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WEIGHTED SPACES OF HOLOMORPHIC 2π-PERIODIC FUNCTIONS ON THE UPPER HALFPLANE

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Dedicated to the memory of Susanne Dierolf

Abstract: We consider spaces of 2π-periodic holomorphic functions f on the upper halfplane G which are bounded by a weighted sup-norm $\sup_{w\in G}|f(w)|v(w)$. Here $v : G \to]0,\infty[$ is a function which depends essentially only on $\operatorname{Im} w$, $w \in G$, and satisfies $\lim_{t\to 0} v(it) = 0$. We give a complete isomorphic classification of such spaces and investigate composition operators and the differentiation operator between them.

Keywords: weighted spaces, holomorphic periodic functions, halfplane, differentiation operators, composition operators.

1. Introduction

Let $O \subset \mathbb{C}$ be an open subset and $v : O \to [0,\infty[$ a given function. Then we consider, for $f : O \to \mathbb{C}$, the weighted sup-norm

$$\|f\|_v = \sup_{z\in O} |f(z)|v(z)$$

and the spaces

$$Hv(O) = \{f : O \to \mathbb{C} \text{ holomorphic} : \|f\|_v < \infty\}$$

and

$$Hv_0(O) = \{f \in Hv(O) : |f(z)|v(z) \text{ vanishes at } \infty\}.$$

(Here $|f|v$ vanishes at ∞ if for any $\epsilon > 0$ there is a compact subset $K \subset O$ such that $|f(z)|v(z) < \epsilon$ for all $z \in O \setminus K$.)

Assume that $\lim_{\operatorname{dist}(z,\partial O)\to 0} v(z) = 0$, $v(z) > 0$ for all $z \in O$ and v is continuous. Then, for a holomorphic function f, $f \in Hv(O)$ is equivalent to the growth condition $|f(z)| = O(1/v(z))$ as $\operatorname{dist}(z,\partial O) \to 0$ while $f \in Hv_0(O)$ is equivalent to $|f(z)| = o(1/v(z))$ as $\operatorname{dist}(z,\partial O) \to 0$.

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A lot of research has been done for these spaces in the case $O = \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ while very little is known for open half spaces ([11, 12, 3, 13]). In our paper we focus on

$$G = \{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}.$$

We study the following class of weights

Definition 1.1. *Let v be a continuous function on G satisfying $v(w) > 0$ for all $w \in G$. Assume that there is a constant $C > 0$ such that*

$$\lim_{r\to 0} v(ir) = 0 \quad \textit{and} \quad v(w_1) \leqslant Cv(w_2) \qquad \textit{whenever} \quad 0 < \mathrm{Im}(w_1) \leqslant \mathrm{Im}(w_2).$$

Then v is called a standard weight.

A standard weight always satisfies

$$\frac{1}{C}v(w_1) \leqslant v(w_2) \leqslant Cv(w_1) \qquad \text{whenever} \quad \mathrm{Im}(w_1) = \mathrm{Im}(w_2)$$

which is a consequence of the definition. This implies that the translation operators T_a, $a \in \mathbb{R}$, with $(T_a f)(w) = f(w + a)$, $w \in G$, are uniformly bounded on $Hv(G)$ as well as on $Hv_0(G)$.

Example 1.1. Let $\beta > 0 > \gamma$ and put

$$v_1(w) = (\mathrm{Im}(w))^{\beta}, \qquad v_2(w) = \min(v_1(w), 1), \qquad v_3(w) = e^{-1/\mathrm{Im}(w)},$$

$$v_4(w) = \begin{cases} (1 - \log(\mathrm{Im}(w)))^{\gamma} & \text{if } \mathrm{Im}(w) \leqslant 1 \\ 1 & \text{if } \mathrm{Im}(w) > 1 \end{cases}$$

Then all these weights are standard weights. Moreover, if $\gamma : \mathbb{R} \to \mathbb{R}$ is a continuous function with $0 < \inf_{t\in\mathbb{R}} \gamma(t) \leqslant \sup_{t\in\mathbb{R}} \gamma(t) < \infty$ and v is a standard weight then $\hat{v}(w) = \gamma(Re(w))v(w)$ is a standard weight, too.

A result due to Stanev ([13]) shows that, for any standard weight v, $Hv(G) \neq \{0\}$ is equivalent to the existence of constants $a, b > 0$ such that $v(w) \leqslant ae^{b\,\mathrm{Im}(w)}$ for all $w \in G$. Moreover, according to [8, 1], $Hv(G)$ is isometrically isomorphic to $Hv_0(G)^{**}$ and $id : Hv_0(G) \to Hv(G)$ corresponds to the canonical injection $Hv_0(G) \to Hv_0(G)^{**}$.

To obtain results about the Banach spaces $Hv_0(G)$ and $Hv(G)$ one could apply the Möbius transform $\alpha : \mathbb{D} \to G$ with $\alpha(z) = (1 + z)(1 - z)^{-1}i$, $z \in \mathbb{D}$, and study the weight $v \circ \alpha$ on $\mathbb{D}$. However this weight is non-radial and $\lim_{z\to 1}(v \circ \alpha)(z)$ does not exist. This creates extreme difficulties transfering the known results about the isomorphic classification of weighted spaces on $\mathbb{D}$ for radial weights vanishing on the boundary to $Hv_0(G)$ and $Hv(G)$.

The situation is different if we restrict ourselves to 2π-periodic functions. Put

$$Hv^{2\pi}(G) = \{f \in Hv(G) : f(w) = f(w + 2\pi) \ \text{ for all } \ w \in G\}$$

and $Hv_0^{2\pi}(G) = Hv^{2\pi}(G) \cap Hv_0(G)$. In the following we show that $Hv_0^{2\pi}(G)$ and $Hv^{2\pi}(G)$ are isomorphic to $H\tilde{v}_0(\mathbb{D})$ and $H\tilde{v}(\mathbb{D})$ for suitable weights $\tilde{v}$ on $\mathbb{D}$ and we discuss some consequences. For example we determine all isomorphism classes of $Hv^{2\pi}(G)$ and $Hv_0^{2\pi}(G)$. Then we investigate the differentiation operator and composition operators on $Hv^{2\pi}(G)$.

2. The isomorphism theorem

Let v be a standard weight on G with $Hv(G) \neq \{0\}$. Let b_v be the smallest integer such that $\sup_{w\in G} e^{-b_v \operatorname{Im}(w)} v(w) < \infty$. Since $v(ir)$, $r \in \mathbb{R}$, is almost increasing this means that $b_v \geqslant 0$. Moreover $b_v = 0$ if and only if v is bounded.

For the examples in section 1 we obtain $b_{v_1} = 1$, $b_{v_2} = b_{v_3} = b_{v_4} = 0$.

Proposition 2.1. *For each $f \in Hv^{2\pi}(G)$ there exist $\gamma_k \in \mathbb{C}$ with $f(w) = \sum_{k=b_v}^{\infty} \gamma_k e^{ikw}$ where the series converges uniformly on compact sets.*

Proof. Put $\tau(z) = -i \log z$, $z \in \mathbb{D} \setminus \{0\}$. Then $\tau(z) \in G$. We claim that, for $f \in Hv^{2\pi}(G)$, the function $(f \circ \tau)(z) = f(-i \log z)$ is holomorphic on $\mathbb{D} \setminus \{0\}$.

To see this consider $\widetilde{\log} z = \log|z| + i\widetilde{\arg} z$ where $\widetilde{\arg} z$ is the argument of z with $\widetilde{\arg} \in [0, 2\pi[$. Then we have

$$\widetilde{\log} z = \begin{cases} \log z & \text{if } \ 0 \leqslant \arg z < \pi \\ \log z + 2\pi i & \text{if } \ -\pi \leqslant \arg z < 0 \end{cases}.$$

Since f is 2π-periodic we obtain $f(-i \log z) = f(-i\widetilde{\log} z)$. log is holomorphic in $z \in \mathbb{D}$ if $\operatorname{Im}(z) \neq 0$ or $\operatorname{Im}(z) = 0$ and $Re(z) > 0$. $\widetilde{\log}$ is holomorphic in $z \in \mathbb{D}$ if $\operatorname{Im}(z) \neq 0$ or $\operatorname{Im}(z) = 0$ and $Re(z) < 0$. Hence $f \circ \tau = f \circ \tilde{\tau}$ is holomorphic on $\mathbb{D} \setminus \{0\}$ where $\tilde{\tau}(z) = -i\widetilde{\log}(z)$.

We find $\gamma_k \in \mathbb{C}$ such that $f(-i \log z) = \sum_{k=-\infty}^{\infty} \gamma_k z^k$, $z \in \mathbb{D} \setminus \{0\}$. Since $-i \log(e^{iw}) = w$ for all $w \in G$ with $-\pi \leqslant \operatorname{Re} w < \pi$ we obtain $f(w) = \sum_{k=-\infty}^{\infty} \gamma_k e^{ikw}$. The latter equality holds for all $w \in G$ since f and e^{ikw} are 2π-periodic.

Moreover we have

$$\gamma_k e^{ikw} = \frac{1}{2\pi} \int_0^{2\pi} f(w+x) e^{-ikx} dx.$$

This function must be an element of $Hv(G)$ since we have

$$\begin{aligned} \left|\gamma_k e^{ikw}\right| v(w) &= \frac{1}{2\pi} \left| \int_0^{2\pi} f(w+x) e^{-ikx} dx \right| v(w) \leqslant \sup_{0 \leqslant x \leqslant 2\pi} |f(w+x)| v(w) \\ &\leqslant C \sup_{0 \leqslant x \leqslant 2\pi} |f(w+x)| v(w+x) \end{aligned}$$

for some universal constant $C > 0$. This implies

$$|\gamma_k e^{ikw}| v(w) = |\gamma_k| e^{-k \operatorname{Im}(w)} v(w) \leqslant C \|f\|_v \qquad \text{for all } \ k.$$

By definition of b_v the function $e^{-k\,\mathrm{Im}(w)}v(w)$ is unbounded whenever $k < b_v$. Hence $\gamma_k = 0$ for all $k < b_v$. This proves Proposition 2.1. ■

Proposition 2.1 implies that, for any $f \in Hv^{2\pi}(G)$, the function $z^{-b_v}f(-i\log z)$ can be holomorphically extended in $z = 0$.

Theorem 2.1. *Put* $\tilde{v}(z) = \begin{cases} |z|^{b_v}v(-i\log|z|), & z \neq 0 \\ 0, & z = 0 \end{cases}$. *Then $\tilde{v}$ is a radial weight on $\mathbb{D}$, i.e. $\tilde{v}(z) = \tilde{v}(|z|)$ for all $z \in \mathbb{D}$. Moreover, $\lim_{|z|\to 1}\tilde{v}(z) = 0$. The operator S with $(Sf)(z) = z^{-b_v}f(-i\log z)$, $z \in \mathbb{D}$, maps $Hv^{2\pi}(G)$ (and $Hv_0^{2\pi}(G)$, resp.) isomorphically onto $H\tilde{v}(\mathbb{D})$ (onto $H\tilde{v}_0(\mathbb{D})$, resp.).*

Proof. Since v is a standard weight v is equivalent to $v_1(w) = v(\mathrm{Im}(w)i)$. If $f \in Hv^{2\pi}(G)$ then there are γ_k with $f(w) = \sum_{k=b_v}^{\infty}\gamma_k e^{ikw}$. With $z = e^{iw}$ we obtain

$$|f(w)|v(\mathrm{Im}(w)i) = \left|\sum_{k=b_v}^{\infty}\gamma_k z^k\right| v(-i\log|z|) = |(Sf)(z)|\tilde{v}(z).$$

This implies $\|Sf\|_{\tilde{v}} = \|f\|_{v_1}$.

Conversely, if $g \in H\tilde{v}(\mathbb{D})$ then put $f(w) = e^{ib_v w}g(e^{iw})$, $w \in G$. Then f is 2π-periodic. Moreover, $|f(w)|v(\mathrm{Im}(w)i) = |z|^{b_v}g(z)v(-i\log|z|)$ with $z = e^{iw}$. So $\|f\|_{v_1} = \|g\|_{\tilde{v}}$ which implies $f \in Hv^{2\pi}(G)$. Finally we have $Sf = g$. This shows that S is surjective.

The proof for $Hv_0^{2\pi}(G)$ instead of $Hv^{2\pi}(G)$ is the same. ■

The weight $\tilde{v}$ of Theorem 2.1 is bounded on $\mathbb{D}$ since $\sup_{w\in G} e^{-b_v\,\mathrm{Im}(w)}v(w) < \infty$ but not continuous in general. We show that it can be replaced by a weight u on $\mathbb{D}$ with more suitable properties.

Lemma 2.1. *Let*

$$u(z) = \begin{cases} \sup_{|z|\leqslant t<1} v(-i\log t), & \text{if } |z| \geqslant \frac{1}{2} \\ \sup_{1/2\leqslant t<1} v(-i\log t), & \text{if } |z| \leqslant \frac{1}{2} \end{cases}.$$

Then u is a continuous radial weight on $\mathbb{D}$ with $\lim_{|z|\to 1}u(z) = 0$ and $u(t) \leqslant u(s)$ whenever $0 \leqslant s \leqslant t < 1$. $\tilde{v}|_{[2^{-1},1[}$ and $u|_{[2^{-1},1[}$ are equivalent. Moreover, $\|\cdot\|_{\tilde{v}}$ and $\|\cdot\|_u$ are equivalent (on holomorphic functions).

Proof. Since v is a standard weight we obtain $\lim_{|z|\to 1}\tilde{v}(z) = 0$ and $\tilde{v}(t) \leqslant C\tilde{v}(s)$ whenever $1/2 \leqslant s \leqslant t < 1$. Here C is a universal constant. Hence $\lim_{|z|\to 1}u(z) = 0$ and $\tilde{v}|_{[2^{-1},1[}$ and $u|_{[2^{-1},1[}$ are equivalent. By definition of u we have $u(t) \leqslant u(s)$ whenever $0 \leqslant s \leqslant t < 1$. Since $v(-i\log|z|)$ is continuous if $1/2 \leqslant |z| < 1$ we conclude that u is continuous on $\mathbb{D}$.

Finally we show that $\|\cdot\|_u$ and $\|\cdot\|_{\tilde{v}}$ are equivalent. To this end consider $d = \sup_{z\in\mathbb{D}} \tilde{v}(z)$. Then the maximum principle implies

$$\|f\|_{\tilde{v}} \leqslant \max\left(\sup_{1/2\leqslant|z|<1} |f(z)|\tilde{v}(z), \sup_{|z|=1/2} |f(z)|\tilde{v}(\frac{1}{2})\cdot\frac{d}{\tilde{v}(1/2)}\right) \leqslant \frac{d}{\tilde{v}(\frac{1}{2})}\|f\|_{\tilde{v}}$$

for any holomorphic f. A similar inequality holds for $\|f\|_u$. Since $u|_{[2^{-1},1[}$ and $\tilde{v}|_{[2^{-1},1[}$ are equivalent we obtain that $\|\cdot\|_{\tilde{v}}$ and $\|\cdot\|_u$ are equivalent. ■

Corollary 2.1. *S with $(Sf)(z) = z^{-b_v} f(-i\log z)$, $z\in\mathbb{D}$, maps $Hv^{2\pi}(G)$ (and $Hv_0^{2\pi}(G)$, resp.) isomorphically onto $Hu(\mathbb{D})$ (onto $Hu_0(\mathbb{D})$, resp.).*

3. The isomorphism classes

Let

$$H_\infty = \{f : \mathbb{D}\to\mathbb{C} \text{ holomorphic} : f \text{ bounded}\}$$

and $H_n = \{f : f \text{ a polynomial of degree} \leqslant n\}$ be endowed with the sup-norm on $\mathbb{D}$. Moreover let v be a standard weight on G such that $Hv(G)\neq\{0\}$.

Theorem 3.1. *Either $Hv^{2\pi}(G)$ is isomorphic to l_∞ and $Hv_0^{2\pi}(G)$ is isomorphic to c_0, or $Hv^{2\pi}(G)$ is isomorphic to H_∞ and $Hv_0^{2\pi}(G)$ is isomorphic to $(\sum_{n=0}^\infty \oplus H_n)_{(0)}$.*

Proof. Consider the weights $\tilde{v}$ and u of Theorem 2.1 and Lemma 2.1. According to Corollary 2.1, $Hv^{2\pi}(G)$ and $Hu(\mathbb{D})$ are isomorphic as well as $Hv_0^{2\pi}(G)$ and $Hu_0(\mathbb{D})$. In [10] it was shown that $Hu(\mathbb{D})$ is isomorphic to l_∞ and $Hu_0(\mathbb{D})$ is isomorphic to c_0 or $Hu(\mathbb{D})$ is isomorphic to H_∞ and $Hu_0(\mathbb{D})$ is isomorphic to $(\sum_{n=0}^\infty \oplus H_n)_{(0)}$. ■

Corollary 3.1. *$Hv_0^{2\pi}(G)$ has a Schauder basis.*

Proof. This follows from Theorem 3.1 since it is well-known that c_0 as well as $(\sum_{n=1}^\infty \oplus H_n)_{(0)}$ have Schauder bases ([2, 5]). ■

Since $c_0^{**} = l_\infty$ and $(\sum_{n=1}^\infty \oplus H_n)_{(0)}^{**} = H_\infty$ ([15]) Theorem 3.1 also implies that $Hv^{2\pi}(G)$ is always isomorphic to $(Hv_0^{2\pi}(G))^{**}$.

For a given weight v it is possible to decide, using special properties of u, which case of Theorem 3.1 holds (according to [10]). However this is technically complicated. Here we only consider the special case of moderately increasing weights.

At first we note the following

Lemma 3.1. *Let u be the weight of Lemma 2.1. Then*

$$\sup_n \frac{v(i2^{-n})}{v(i2^{-n-1})} < \infty \tag{1}$$

if and only if

$$\sup_n \frac{u(1-2^{-n})}{u(1-2^{-n-1})} < \infty. \tag{2}$$

Moreover

$$\inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{v(i2^{-n-k})}{v(i2^{-n})} < 1 \tag{3}$$

if and only if

$$\inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{u(1-2^{-n-k})}{u(1-2^{-n})} < 1. \tag{4}$$

Proof. The equivalence of (1) and (2) follows from an elementary calculation. To prove the equivalence of (3) and (4), observe at first that (3) is easily seen to be equivalent to

$$\inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{\tilde{v}(1-2^{-n-k})}{\tilde{v}(1-2^{-n})} < 1. \tag{5}$$

Hence we find $q < 1$ and $k \in \mathbb{N}$ such that

$$\frac{\tilde{v}(1-2^{-n-k})}{\tilde{v}(1-2^{-n})} < q$$

for large enough n. This implies

$$\frac{\tilde{v}(1-2^{-n-km})}{\tilde{v}(1-2^{-n})} < q^m$$

for all m and large enough n. Using the fact that $u|_{[2^{-1},1[}$ and $\tilde{v}|_{[2^{-1},1[}$ are equivalent we see that, for large enough m,

$$\sup_{n\geqslant n_0} \frac{u(1-2^{-n-km})}{u(1-2^{-n})} < 1$$

for some n_0. This implies (4). A similar argument shows that (4) implies (5). ■

Proposition 3.1. *Assume that*

$$\sup_n \frac{v(i2^{-n})}{v(i2^{-n-1})} < \infty.$$

Then $Hv^{2\pi}(G)$ is isomorphic to l_∞ if and only if

$$\inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{v(i2^{-n-k})}{v(i2^{-n})} < 1.$$

Proof. According to [9] (4) is equivalent to the fact that $Hu(\mathbb{D})$ is isomorphic to l_∞ provided that (2) holds. Then the proposition follows from Corollary 2.1. ■

The examples v_1, v_2 and v_4 are easily seen to satisfy the condition (1) of Lemma 3.1 v_1 and v_2 also satisfy (3) while v_4 does not satisfy (3). Hence $Hv_k^{2\pi}(G)$ is isomorphic to l_∞ for $k = 1, 2$ while $Hv_4^{2\pi}(G)$ is isomorphic to H_∞. v_3 does not satisfy (1). However, one can easily show that here $\|\cdot\|_u$ is equivalent to $\|\cdot\|_{\hat{v}}$ where $\hat{v}(z) = e^{-1/(1-|z|)}$. It is known that $H\hat{v}(\mathbb{D})$ is isomorphic to l_∞ ([10]). Hence $H_{v_3}^{2\pi}(G)$ is isomorphic to l_∞.

Finally, we remark that $Hv^{2\pi}(G)$ is a complemented subspace of $Hv(G)$ if $Hv^{2\pi}(G)$ is isomorphic to l_∞ (since l_∞ is complemented in any superspace).

4. Some operators on $Hv^{2\pi}(G)$

We start noting a simple but useful lemma concerning radial weights on $\mathbb{D}$.

Lemma 4.1. *Let $q : \mathbb{D} \to [0, \infty[$ be a radial weight such that*

$$0 < q\left(\frac{1}{2}\right) \leqslant \sup_{|z|\leqslant 1/2} q(z) < \infty.$$

(a) *Then*

$$\sup_{1/2\leqslant|z|<1} |f(z)|q(z) \leqslant \|f\|_q \leqslant \left(\frac{\sup_{|z|\leqslant 1/2} q(z)}{q(1/2)}\right) \sup_{1/2\leqslant|z|<1} |f(z)|q(z)$$

for every holomorphic $f : \mathbb{D} \to \mathbb{C}$;

(b) $|\alpha_k| \leqslant 2^k/q(1/2)\|f\|_q$, $k = 0, 1, \ldots$, *whenever* $f(z) = \sum_{j=0}^\infty \alpha_j z^j$.

Proof. (a) is a consequence of the maximum principle. (b) follows from the fact that $|\alpha_k|2^{-k} \leqslant \sup_{|z|=1/2} |f(z)|$. ■

Now, again, let v be a standard weight on G such that $Hv(G) \neq \{0\}$. We put

$$Hv^{2\pi,0}(G) = \left\{ f \in Hv^{2\pi}(G) : \text{ there are } \gamma_k \text{ with } f(w) = \sum_{k\geqslant 1} \gamma_k e^{ikw} \right\}$$

If $b_v = 0$ then, according to Proposition 2.1, $Hv^{2\pi,0}(G)$ is a 1-codimensional subspace of $Hv^{2\pi}(G)$. If $b_v > 0$ then we have $Hv^{2\pi,0}(G) = Hv^{2\pi}(G)$. In any case,

$$Hv^{2\pi,0}(G) = \left\{ f \in Hv^{2\pi}(G) : \lim_{\mathrm{Im}(w)\to\infty} f(w) = 0 \right\}$$

At first we deal with the differentiation operator $Df = f'$. Of course, Df is 2π-periodic if f is 2π-periodic. Notice that a constant function different from zero is never the derivative of a 2π-periodic function. For the next theorem we want to assume that any $r_0 \in [0, 1[$ is a global maximum point of the function $r \to r^n u(r)$ for some $n > 0$ where u is the weight of Lemma 2.1. (This is no restriction, according to [7] one finds a weight u_0 satisfying this such that $\|\cdot\|_u$ and $\|\cdot\|_{u_0}$ are equivalent.)

Theorem 4.1. *Put* $v_1(w) = \left(1 - e^{-\operatorname{Im}(w)}\right)v(w)$, $w \in G$.

(a) *The following are equivalent*

(i) *Whenever* $f \in Hv^{2\pi}(G)$ *then* $Df \in Hv_1^{2\pi}(G)$;

(ii) D *is a bounded operator from* $Hv^{2\pi}(G)$ *into* $Hv_1^{2\pi}(G)$;

(iii) $\sup_n \dfrac{v(i2^{-n})}{v(i2^{-n-1})} < \infty$.

In this case, $DHv^{2\pi}(G) \subset Hv_1^{2\pi,0}(G)$.

(b) *The following are equivalent*

(i) $DHv^{2\pi}(G) = Hv_1^{2\pi,0}(G)$;

(ii) $\sup_n \dfrac{v(i2^{-n})}{v(i2^{-n-1})} < \infty$ *and* $\inf_k \limsup_{n\to\infty} \dfrac{v(i2^{-n-k})}{v(i2^{-n})} < 1$.

Proof. The equivalence of (a), (i) and (ii), is a consequence of the closed graph theorem.

Let S be the operator of Theorem 2.1. Let us take $f \in Hv^{2\pi}(G)$, say $f(w) = \sum_{k=b_v}^{\infty} \alpha_k e^{ikw}$. Then

$$(Sf)(z) = \sum_{k=0}^{\infty} \alpha_{k+b_v} z^k, \qquad (Df)(w) = i \sum_{k=b_v}^{\infty} \alpha_k k e^{ikw}$$

and

$$(SDf)(z) = i \sum_{k=0}^{\infty} \alpha_{k+b_v}(k+b_v) z^k.$$

Let $\tilde{D}$ be the differentiation operator on the holomorphic functions over $\mathbb{D}$ and let M_g be the multiplication with the function g over $\mathbb{D}$. Then we have

$$SDf = iM_{z^{-b_v+1}} \tilde{D} M_{z^{b_v}} Sf. \tag{6}$$

Let u be the weight of Lemma 2.1 corresponding to v. Moreover let $u_1(z) = (1-|z|)u(z)$, $z \in \mathbb{D}$. Finally put

$$\tilde{v}_1(z) = \begin{cases} (1-|z|)|z|^{b_v} v(-i \log|z|), & z \neq 0 \\ 0, & z = 0 \end{cases}.$$

By definition of v_1 we have $b_v = b_{v_1}$. In view of Lemma 2.1 we see that $\|\cdot\|_{\tilde{v}_1}$ and $\|\cdot\|_{u_1}$ are equivalent. Hence S also maps $Hv_1^{2\pi}(G)$ isomorphically onto $Hu_1(\mathbb{D})$. It follows from Lemma 4.1(a) that M_{z^m} is an isomorphism from $Hu(\mathbb{D})$ onto a m-codimensional subspace of $Hu(\mathbb{D})$ provided that m is a non-negative integer. We have $Hu(\mathbb{D}) = M_{z^m} Hu(\mathbb{D}) \oplus E_m$ where E_m consists of all polynomials of degree less than m.

(6) implies that D is bounded if and only if $\tilde{D}$ is bounded as an operator from $A := \{M_{z^{b_v}} g : g \in Hu(\mathbb{D})\}$ into $Hu_1(\mathbb{D})$. Since $Hu(\mathbb{D}) = A \oplus E_{b_v}$ and $\tilde{D}|_{E_{b_v}}$ is

bounded in view of Lemma 4.1(b) this is equivalent to the boundedness of $\tilde{D}$ as an operator on $Hu(\mathbb{D})$. But the latter fact is equivalent to

$$\sup_n \frac{u(1-2^{-n})}{u(1-2^{-n-1})} < \infty$$

([7], Theorem 3.2). Now Lemma 3.1 concludes the proof of (a), $(ii) \Leftrightarrow (iii)$. The remaining assertion of (a) is obvious.

To prove (b) assume at first $b_v = b_{v_1} > 0$. Then $h \in Hv_1^{2\pi,0}(G)$ if and only if $Sh \in Hu_1(\mathbb{D})$. The existence of $f \in Hv^{2\pi}(G)$ with $Df = h$ is equivalent to $Sh = iM_{z^{-b_v+1}}\tilde{D}M_{z^{b_v}}Sf$ with $Sf \in Hu(\mathbb{D})$ (see (6)) and to $M_{z^{b_v-1}}Sh = i\tilde{D}M_{z^{b_v}}Sf$. The latter problem is equivalent to the existence of $g \in B := \{M_{z^{b_v}}k : k \in Hu(\mathbb{D})\}$ with $\tilde{D}g = l$ for any $l \in C := \{M_{z^{b_v-1}}k : k \in Hu_1(\mathbb{D})\}$. Since B is finite codimensional in $Hu(\mathbb{D})$ and C is finite codimensional in $Hu_1(\mathbb{D})$ this is equivalent to the existence of $g \in Hu(\mathbb{D})$ with $\tilde{D}g = l$ for any $l \in Hu_1(\mathbb{D})$. Indeed, $Hu_1(\mathbb{D}) = C \oplus E_{b_v-1}$ and, for any $l \in E_{b_v-1}$, we find $g \in Hu(\mathbb{D})$ with $\tilde{D}g = l$. In other words, (b)(i) is equivalent to the condition, $g \in Hu(\mathbb{D})$ if and only if $\tilde{D}g \in Hu_1(\mathbb{D})$. This property is equivalent to

$$\sup_n \frac{u(1-2^{-n})}{u(1-2^{-n-1})} < \infty \qquad \text{and} \qquad \inf_k \limsup_{n\to\infty} \frac{u(1-2^{-n-k})}{u(1-2^{-n})} < 1$$

(see [7]). Again, Lemma 3.1 concludes the proof of (b) in this case.

If $b_v = b_{v_1} = 0$ then $h \in Hv_1^{2\pi,0}(G)$ if and only if $Sh \in M_z Hu_1(\mathbb{D})$. Then the preceding proof works as well here with $B = Hu(\mathbb{D})$ and $C = Hu_1(\mathbb{D})$. ∎

There is no standard weight v on G such that $D(Hv^{2\pi}(G)) \subset Hv^{2\pi}(G)$. Indeed, otherwise D would be a bounded endomorphism on $Hv^{2\pi}(G)$ (with the closed graph theorem). But we have, for $f_k(w) = e^{ikw}$, $\|Df_k\|_v = k\|f_k\|_v$, where $k = b_v, b_v+1, \ldots$.

Finally, we turn to composition operators. Let $\varphi : G \to G$ be a 2π-periodic holomorphic function. (For example, $\varphi(w) = i + e^{inw}$ or $\varphi(w) = -(i+e^{inw})^{-1}$ for some $n \in \mathbb{N}$.)

We put $C_\varphi(f) = f \circ \varphi$. Then we have

Theorem 4.2. *Let v and $\hat{v}$ be two standard weights on G such that $Hv(G) \neq \{0\} \neq H\hat{v}(G)$. Then C_φ is a bounded operator from $Hv^{2\pi}(G)$ into $H\hat{v}^{2\pi}(G)$ if and only if*

$$\sup_n \frac{\|e^{in\varphi(w)}\|_{\hat{v}}}{\|e^{inw}\|_v} < \infty$$

Furthermore, the following are equivalent

(i) $C_\varphi : Hv^{2\pi}(G) \to H\hat{v}^{2\pi}(G)$ *is compact;*
(ii) $C_\varphi : Hv^{2\pi}(G) \to H\hat{v}^{2\pi}(G)$ *is weakly compact;*
(iii) $\displaystyle \lim_{n\to\infty} \frac{\|e^{in\varphi(w)}\|_{\hat{v}}}{\|e^{inw}\|_v} = 0.$

Proof. Let u, $\hat{u}$ be the weights on $\mathbb{D}$ of Lemma 2.1 corresponding to v, $\hat{v}$. Moreover, put

$$(Sf)(z) = z^{-b_v} f(-i\log z) \qquad \text{and} \qquad (\hat{S}f)(z) = z^{-b_{\hat{v}}} f(-i\log z).$$

We obtain, if $f \in Hv^{2\pi}(G)$,

$$(\hat{S}C_\varphi f)(z) = z^{-b_{\hat{v}}} f(\varphi(-i\log z)). \tag{7}$$

Put $\tilde{\varphi}(z) = e^{i\varphi(-i\log z)}$, $z \in \mathbb{D}$. Then $\tilde{\varphi}$ is holomorphic on $\mathbb{D}\setminus\{0\}$ since φ is 2π-periodic. Furthermore, $\tilde{\varphi}$ maps $\mathbb{D}\setminus\{0\}$ into $\mathbb{D}$. This implies that $\tilde{\varphi}$ is bounded in a neighbourhood of zero. Hence $\tilde{\varphi}$ can be holomorphically extended in zero. The extension will be called $\tilde{\varphi}$ again. The maximum principle ensures that then $\tilde{\varphi}(\mathbb{D}) \subset \mathbb{D}$. Since $(Sf)(\tilde{\varphi}(z)) = e^{-ib_v\varphi(-i\log z)} f(\varphi(-i\log z))$, (7) implies

$$(\hat{S}C_\varphi S^{-1})(Sf)(z) = \tilde{\varphi}(z)^{b_v} z^{-b_{\hat{v}}} (Sf)(\tilde{\varphi}(z)).$$

Applying the multiplication operator M_{z^m} we see that

$$M_{z^{b_{\hat{v}}}}(\hat{S}C_\varphi S^{-1})(Sf)(z) = (M_{z^{b_v}}Sf)(\tilde{\varphi}(z)). \tag{8}$$

Let $\tilde{C}_{\tilde{\varphi}}$ be defined by $(\tilde{C}_{\tilde{\varphi}}g)(z) = g(\tilde{\varphi}(z))$, $z \in \mathbb{D}$, for a holomorphic function g on $\mathbb{D}$. Then (8) implies

$$M_{z^{b_{\hat{v}}}}\hat{S}C_\varphi S^{-1} = \tilde{C}_{\tilde{\varphi}} M_{z^{b_v}}, \tag{9}$$

where M_{z^m}, for $m \in \mathbb{N}$, is an isomorphism onto the finite codimensional subspace $M_{z^{b_v}}Hu(\mathbb{D})$. Since $|\tilde{\varphi}| \leqslant 1$ we have $\|\tilde{\varphi}^k\|_{\hat{u}} \leqslant \sup_{z\in\mathbb{D}} \hat{u}(z)$ for all k. Hence $\tilde{C}_{\tilde{\varphi}}$ is always continuous on the complement of $M_{z^{b_v}}Hu(\mathbb{D})$. We obtain, $\tilde{C}_{\tilde{\varphi}}$ is continuous on $Hu(\mathbb{D})$ if and only if $\tilde{C}_{\tilde{\varphi}}M_{z^{b_v}} = M_{z^{b_{\hat{v}}}}\hat{S}C_\varphi S^{-1}$ is continuous. Since $M_{z^{b_{\hat{v}}}}$ is an (into-)isomorphism on $H\hat{u}(\mathbb{D})$ we see that $\hat{S}C_\varphi S^{-1}$ and hence C_φ is bounded if and only if $\tilde{C}_{\tilde{\varphi}}$ is bounded. It follows from [4] (see also [3], Proposition 2.1) that this is equivalent to

$$\sup_n \frac{\|\tilde{\varphi}(z)^n\|_{\hat{u}}}{\|z^n\|_u} < \infty.$$

Finally, with $z = e^{iw}$, $w \in G$, this holds if and only if

$$\sup_n \frac{\|e^{in\varphi(w)}\|_{\hat{v}}}{\|e^{inw}\|_v} < \infty.$$

Similarly, (9) implies that C_φ is (weakly) compact if and only if $\tilde{C}_{\tilde{\varphi}}$ is (weakly) compact. It follows from [4] (see also [3], Proposition 2.1) that compactness and weak compactness for $\tilde{C}_{\tilde{\varphi}}$ are the same. Moreover, this is equivalent to

$$\lim_{n\to\infty} \frac{\|\tilde{\varphi}(z)^n\|_{\hat{u}}}{\|z^n\|_u} = 0$$

which in turn is equivalent to

$$\lim_{n\to\infty} \frac{\|e^{in\varphi(w)}\|_{\hat{v}}}{\|e^{inw}\|_v} = 0. \qquad \blacksquare$$

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