi_0(Nq)}(x)) = \mathrm{Res}_{\Phi_0(N)/\Phi_0(Nq)}(T_p(x)) \qquad \text{if } p \neq q,$$

$$U_p(x|_{\eta_p}) = p\psi(p)\mathrm{Res}_{\Phi_0(N)/\Phi_0(Np)}(x), \tag{3}$$

$$U_p(\mathrm{Res}_{\Phi_0(N)/\Phi_0(Np)}(x)) = \mathrm{Res}_{\Phi_0(N)/\Phi_0(Np)}(T_p(x)) - x|_{\eta_p}. \tag{4}$$

Proposition 3.2. *Let p be a prime not dividing N, and $x_1, x_2 \in H^1(\Phi_0(N), k(\tilde{\psi}))$. Suppose that*

$$Res_{\Phi_0(N)/\Phi_0(Np)}(x_1) = x_2|_{\eta_p}. \tag{5}$$

Then

(a) $T_p(x_2) = (p+1)x_1$.
(b) $T_p(x_1) = \psi(p)(p+1)x_2$.
(c) *The operator* $T_p^2 - \psi(p)(p+1)^2$ *annihilates* x_1 *and* x_2.

Proof. We obtain (a) directly applying corestriction to (5). Applying U_p and then the corestriction to (5), we get $pT_p(x_1) = p\psi(p)(p+1)x_2$ and thus (b), since $p \neq \ell$. Claim (c) is an immediate consequence of (a) and (b). ■

Corollary 3.1. *Let M be an integer and suppose that (5) holds for every prime p not dividing $MN\Delta$. Then the annihilator of x_1 (and x_2) is contained in an Eisenstein ideal.*

If U is an open compact subgroup of $B_{\mathbf{A}}^{\times,\infty}$, we put $\Phi(U) = GL_2^+(\mathbf{R})U \cap B^\times$. Let M be an integer and define the compact open subgroup of $B_{\mathbf{A}}^{\times,\infty}$

$$U(M) = \prod_{p \nmid M} R_p^\times \times \prod_{p \mid M} (1 + MR_p).$$

Definition 3.1. *A subgroup Ψ of $\Phi_0(M)$ is a congruence subgroup if it contains $\Phi(U(M))$ for some integer M.*

Proposition 3.3. *Let Ψ be a congruence subgroup of $\Phi_0(N)$. Then*

(a) *the kernel of the restriction*

$$\mathrm{Res}_{\Phi_0(N)/\Psi} : H_1(\Phi_0(N), k(\tilde{\psi})) \to H_1(\Psi, k(\tilde{\psi}))$$

is stable for Hecke operators T_p, for almost every prime p.

(b) *let $\mathfrak{m}$ be a non Eisenstein maximal ideal of $\mathbf{T}^{\hat{\psi}}(N)$. Then the restriction*

$$\mathrm{Res}_{\Phi_0(N)/\Psi} : H^1(\Phi_0(N), k(\tilde{\psi}))_{\mathfrak{m}} \longrightarrow H^1(\Psi, k(\tilde{\psi}))$$

is injective.

Proof.

(a) Suppose that $\Psi \supseteq \Phi(U(M))$ and let p be a prime not dividing $N\ell M$. By strong approximation we can decompose $\eta_p = \delta_p g_\infty u$ with $\delta_p \in B^\times$, $g_\infty \in \mathrm{GL}_2^+(\mathbf{R})$ and $u \in V_0(Np)$. Observe that since $p \nmid M$, Ψ is dense in $\Phi_0(N)$ with respect to the p-adic topology, so that the map $\Psi \to \Phi_0(N)/\Phi_0(Np)$ is surjective. Then we can choose in Ψ a set of representatives $\{h_1, ..., h_s\}$ of right cosets of $\Phi_0(Np)$ in $\Phi_0(N)$, and we get a decomposition of double cosets $\Phi_0(N)\delta_p\Phi_0(N) = \coprod_i \Phi_0(N)\delta_p h_i$ and $\Psi\delta_p\Psi = \coprod_i \Psi\delta_p h_i$. Let ξ be a cocycle representing a cohomology class in $\mathrm{H}^1(\Phi_0(N), k(\tilde{\psi}))_{\mathfrak{m}}$ which restricts to zero on Ψ, and let $\gamma \in \Psi$. Write $\delta_p h_i \gamma = \gamma_i \delta_p h_{j(i)}$ with $\gamma_i \in \Psi$. By definition $(T_p\xi)(\gamma) = \sum_i \psi(\delta_p h_i)\xi(\gamma_i) = 0$.

(b) We can suppose that $\Psi = \Phi(U(M))$ with $N\ell$ dividing M, so that $\tilde{\psi}$ is trivial over Ψ. Let ξ be a cocycle representing a cohomology class in $\mathrm{H}^1(\Phi_0(N), k(\tilde{\psi}))_{\mathfrak{m}}$ which restricts to zero on Ψ and let $\gamma \in \Phi_0(Np)$. Since Ψ is normal in $\Phi_0(N)$, we can write $h_i\gamma = \gamma h'_i h_{j(i)}$, with $h'_i \in \Psi \cap \Phi_0(Np)$; then $\delta_p h'_i \delta_p^{-1} \in \Psi$ and $(T_p\xi)(\gamma) = \psi(\delta_p)\sum_i \xi(\delta_p \gamma h'_i \delta_p^{-1}) = \psi(\delta_p)\sum_i \xi(\delta_p\gamma\delta_p^{-1}) = (p+1)\xi|_{\eta_p}(\gamma)$. Then we get $\mathrm{Res}_{\Phi_0(N)/\Phi_0(Np)}(T_p\xi) = (p+1)\xi|_{\eta_p}$. By Proposition 3.2b) $T_p^2 - \psi(p)(p+1)^2$ annihilates ξ, so it must belong to $\mathfrak{m}$ unless $\xi = 0$. But $\mathfrak{m}$ is non Eisenstein, so it cannot contain $T_p^2 - \psi(p)(p+1)^2$ for almost every p. Therefore ξ must be zero. ∎

Let us consider the amalgamated product:

$$\Phi := \Phi_0(N) *_{\Phi_0(Nq)} \delta_q^{-1}\Phi_0(N)\delta_q. \tag{6}$$

If G is a subgroup of B, we shall denote by $G^{(1)}$ the subgroup of elements of reduced norm 1. The following proposition describes the structure of Φ:

Proposition 3.4. *Let $R(N)$ be an Eichler order of level N in B and let q be a prime not dividing N. Then $\Phi = \left(R(N) \otimes_{\mathbf{Z}} \mathbf{Z}\left[\frac{1}{q}\right]\right)^{(1)}$.*

Proof. If q is a prime number not dividing ΔN then $R_q(N)^{(1)} \cong SL_2(\mathbf{Z}_q)$ and, by [Ser80] p.79

$$R_q(N)^{(1)} *_{R_q(Nq)^{(1)}} \delta_q^{-1} R_q(N)^{(1)}\delta_q \cong B_q^{(1)}.$$

Since $\Phi_0(N) \cap R_q(Nq)^{(1)} = \delta_q^{-1}\Phi_0(N)\delta_q \cap R_q(Nq)^{(1)} = \Phi_0(Nq)$, by Proposition 3. §1.3 of [Ser80], there is an injection $\Phi \hookrightarrow B_q^{(1)}$. The universal property of the amalgamated product then assures that there is an inclusion $\Phi \subseteq \left(R(N) \otimes_{\mathbf{Z}} \mathbf{Z}\left[\frac{1}{q}\right]\right)^{(1)}$. By strong approximation, $\Phi_0(N)$ is dense in $R_q(N)^{(1)}$ with respect to the q-adic topology, so that Φ is dense in $B_q^{(1)}$, and therefore in $\left(R(N) \otimes_{\mathbf{Z}} \mathbf{Z}\left[\frac{1}{q}\right]\right)^{(1)}$. But $\Phi_0(N) = \mathbf{SL}_2(\mathbf{Z}_q) \cap \left(R(N) \otimes_{\mathbf{Z}} \mathbf{Z}\left[\frac{1}{q}\right]\right)^{(1)}$, so that Φ is also open (and thus closed) in $\left(R(N) \otimes_{\mathbf{Z}} \mathbf{Z}\left[\frac{1}{q}\right]\right)^{(1)}$. This implies $\Phi = \left(R(N) \otimes_{\mathbf{Z}} \mathbf{Z}\left[\frac{1}{q}\right]\right)^{(1)}$. ∎

If the amalgama Φ has the subgroup congruence property, then the same approach employed by Ribet and Diamond in the classical case is applicable to the quaternionic case:

Theorem 3.1. *Assume that Φ has the subgroup congruence property. Then Conjecture 1.1 is true.*

Proof. It suffices to show that if (x, y) is in the kernel of $\alpha_{\mathfrak{m}}$ then x restricts to zero in some congruence subgroup of $\Phi_0(N)$. If this is the case, we would have $y|\eta_q = 0$, and applying the operator U_p this yields $\mathrm{Res}_{\Phi_0(N)/\Phi_0(Nq)}(y) = 0$, and this implies $x = 0, y = 0$ by Proposition 3.3.b). We can consider the the Lyndon exact sequence

$$H^1(\Phi, k(\tilde{\psi})) \longrightarrow H^1(\Phi_0(N), k(\tilde{\psi}))^2 \xrightarrow{\alpha \otimes k} H^1(\Phi_0(Nq), k(\tilde{\psi}))$$

[Ser80, II,2.8]. Then (x, y) comes from an element $z \in H^1(\Phi, k(\tilde{\psi}))$. Choose a subgroup Ψ' of finite index in Φ such that z restricts to zero in Ψ'. By our assumption, Ψ' is a congruence subgroup, so that x restricts to zero in $\Psi = \Psi' \cap \Phi_0(N)$ which is a congruence subgroup of $\Phi_0(N)$. ■

Unfortunately the congruence subgroup property for q-arithmetic subgroups of $B^\times$ is conjectured, but not known. A complete reference about the present status of the congruence subgroup problem is [Rap99].

Let Σ be the set of places $\{q, \infty\}$ with q a prime number as in the previous sections; we observe that Σ does not contain any anisotropic nonarchimedean place for $B^{(1)}$. Let us consider on $B^{(1)}$ the Σ-arithmetic topology τ_a (generated by all subgroups of finite index in Φ) and τ_c the Σ-congruence topology (generated by the Σ-congruence subgroups in Φ) (see, for example [Ser67]). We consider the natural continuous homomorphism $\pi : (\overline{B^{(1)}}, \tau_a) \to (\overline{B^{(1)}}, \tau_c)$ where $(\overline{B^{(1)}}, \tau_a)$ and $(\overline{B^{(1)}}, \tau_c)$ are the completions of $B^{(1)}$ with respect to τ_a and τ_c. The kernel of π is the so called Σ-congruence kernel, denoted $C(\Sigma)$ and its size measures deviation from the affirmative solution of the congruence subgroup problem.

Since the normal subgroup structure of $B^{(1)}$ is given by the Platonov-Margulis conjecture (as proved in [Seg99]), by Theorem 1 of [Rap99], if $C(\Sigma)$ is central then Φ has the congruence subgroup property. Thus, Conjecture 1.1 would be a consequence of the following:

Conjecture 3.1. *The Σ-congruence kernel $C(\Sigma)$ is central.*

As pointed out in [Rap99], this last open problem still does not have a uniform approach.

By Theorem 5 of [Rap99], if Φ has the bounded generation (BG) (i.e. there are elements $\gamma_1, ..., \gamma_t \in \Phi$ such that $\Phi = \langle\gamma_1\rangle ... \langle\gamma_t\rangle$ where $\langle\gamma_i\rangle$ is the cyclic subgroup generated by γ_i), then the Σ-congruent kernel is central. Unfortunately (BG) is not established for Φ but only conjectured [Rap99].

Our optimism is renforced by the fact that the problem of showing sufficient conditions for the centrality of the Σ-congruence kernel for general algebraic groups, is the subject of research of many mathematicians, see for example [PR08], [PR96], [PR83].

4. Some consequences of the conjecture

By Jacquet-Langlands correspondence and Matsushima-Shimura isomorphism [MS63], some modular forms can be reinterpreted as elements of the cohomology of Shimura curves (for details see [MT99], [Ter03]). Therefore one can use quaternionic cohomological modules in the construction of Taylor-Wiles systems for certain deformation problems. This has be done for example in [Ter03], [Cia09].

In this section we will describe two interesting consequences of Conjecture 1.1.

Let S be a finite set of rational primes which does not divide $N\Delta$. Keeping the same notations as in the previous sections, let $S_2(V_1(N,S))$ be the space of quaternionic automorphic forms right invariant by $V_1(N,S)$ [Hid88]; let $S_2(V_0(NS),\widehat{\psi})$ be the subspace of $S_2(V_1(N,S))$ consisting of the φ such that

$$\varphi(gk)=\widehat{\psi}(k)\varphi(g) \qquad \text{for any} \quad k\in V_0(NS), \quad g\in B_{\mathbf{A}}^{\times}.$$

The Jacquet-Langlands correspondence establishes an isomorphism

$$JL: S_2(V_0(NS),\widehat{\psi})\tilde{\to} V_{\widehat{\psi}}$$

where $V_{\widehat{\psi}}$ is a direct summand of $S_2(\Gamma_0(S\Delta')\cap\Gamma_1(N\ell^2))$ consisting of classical modular forms of level 2, nebentypus ψ, special at primes dividing Δ' and supercuspidal of type $\tau=\chi\oplus\chi^{\sigma}$ at ℓ. By the Matsushima-Shimura isomorphism, one can see such forms in the ℓ-adic cohomology of a Shimura curve, as we shall show in details in the following.

4.1. The Hecke structure of the cohomology of Shimura curves

Let us fix $f\in S(\Gamma_0(N\Delta'\ell),\psi)$ a modular newform of weight 2, level $N\Delta'\ell$, supercuspidal of type τ at ℓ, special at primes dividing Δ' and with Nebentypus ψ of order prime to ℓ. Let ρ be the Galois representation associated to f and let $\overline{\rho}$ be its reduction modulo ℓ; we impose the following conditions on $\overline{\rho}$:

$$\overline{\rho} \text{ is absolutely irreducible;} \tag{7}$$

$$\text{if } p|N \text{ then } \overline{\rho}(I_p)\neq 1; \tag{8}$$

$$\text{if } p|\Delta' \text{ and } p^2\equiv 1 \bmod \ell \text{ then } \overline{\rho}(I_p)\neq 1; \tag{9}$$

$$\mathrm{End}_{\overline{\mathbf{F}}_\ell[G_\ell]}(\overline{\rho}_\ell)=\overline{\mathbf{F}}_\ell. \tag{10}$$

$$\text{if } \ell=3,\ \overline{\rho} \text{ is not induced from a character of } \mathbf{Q}(\sqrt{-3}). \tag{11}$$

We denote by Δ_1 the product of primes $p|\Delta'$ such that $\overline{\rho}(I_p)\neq 1$ and by Δ_2 the product of primes $p|\Delta'$ such that $\overline{\rho}(I_p)=1$ so that we are assuming that $p^2\not\equiv 1 \bmod \ell$ if $p|\Delta_2$. We say that the representation ρ is of **type** $(S,\mathrm{sp},\tau,\psi)$ if ρ is a deformation of $\bar{\rho}$ and the following conditions hold:

(a) ρ is unramified outside $N\Delta S$.
(b) if $p|\Delta_1 N$ then $\rho(I_p)\simeq\overline{\rho}(I_p)$;
(c) if $p|\Delta_2$ then ρ_p satisfies the condition $\mathrm{tr}(\rho(F))^2=\psi(p)(p+1)^2$ for a lift F of Frob_p in G_p;
(d) ρ_ℓ is weakly of type τ (see [CDT99]);
(e) $\det(\rho)=\epsilon\psi$, where $\epsilon: G_{\mathbf{Q}}\to\mathbf{Z}_\ell^{\times}$ is the cyclotomic character.

As proved in [Ter03] or [Cia09], this is a deformation condition. Let $\mathcal{R}_S$ be the universal deformation ring parametrizing representations of type $(S, \mathrm{sp}, \tau, \psi)$ with residual representation $\overline{\rho}$.

We say that a modular eigenform is of type (S, sp, τ, ψ) if the Galois representation associated ρ_f is of type (S, sp, τ, ψ).

Let $\mathcal{B}_S$ denote the set of newforms h of level dividing $N\Delta' S\ell^2$ which are of type (S, sp, τ, ψ) and for each $h \in \mathcal{B}_S$ let $\mathcal{O}_h$ be the extension of $\mathcal{O}$ generated by the Hecke eigenvalues of h. Let $\mathbf{T}_S^{\psi}$ be the Hecke algebra acting on the space of modular forms of type $(S, \mathrm{sp}, \tau, \psi)$. Let $\mathbf{T}_S$ denote the sub-$\mathcal{O}$- algebra of $\prod_{h \in \mathcal{B}_S} \mathcal{O}_h$ generated by the elements $T_p = (a_p(h))_{h \in \mathcal{B}_S}$ for $p \not| N\Delta' S\ell$. Similarly to [Ter03, Proposition 3.3], we can see that there is a surjective homomorphism $\mathcal{R}_S \to \mathbf{T}_S$. By combining the Jacquet-Langlands correspondence with the discussion in Section 4.2 of [DDT], we know that $\mathbf{T}_S$ is isomorphic to the localization in a maximal ideal $\mathfrak{m}_S$ of the full Hecke algebra $\mathbf{T}^{\hat{\psi}}(N, S)$ acting on $H^1(\mathbf{X}_1(N, S), \mathcal{O})^{\hat{\psi}}$, and that $\mathfrak{m}_S$ lies over $\mathfrak{m} = \mathfrak{m}_{\emptyset}$. We define $\mathcal{M}_S = H^1(\mathbf{X}_1(N, S), \mathcal{O})^{\hat{\psi}}_{\mathfrak{m}_S}$. As explained in Section 4.2 each $\mathcal{M}_S$ is endowed by a $\mathbf{T}_S$-bi-linear, $\mathcal{O}$-perfect pairing $\langle \cdot, \cdot \rangle_S$.

For simplicity, the following application of our main conjecture is stated for sets S of primes not congruent to ± 1 modulo ℓ:

Theorem 4.1. *Suppose that Conjecture 1.1 is true. Let S be a set of primes q not dividing $N\Delta$ such that $q^2 \not\equiv 1 \bmod \ell$. Then*

1. *$\mathcal{R}_S \to \mathbf{T}_S$ is an isomorphism of complete intersection rings;*
2. *$\mathcal{M}_S$ is a free $\mathbf{T}_S$-module.*

The proof relies over a theorem of Diamond [Dia97, Theorem 2.4], that we briefly illustrate below.

We assume that $\mathcal{O}$ is a complete discrete valuation ring with uniformizer λ and residue field k. Fix complete local noetherian $\mathcal{O}$-algebras R, R' with $\mathcal{O}$-algebra homomorphisms $\varphi : R' \to R$, $\pi : R \to \mathcal{O}$; define $\pi' = \pi \circ \varphi$.

Let M (resp. M') be an R (resp. R') module, finitely generated and free over $\mathcal{O}$; we can view M as an R'-module via φ. Suppose that there exists an injective R'-homomorphism $\alpha : M \to M'$ with cokernel torsion free. Define $T = R/Ann_R(M), T' = R'/Ann_{R'}(M')$; then φ induces a surjective map $T' \to T$.

Suppose that $\mathfrak{p} = \mathrm{Ker}(\pi)$ is in the support of M and let $\mathfrak{p}' = \varphi^{-1}(\mathfrak{p})$. Let I (resp. I') be the annihilator of $\mathfrak{p}$ (resp. $\mathfrak{p}'$) in T (resp. T'). Then α induces an isomorphism $M[\mathfrak{p}] \simeq M'[\mathfrak{p}']$, which are both free $\mathcal{O}$-modules of finite rank d. Define

$$\Omega = \frac{M}{M[\mathfrak{p}] + M[I]}$$

$$\Omega' = \frac{M'}{M'[\mathfrak{p}'] + M'[I']}$$

According to [Dia97, Theorem 2.4], if Ω has finite lenght over $\mathcal{O}$, then the following statements are equivalent:

(i) $rk_{\mathcal{O}}(M) \leqslant d \cdot rk_{\mathcal{O}}(T)$ and $length_{\mathcal{O}}(\Omega) \geqslant d \cdot \mathrm{lenght}_{\mathcal{O}}(\mathfrak{p}/\mathfrak{p}^2)$;
(ii) R is a complete intersection and M is free (of rank d) over R.

Diamond theorem allows, in some particular case, to deduce property (ii) for M', R' once it is proved for M, R. We briefly explain this technique in the following.

Suppose moreover that M (resp. M') is equipped with a T (resp. T') bi-linear, $\mathcal{O}$-perfect pairing $\langle \cdot, \cdot \rangle$ (resp. $\langle \cdot, \cdot \rangle'$) and let $\beta : M' \to M$ be the transpose of α w.r.t. these pairings. It is easy to see that Ω (resp. Ω') is isomorphic to the cokernel of the map $M[\mathfrak{p}] \to \mathrm{Hom}(M[\mathfrak{p}], \mathcal{O})$ (resp. $M'[\mathfrak{p}'] \to \mathrm{Hom}(M'[\mathfrak{p}'], \mathcal{O})$, so that Ω and Ω' have finite lenght over $\mathcal{O}$. Suppose that R is complete intersection and M is free over R. Suppose moreover that

$$rk_{\mathcal{O}}(\Omega') - rk_{\mathcal{O}}(\Omega) \geqslant d \cdot (\mathrm{lenght}_{\mathcal{O}}(\mathfrak{p}'/\mathfrak{p}'^2) - \mathrm{lenght}_{\mathcal{O}}(\mathfrak{p}/\mathfrak{p}^2)) \tag{12}$$

Then by Diamond theorem we can conclude that R' is complete intersection and M' is free over R'.

If $x_1, ..., x_d$ is a $\mathcal{O}$-basis for $M[\mathfrak{p}]$ it is easy to see that

$$rk_{\mathcal{O}}(\Omega) = v_\lambda \det(\langle x_i, x_j \rangle)$$

and the same holds for Ω' and any basis $x'_1, ..., x'_d$ of $M'[\mathfrak{p}']$. Since $\alpha(x_1), ..., \alpha(x_d)$ is a basis of $M'[\mathfrak{p}']$, then:

$$rk_{\mathcal{O}}(\Omega') = v_\lambda (\det(\langle \alpha(x_i), \alpha(x_j) \rangle))$$

If the composition $\beta \circ \alpha$ over $M[\mathfrak{p}]$ is the multiplication by an element c of $\mathcal{O}$, we have:

$$\begin{aligned} rk_{\mathcal{O}}(\Omega') &= v_\lambda (\det(\langle x_i, \beta \circ \alpha(x_j) \rangle)) \\ &= v_\lambda \left(\det(\langle x_i, x_j \rangle) c^d\right) = d \cdot v_\lambda(c) + rk_{\mathcal{O}}(\Omega) \end{aligned}$$

On the other hand $\mathrm{Ker}(\varphi) \subseteq \mathfrak{p}'$ and its image K in $\mathfrak{p}'/\mathfrak{p}'^2$ is an $\mathcal{O}$-module which generates the kernel of the map $\mathfrak{p}'/\mathfrak{p}'^2 \to \mathfrak{p}/\mathfrak{p}^2$ induced by φ. Therefore

$$\mathrm{lenght}_{\mathcal{O}}(\mathfrak{p}'/\mathfrak{p}'^2) - \mathrm{lenght}_{\mathcal{O}}(\mathfrak{p}/\mathfrak{p}^2) = \mathrm{lenght}_{\mathcal{O}}(K),$$

and (12) will be proved if we can show that

$$v_\lambda(c) \geqslant \mathrm{lenght}_{\mathcal{O}}(K). \tag{13}$$

4.2. A duality result on the cohomology of Shimura curves

In this section we prove the existence of a bilinear perfect Hecke-equivariant pairing $\langle \cdot, \cdot \rangle_S$ on $H^1(\mathbf{X}_0(N, S), \mathcal{O}(\widehat{\psi}))$; it induces an isomorphism $\mathcal{M}_S \to \mathrm{Hom}_{\mathcal{O}}(\mathcal{M}_S, \mathcal{O})$ as Hecke modules; this is one of the ingredients of the proof of Theorem 4.1 illustrated above.

The $\mathcal{O}-$linear map $b : \mathcal{O}(\widehat{\psi}) \otimes_{\mathcal{O}} \mathcal{O}(\widehat{\psi}^{-1}) \to \mathcal{O}$ allows to consider the cup product:

$$b_S^\times : H^1(\mathbf{X}_0(N,S), \mathcal{O}(\widehat{\psi})) \otimes_{\mathcal{O}} H^1(\mathbf{X}_0(N,S), \mathcal{O}(\widehat{\psi}^{-1})) \xrightarrow{\cup} H^2(\mathbf{X}_0(N,S), \mathcal{O})) \simeq \mathcal{O}$$

Proposition 4.1. *The pairing $b_S^\times$ is perfect.*

Proof. It suffices to show that the cup product

$$H^1(\mathbf{X}_0(N,S), k(\widehat{\psi})) \otimes_{\mathcal{O}} H^1(\mathbf{X}_0(N,S), k(\widehat{\psi}^{-1})) \to k$$

is non-singular.

Under our hypothesis $\Delta \neq 1$, the group $\Phi_0(N,S)$ is co-compact in $SL_2(\mathbf{R})$ (see for example [Vig80]); then by Poincaré duality

$$H^1(\mathbf{X}_0(N,S), k(\widehat{\psi})) \simeq H_1(\mathbf{X}_0(N,S), k(\widehat{\psi})).$$

This isomorphism identifies cup and cap products; the cap product is non-singular by Theorem 10.5, §V of [Bre97]. ■

Let ω_ℓ be an element in $R_\ell^\times$ such that $\omega_\ell^2 = \ell$ and $\chi(\omega_\ell \alpha \omega_\ell^{-1}) = \chi^{-1}(\alpha)$ for every $\alpha \in R_\ell^\times$. Let us consider the element ω_{NS} of $B_{\mathbf{A}}^{\times,\infty}$ defined as follows: $\omega_{NS,\nu} = 1$ if $\nu \not| NS\ell$, $\omega_{NS,\ell} = \omega_\ell$, $\omega_{NS,\nu} = i_\nu^{-1} \begin{pmatrix} 0 & -1 \\ NS & 0 \end{pmatrix}$ if $\nu | NS$.

By strong approximation, we can write

$$\omega_{NS} = \delta g_\infty u \tag{14}$$

with $\delta \in B^\times$, $g_\infty \in GL_2^+(\mathbf{R})$ and $u \in V_0(NS)$. We define a map

$$\theta_S : H^1(\Phi_0(NS), \mathcal{O}(\widetilde{\psi})) \to H^1(\Phi_0(NS), \mathcal{O}(\widetilde{\psi}^{-1}))$$

as follows: let ξ be a cocycle representing the cohomology class x in $H^1(\Phi_0(NS), \mathcal{O}(\widetilde{\psi}))$, then $\theta_S(\xi)$ is represented by the cocycle $\xi'(g) = \widehat{\psi}(u)\xi(\delta g \delta^{-1})$.

Observe that for all $\gamma \in \Phi_0(NS)$ the character satisfies: $\widetilde{\psi}(\delta\gamma\delta^{-1}) = \widetilde{\psi}(\gamma)^{-1}$. It is immediate to verify that the map θ_S is a well defined isomorphism of abelian groups.

Let $g_p \in B^\times$ be such that the double coset $\Phi_0(NS) g_p \Phi_0(NS)$ defines the Hecke operator T_p (see for example [Ter03]). Let us define on $H^1(\Phi_0(NS), \mathcal{O}(\widetilde{\psi}^{-1}))$, $\widetilde{T}_p$ as the operator defined by the double coset

$$\Phi_0(NS)\delta^{-1} g_p \delta \Phi_0(NS).$$

Notice that for a different decomposition $\omega_{NS} = \delta' g'_\infty u'$ we have $\delta^{-1}\delta' \in \Phi_0(NS)$, so that there is an equality of double cosets

$$\Phi_0(NS)\delta^{-1} g_p \delta \Phi_0(NS) = \Phi_0(NS)\delta'^{-1} g_p \delta' \Phi_0(NS).$$

Lemma 4.1. *The following equalities hold:*

(a) $\theta_S(T_p x) = \widetilde{T}_p(\theta_S(x))$ *for all* p *and* $x \in H^1(\Phi_0(NS), \mathcal{O}(\widetilde{\psi}))$;

(b) $b_S^\times \times (T_p x, y) = b_S^\times(x, \widetilde{T}_p y)$ *for all* $x \in H^1(\Phi_0(NS), \mathcal{O}(\widetilde{\psi}))$, $y \in H^1(\Phi_0(NS), \mathcal{O}(\widetilde{\psi}^{-1})$.

(c) *If* p *does not divide* NS *then* $\widetilde{T}_p = \psi(p) T_p$

Proof. (a) Decompose $\Phi_0(NS) g_p \Phi_0(NS) = \coprod_i \Phi_0(NS) h_i$, with $h_i \in B^\times$. Let ξ be a cocycle representing the cohomology class x in $H_1(\Phi_0(NS), \mathcal{O}(\widetilde{\psi}))$.

For $\gamma \in \Phi_0(NS)$, write

$$h_i \delta \gamma \delta^{-1} = \gamma_i h_{j(i)}; \tag{15}$$

so:

$$\begin{aligned}\theta_S(T_p(\xi))(\gamma) &\overset{\text{Def. of } \theta_S}{=} \widehat{\psi}(u)(T_p(\xi))(\delta\gamma\delta^{-1}) \\ &\overset{\text{Def. of } T_p}{=} \widehat{\psi}(u) \sum_i \psi(h_i)\xi(\gamma_i)\end{aligned}$$

where $\psi(h_i)$ is defined as follows: first we observe that the elements $h_i \in V_0(NS)_q$ for all place q, where $V_0(NS)_q$ is the local component at q of $V_0(NS)$. Then

$$\psi(h_i) = \prod_{\substack{q|N \\ q \nmid p\ell}} \psi_q(h_i^{-1}) \times \chi(h_i^{-1}).$$

Since $\delta^{-1}\Phi_0(NS)\delta = \Phi_0(NS)$, we have that $\Phi_0(NS)\delta = \delta\Phi_0(NS)$. Then:

$$\begin{aligned}\Phi_0(NS)\delta^{-1} g_p \delta \Phi_0(NS) &= \delta^{-1}\Phi_0(NS) g_p \Phi_0(NS)\delta \\ &= \coprod_i \delta^{-1}\Phi_0(NS) h_i \delta \\ &= \coprod_i \Phi_0(NS)\delta^{-1} h_i \delta.\end{aligned} \tag{16}$$

For $\gamma \in \Phi_0(NS)$ write

$$\delta^{-1} h_i \delta \gamma \overset{by (15)}{=} \delta^{-1} h_i \delta\gamma\delta^{-1}\delta = \delta^{-1}\gamma_i h_{j(i)}\delta = \delta^{-1}\gamma_i\delta\ \delta^{-1} h_{j(i)}\delta$$

$$\begin{aligned}\widetilde{T}_p(\theta_S(\xi))(\gamma) &= \sum_i \psi^{-1}(\delta^{-1}h_i\delta)\theta_S(\xi)(\delta^{-1}\gamma_i\delta) \\ &= \sum_i \psi(h_i)\widehat{\psi}(u)\xi(\delta\delta^{-1}\gamma_i\delta\delta^{-1}) \\ &= \sum_i \psi(h_i)\widehat{\psi}(u)\xi(\gamma_i)\end{aligned}$$

(b) Notice that the double coset defining the operator $\tilde{T}_p$ coincides with $\Phi_0(NS)g_p^\iota\Phi_0(NS)$ where $g_p^\iota = \nu(g_p)g_p^{-1}$. It is well known (see for example [Hid93, Chapter 6]) that the operator associated to the double coset $\Phi_0(NS)g_p^\iota\Phi_0(NS)$ is the adjoint of the operator associated to $\Phi_0(NS)g_p\Phi_0(NS)$ under the cup product.

(c) See for example [Shi71] Section 3.4.5. ∎

For $x, y \in H^1(\mathbf{X}_0(N,S), \mathcal{O}(\widehat{\psi}))$ we define

$$\langle x, y\rangle_S = b_S^\times(x, \theta_S(y))$$

Theorem 4.2. *The pairing $\langle .,. \rangle_S$ on $H^1(\mathbf{X}_0(N,S), \mathcal{O}(\widehat{\psi}))$ satisfies the following properties:*

1. *it is perfect;*
2. *it is Hecke-equivariant.*

Proof. 1. By proposition (4.1) the pairing $\langle .,. \rangle_S$ is perfect.
2. It is Hecke-equivariant; in fact for all $x, y \in H^1(\Phi_0(NS), \mathcal{O}(\widetilde{\psi}))$

$$\begin{aligned}\langle T_p x, y\rangle_S = b_S^\times(T_p x, \theta_S(y)) &\overset{Lemma 4.1,b)}{=} b_S^\times(x, \widetilde{T}_p(\theta_S(y))) \\ &\overset{Lemma 4.1,a)}{=} b_S^\times(x, \theta_S(T_p y)) = \langle x, T_p y\rangle_S.\end{aligned}$$ ∎

A straightforward calculation allows to establish the following facts:

Proposition 4.2. *Let $S' = S \cup \{q\}$ where q is a prime number not dividing $\Delta NS\ell$. Then for all $x \in H^1(\Phi_0(N,S), \mathcal{O}(\widetilde{\psi}))$*

(a) $\theta_{S'}(\mathrm{Res}_{\Phi_0(NS)/\Phi_0(NS')}(x)) = (\theta_S(x))|_{\eta_q}$.
(b) $\theta_{S'}(x|_{\eta_q}) = \mathrm{Res}_{\Phi_0(NS)/\Phi_0(NS')}\theta_S(x)$.

Lemma 4.2. *Let $S' = S \cup \{q\}$ where q is a prime number not dividing $\Delta NS\ell$. The following properties hold for all $x, y \in H^1(\Phi_0(NS), \mathcal{O}(\widetilde{\psi}))$:*

(a) $\langle \mathrm{Res}(x), \mathrm{Res}(y)\rangle_{S'} = \langle \psi^{-1}(q)T_q x, y\rangle_S$.
(b) $\langle \mathrm{Res}(x), y|_{\eta_q}\rangle_{S'} = \langle x|_{\eta_q}, \mathrm{Res}(y)\rangle_{S'} = (q+1)\langle x, y\rangle_S$.
(c) $\langle x|_{\eta_q}, y|_{\eta_q}\rangle_{S'} = \langle T_q x, y\rangle_S$.

Proof. The above equalities are easily deduced from definitions by recalling that restriction and corestriction are mutually adjoint with respect to cup product, and using the facts that $b_{S'}^\times(x|_{\eta_q}, y|_{\eta_q}) = (q+1)b_S^\times(x,y)$ and that $T_q(x) = \mathrm{Cor}(x|_{\eta_q})$ (see for example [Shi71]). ∎

4.3. Proof of Theorem 4.1

The proof is by induction on $|S|$. If S is empty, the result has been proven in [Cia09, Ter03] by the construction of a suitable Taylor-Wiles system. Then we assume that the claim is true for S and we deduce if for $S' = S \cup \{q\}$. Suppose firstly that the polynomial

$$X^2 - T_q X + q\psi(q) \tag{17}$$

has distinct roots $a, b \in k$ modulo $\mathfrak{m}_S$. Suppose moreover that $q^2 \not\equiv 1 \bmod \ell$. Then we can suppose that $a \neq qb$ in k. Since $\mathbf{T}_S$ is an Henselian ring, there exists a root $A \in \mathbf{T}_S$ such that A reduces to a modulo $\mathfrak{m}_S$. Define a map $\tilde{\alpha} : \mathcal{M}_S \to \mathcal{M}_{S'}$ by $\tilde{\alpha}(x) = \mathrm{Res}(Ax) - x|_{\eta_q}$. Then $\tilde{\alpha}$ is a map of $\mathbf{T}_{S'}$ modules and by Conjecture 1.1 it is injective with torsion-free cokernel. Let $\tilde{\beta}$ be the transpose of $\tilde{\alpha}$ w.r.t. the pairings $\langle \cdot, \cdot \rangle_S$ and $\langle \cdot, \cdot \rangle_{S'}$; by Lemma 4.2 we compute

$$\langle \tilde{\alpha}(x), \tilde{\alpha}(y) \rangle_{S'} = \langle \psi^{-1}(q) T_q A^2 x, y \rangle_S - 2(q+1)\langle Ax, y \rangle_S + \langle T_q x, y \rangle_S$$

so that (notice that A is a unit in $\mathbf{T}_S$) the composition $\tilde{\beta} \circ \tilde{\alpha}$ is the multiplication by

$$\begin{aligned} c &= \psi^{-1}(q) T_q A^2 - 2(q+1)A + T_q \\ &= \frac{\psi^{-1}(q)}{A}(A^2 - q\psi(q))(A^2 - \psi(q)). \end{aligned}$$

On the other hand, the form f determines maps $\pi_{S'} : \mathbf{T}_{S'} \to \mathcal{O}$ and $\pi_S : \mathbf{T}_S \to \mathcal{O}$ with kernels $\mathfrak{p}_{S'}$ and $\mathfrak{p}_S$ respectively. We want to compute

$$K = \mathrm{Ker}(\mathfrak{p}_{S'}/\mathfrak{p}_{S'}^2 \to \mathfrak{p}_S/\mathfrak{p}_S^2).$$

It is easy to show with the techniques employed in [Cia09, Ter03] that the versal local deformation ring of $\bar{\rho}_q$ without ramification conditions concides with the versal ring with a ramification conditions unless the congruence

$$a_q(f)^2 \equiv \psi(q)(q+1)^2 \bmod \lambda \tag{18}$$

is satisfied. Therefore, K is zero if (18) is not satified. Suppose now that (18) is satisfied. In this case the versal ring without ramification conditions is isomorphic to $\mathcal{O}[[X, Y]]/(XY)$, where X is equal, up to units, to $Trace(\rho(Frob_q))^2 - \psi(q)(q+1)^2$; by imposing the non-ramification condition Y annihilates and the corresponding versal becomes $\mathcal{O}[[X]]$. By the methods of [Cia09, Ter03] it is easy to see that K is the $\mathcal{O}$-module generated by the image of Y in $\mathfrak{p}_{S'}/\mathfrak{p}_{S'}^2$ and that it is isomorphic to $\mathcal{O}/(a_q(f)^2 - \psi(q)(q+1)^2)$. Notice that

$$T_q^2 - \psi(q)(q+1)^2 = A^{-2}(A^2 - q^2\psi(q))(A^2 - \psi(q))$$

and under our hypotheses $v_\lambda(c) = v_\lambda(\pi_{S'}(T_q^2 - \psi(q)(q+1)^2)$. Thus the claim is proved if $a \neq b$. Assume now that $a = b$ and consider the Ihara map

$$\tilde{\alpha} : \mathcal{M}_S^2 \to \mathcal{M}_{S'}$$

Since $q^2 \not\equiv 1 \bmod \ell$, $\mathcal{B}_{S'} = \mathcal{B}_S$ so that $\mathbf{T}_{S'} = \mathbf{T}_S$ and $\tilde{\alpha}$ is an isomorphism. Therefore $\mathcal{M}_{S'}$ is free of rank 4 over $\mathbf{T}_{S'}$. We have $\Omega_{S'} \simeq \Omega_S^2$ so that $\mathrm{lenght}_{\mathcal{O}}(\Omega_{S'}) = 2\,\mathrm{lenght}_{\mathcal{O}}(\Omega_S) = 4\,\mathrm{lenght}_{\mathcal{O}}(\mathfrak{p}/\mathfrak{p}^2)$ so that the map $\mathcal{R}_{S'} \to \mathbf{T}_{S'} = \mathbf{T}_S$ is an isomorphism of complete intersection.

We remark that in the case $a = b$ the full Hecke algebra (with the T_q-operator) $\mathbf{T}_{S',cohom}$ acting on $\mathcal{M}'_S$ is isomorphic to $\mathbf{T}_S[X]/(X^2 - T_q X + q\psi(q))$ so that $\mathcal{M}_{S'}$ is free of rank 2 over $\mathbf{T}_{S',cohom}$.

4.4. A consequence about congruences of modular forms

Under Conjecture 1.1, we can prove the following result about raising the level of modular forms:

Theorem 4.3. *Assume Conjecture 1.1. Let $f = \sum a_n q^n$ be a normalized newform in $S_2(\Gamma_0(N\Delta'\ell^2), \psi)$, supercuspidal of type $\tau = \chi \oplus \chi^\sigma$ at ℓ, special at every prime $p|\Delta'$. Let q be a prime not dividing $N\Delta$ such that $q^2 \not\equiv 1 \bmod \ell$; then there exists $g \in S_2(\Gamma_0(qN\Delta'\ell^2), \psi)$ supercuspidal of type τ at ℓ, special at every prime $p|\Delta'$ such that $f \equiv g \bmod \lambda$ if and only if*

$$a_q^2 \equiv \psi(q)(q+1)^2 \bmod \lambda. \tag{19}$$

Proof. By Theorem 4.1 and its proof we know that Condition (19) is equivalent to the non injectivity of the map $T_{S'} \to T_S$. ■

If we do not make the additional hypotheses on the local type the modular forms, then the above Conjecture is the classical "raising the level" problem that was first addressed in [Rib83a].

References

[Bre97] G. E. Bredon, *Sheaf Theory*, volume 170 of Graduate Texts in Mathematics, Springer, New York, second edition, 1997.

[Car86] H. Carayol, *Sur les représentations ℓ-adiques associées aux formes modulaires de Hilbert*, Ann. Sci. Éc. Norm. Sup. (4^e série) **19** (1986), 409–468.

[CDT99] Brian Conrad, Fred Diamond and R. Taylor, *Modularity of certain potentially Barsotti-Tate Galois representations*, J. Amer. Math. Soc. **12** (1999), 521–567.

[Cia09] M. Ciavarella, *Congruences between modular forms and related modules*, accepted for publication in Functiones et Approximatio, 2009.

[DD97] H. Darmon, R. Diamond and R. Taylor, *Fermat's last theorem*, in Elliptic Curves, Modular Forms and Fermat's Last Theorem (Hong Kong, 1993), (Second Ed.), pages 2–140, Cambridge, MA, 1997, International Press.

[Dia91] F. Diamond, *Congruence primes for cusp forms of weight $k \geqslant 2$*, in Astérisque, Courbes Modulaires et Courbes de Shimura, volume 196-197, pages 205–213, Sociétéé Mathématique de France, 1991.

[Dia97] F. Diamond, *The Taylor-Wiles construction and multiplicity one*, Invent. Math. **128** (1997), 379–391.

[DR97] F. Diamond and K. A. Ribet, *ℓ-adic modular deformations and Wiles's "main conjecture"*, in G. Cornell, H. Silverman, and G. Stevens, editors, Modular Forms and Fermat's Last Theorem, pages 357–371, Springer, 1997.

[DT94] F. Diamond and R. Taylor, *Non-optimal levels for mod ℓ modular representations of* $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, Invent. Math. **115** (1994), 435–462.

[Hid88] H. Hida, *On p-adic Hecke algebras for* GL_2 *over totally real fields*, Ann. of Math. **128** (1988), 295–384.

[Hid93] H. Hida, *Elementary theory of L-functions and Eisenstein series*, Cambridge University Press, 1993.

[Iha73] Y. Ihara, *On modular curves over finite fields*, Proceedings of the International Colloquium on Discrete Subgroups of Lie Groups and Applications to Moduli, pages 161–202, January 1973.

[MS63] Y. Matsushima and G. Shimura, *On the cohomology groups attached to certain vector valued differential forms on the product of the upper half planes*, Ann. of Math. **78** (1963), 417–449.

[MT99] A. Mori and L. Terracini, *A canonical map between Hecke algebras*, Boll. Un. Mat. Ital. **(8) 2-B** (1999), 429–452.

[PR83] G. Prasad and M. S. Raghunathan, *On the congruence subgroup problem: determination of the "metaplectic kernel"*, Invent. Math. **71**, no. 1, (1983), 21–42.

[PR96] G. Prasad and A. S. Rapinchuk, *Computation of the metaplectic kernel*, Publ. Math. IHES **84** (1996), 91–187.

[PR08] G. Prasad and A. S. Rapinchuk, *Developments on the congruence subgroup problem after the work of Bass, Milnor and Serre*, Preprint, 2008.

[Rap99] A. S. Rapinchuk, *The congruence subgroup problem*, Contemporary Mathematics **243** (1999), 175–188.

[Rib83a] K. A. Ribet, *Congruence relations between modular forms*, in Proceedings of the International Congress of Mathematics, pages 503–514, Warsaw, 1983.

[Rib83b] K. A. Ribet, *Mod p Hecke operators and congruences between modular forms*, Inventiones Mathematicae **71** (1983),:193–205.

[Seg99] Y. Segev, *On finite homomorphic images of the multiplicative group of a division algebra*, Annals of Mathematics **149** (1999), 219–251.

[Ser67] J.-P. Serre, *Représentations linéaires des groupes fini*, Hermann, Paris, 1967.

[Ser70] J.-P. Serre, *Le problème des groupes des congruences pour* SL_2, Ann. Math. **92** (1970), 489–527.

[Ser80] J.-P. Serre, *Trees*, Springer-Verlag Berlin Heidelberg New York, 1980.

[Shi71] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Iwanami Shoten and Princeton Univ. Press, 1971.

[Ter03] L. Terracini, *A Taylor-Wiles system for quaternionic Hecke algebras*, Compositio Mathematica **137** (2003), 23–47.

[TW95] R. Taylor and A. Wiles, *Ring-theoretic properties of certain Hecke algebras*, Ann. Of Math. (2) **141**, no. 3, (1995), 553–572.

[Vig80] M.-F. Vignéras, *Arithmétique des algèbres de quaternions*, volume 800. Springer, 1980.

[Wil95] A. Wiles, *Modular elliptic curves and Fermat last Theorem*, Ann. of Math. **141** (1995), 443–551.

Addresses: Miriam Ciavarella: Dipartimento di Matematica, Università degli Studi di Torino, Via Carlo Alberto 8, 10123 Torino, Italy;
Lea Terracini: Dipartimento di Matematica, Università degli Studi di Torino, Via Carlo Alberto 8, 10123 Torino, Italy.

E-mail: miriam.ciavarella@unito.it, lea.terracini@unito.it

Received: 30 July 2009; **revised:** 24 March 2011