Functiones et Approximatio
44.1 (2011), 79–109

REAL ANALYTIC PARAMETER DEPENDENCE OF SOLUTIONS OF DIFFERENTIAL EQUATIONS OVER ROUMIEU CLASSES

Paweł Domański

Dedicated to the memory of Susanne Dierolf

Abstract: We consider the problem of real analytic parameter dependence of solutions of the linear partial differential equation $P(D)u = f$, i.e., the question if for every family $(f_\lambda) \subseteq \mathscr{E}_{\{\omega\}}(\Omega)$ of ultradifferentiable functions of Roumieu type (in particular, of real analytic functions or of functions from Gevrey classes) depending in a real analytic way on $\lambda \in U$, U a real analytic manifold, there is a family of solutions $(u_\lambda) \subseteq \mathscr{E}_{\{\omega\}}(\Omega)$ also depending analytically on λ such that

$$P(D)u_\lambda = f_\lambda \qquad \text{for every } \lambda \in U,$$

where $\Omega \subseteq \mathbb{R}^d$ an open set. We solve the problem for many types of differential operators following a similar method as in the earlier paper of the same author for operators acting on spaces of distributions. We show for an operator $P(D)$ on the space of real analytic functions $\mathscr{A}(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ open convex, that it has real analytic parameter dependence if and only if its principal part $P_p(D)$ has a continuous linear right inverse on the space $C^\infty(\Omega)$ (or, equivalently, on $\mathscr{D}'(\Omega)$). In particular, the property does not depend on the set of parameters U. Surprisingly, in all solved non-quasianalytic cases, it follows that the solution is positive if and only if $P(D)$ has a linear continuous right inverse.

Keywords: analytic dependence on parameters, linear continuous right inverse, linear partial differential operator, convolution operator, linear partial differential equation with constant coefficients, space of real analytic functions, ultradifferentiable functions of Roumieu type, Gevrey classes, functor Proj^1, PLS-space, locally convex space, vector valued equation, solvability

1. Introduction

This paper is a continuation of the author's earlier paper [13] devoted to the problem of real analytic parameter dependence of solutions of linear partial differential equations over the space of distributions and ultradistributions of Beurling type. It is also related to our joint works [3] and [5] with Bonet. This time we consider the same problem for operators acting on the space of real analytic functions $\mathscr{A}(\Omega)$ or, more generally, of Roumieu ultradifferentiable functions $\mathscr{E}_{\{\omega\}}(\Omega)$.

The research of the author was supported in years 2007-2010 by Ministry of Science and Higher Education, Poland, grant no. NN201 2740 33. Part of the research was done during author's stay at Bergische Universität Wuppertal supported by Alexander von Humboldt Stiftung.

2010 Mathematics Subject Classification: primary: 35B30, 46E10; secondary: 35E20, 32U05, 46A63, 46A13, 46F05, 46M18

The best known spaces in the Roumieu scale are Gevrey classes $\Gamma_{\{p\}}$ (see [46]). For the motivation and history of the problem see the introduction of the paper [13] and, for instance, [11]. It is also explained there that in order to cover interesting examples we have to define real analytically dependent families $\lambda \mapsto f_\lambda \in X$ as those that for every linear continuous functional φ on X the function $\lambda \mapsto \varphi(f_\lambda) \in \mathbb{C}$ is real analytic.

We characterize (Theorem 3.3) arbitrary linear operators $T : \mathscr{E}_{\{\omega\}}(\Omega) \to \mathscr{E}_{\{\omega\}}(\Omega)$ such that there is an analytic parameter dependence of solutions u of the equation $Tu = f$ in terms of linear topological properties of the kernel of T. As we will see our analytic parameter dependence problem is nothing else but the problem of surjectivity of $T \otimes \mathrm{id}_{\mathscr{A}(U)}$, so we can apply the theory developed in [13]. Then using the Fundamental Principle and ideas of Meise, Taylor and Vogt, we solve the problem completely for many linear partial differential operators with constant coefficients. The most interesting result seems to be Theorem 6.1 without analogue in [13] which shows that $P(D) : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ open convex, has an analytic parameter dependence if and only if the principal part $P_p(D)$ has a linear continuous right inverse as an operator $P_p(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ (or, equivalently, on all non-quasianalytic classes $\mathscr{E}_{\{\omega\}}(\Omega)$). The latter condition is well evaluated and checked for plenty of concrete examples (see [32], [35], [36], [9], see also a summary in [13, Theorem A]) which implies interesting consequences for our problem, see Corollary 6.4 and 6.5. There is a surprising consequence that the problem does not depend in that case on the set U of parameters (Theorem 6.1). In general, the only important feature of U is if it has a non-compact connected component or not (Corollary 5.3). Thus we can always test the problem for $U = \mathbb{R}$ and $U = \mathbb{T}$.

We show that for operators $P(D)$ of two variables (Theorem 5.9), for homogeneous operators (Corollary 5.6) and for operators of order $\leqslant 2$ with $\omega(t) = o(t^{1/2})$ (Theorem 5.8), there is an analytic parameter dependence if and only if there is a continuous linear right inverse for $P(D)$ at least in the non-quasianalytic case. Moreover, we show that for elliptic operators there is never an analytic parameter dependence of solutions (Corollary 5.4). Summarizing, on non-quasianalytic classes $\mathscr{E}_{\{\omega\}}(\mathbb{R}^d)$ Laplace and heat operator never have an analytic parameter dependence while wave equation has it always. The proofs are analogous but not identical to [13], in order to avoid repetitions we often refer to that paper and point out only the differences in proofs.

2. Preliminaries

We could refer to the paper [13] but for the convenience of the reader we repeat the most important notions. Let us recall that Fréchet space means a complete metrizable locally convex space. An operator denotes a linear continuous map and by $L(X, Y)$ we denote the space of all such operators $T : X \to Y$. By Ω we always denote an open subset of $\mathbb{R}^d$ with a fixed but arbitrary compact exhaustion

$$K_1 \Subset K_2 \Subset \cdots \Subset K_N \Subset \cdots \Subset \Omega, \qquad \Omega = \bigcup_{N \in \mathbb{N}} K_N.$$

If Ω is convex then K_N will be chosen convex as well. Throughout the paper we always assume that real analytic manifolds are Hausdorff, paracompact and have countable bases of topology. $H(V)$ denotes the space of holomorphic functions on V.

By $B(x,r)$ we denote a ball of radius r and center x. If $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ then $|z| := \left(\sum_{j=1}^d |z_j|^2\right)^{1/2}$. By $\mathbb{T}$ we denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. The symbols $o(\cdot)$ and $O(\cdot)$ are understood as symbols at infinity.

Let us recall the notion of PLS-spaces which is a very convenient framework for our considerations. For a review on PLS-spaces see [12].

An *LS-space* (called also a DFS-space) is a locally convex inductive limit of a sequence of Banach spaces with compact linking maps or, equivalently, a strong dual of a Fréchet-Schwartz space. All LS-spaces are reflexive, Schwartz, separable and every closed subspace and separated quotient of an LS-space is LS as well. For instance, the space of distributions of compact support $\mathscr{E}'(\Omega)$ is an LS-space.

A projective limit of a sequence of LS-spaces is called a *PLS-space*. Every closed subspace and complete separated quotient of a PLS-space is PLS as well. The class of PLS-spaces is the smallest class of locally convex spaces containing Fréchet-Schwartz spaces, their duals and closed with respect of taking countable products and closed subspaces. PLS-spaces are separable, webbed and complete. They are reflexive if and only if they are barrelled or, equivalently, (ultra)bornological. Examples of PLS-spaces are spaces of holomorphic functions $H(V)$, smooth functions $C^\infty(\Omega)$, distributions $\mathscr{D}'(\Omega)$, real analytic functions $\mathscr{A}(U)$, ultradistributions in the sense of Beurling $\mathscr{D}'_{(\omega)}(\Omega)$, ultradifferentiable functions in the sense of Roumieu $\mathscr{E}_{\{\omega\}}(\Omega)$ and Köthe type PLS-spaces $\Lambda(A)$ (see [12]).

If instead of LS-spaces we take LN-spaces (i.e., locally convex inductive limits of sequences of Banach spaces with nuclear linking maps, called also DFN-spaces since they are strong duals of nuclear Fréchet spaces), we get PLN-spaces. All the above mentioned PLS-spaces except some Köthe spaces $\Lambda(A)$ are PLN-spaces.

If X is a PLS-space we write it throughout the paper as $X = \operatorname{proj}_{N\in\mathbb{N}} X_N$, where $X_N = \operatorname{ind}_{n\in\mathbb{N}} X_{N,n}$ is an LS-space, $(X_{N,n}, \|\cdot\|_{N,n})$ are Banach spaces, $i_N^{N+1} : X_{N+1} \to X_N$, $i_N : X \to X_N$ are linking maps. We denote

$$\|f\|^*_{N,n} := \sup_{\|x\|_{N,n}\leqslant 1} |f(x)|$$

the dual norm. The same conventions will be used for other PLS-spaces Y, E, etc.

The spaces mentioned above are described, for instance, in [3, pp. 333–336]. For the sake of completeness we recall the definition of $\mathscr{D}'_{(\omega)}(\Omega)$ and $\mathscr{E}_{\{\omega\}}(\Omega)$ from [8] since these spaces are extensively used in the paper.

We will use the classical multiindex notation for $\alpha = (\alpha_1, \ldots, \alpha_d)$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $f^{(\alpha)}(x) = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \cdots \cdots \partial^{\alpha_d} x_d}$. We introduce a *weight* $\omega : [0,\infty[\to [0,\infty[$ to be a continuous increasing function satisfying the following conditions:

(α) $\omega(2t) = O(\omega(t))$;
(β) $\omega(t) = O(t)$;

(γ) $\log t = o(\omega(t))$;
(δ) φ is a convex function, $\varphi(t) := \omega(e^t)$.

We extend ω to $\mathbb{C}^d$ by $\omega(z) := \omega(|z|)$. Now, we define the Roumieu class of ultradifferentiable functions $\mathscr{E}_{\{\omega\}}(\Omega)$ [8] as

$$\mathscr{E}_{\{\omega\}}(\Omega) := \{f \in C^\infty(\Omega) : \ \forall N \in \mathbb{N} \ \exists m \in \mathbb{N} : \|f\|_{N,m} < \infty\},$$

where

$$\|f\|_{N,m} := \sup_{x \in K_N} \sup_{\alpha \in \mathbb{N}^d} |f^{(\alpha)}(x)| \exp\left(-\frac{1}{m}\varphi^*\left(|\alpha| m\right)\right),$$
$$\varphi^*(t) := \sup_{x \geqslant 0}(xt - \varphi(t)) \qquad \text{the Young conjugate of } \varphi,$$

For $\omega(t) = t$ we get the space of real analytic functions $\mathscr{A}(\Omega)$. Its topology is given as

$$\mathscr{A}(\Omega) = \operatorname{proj}_{N \in \mathbb{N}} \ H(K_N),$$

where $H(K_N)$ is the space of germs of holomorphic functions over K_N,

$$H(K_N) := \operatorname{ind}_{n \in \mathbb{N}} \ H^\infty(U_{N,n}),$$

where $(U_{N,n})_{n \in \mathbb{N}}$ is a basis of complex neighbourhoods of K_N (see [30], [14], [12], [16]). Analogously, we define topology on $\mathscr{A}(\Omega)$ where Ω is a real analytic manifold. Both $\mathscr{A}(\Omega)$ and $\mathscr{E}_{\{\omega\}}(\Omega)$ are PLN-spaces (see [12] and [3, p. 335]). The space $\mathscr{A}(\Omega)$ is neither metrizable nor dual metrizable and it has a complicated structure: for instance, it has no Schauder basis [16].

If
$$\int_0^\infty \frac{\omega(t)}{1+t^2} dt = \infty$$
then the class (and the weight) is *quasianalytic* (i.e., there are no elements with compact support in $\mathscr{E}_{\{\omega\}}(\Omega)$). Otherwise the class is *non-quasianalytic.*

For $\mathscr{D}'_{(\omega)}(\Omega)$ we consider only non-quasianalytic weights , $\omega(t) = o(t)$, or $\omega(t) := \log(2 + |t|)$ which we also call weight in this context although it does not satisfy the condition (γ). If $\Omega \subseteq \mathbb{R}^d$ open then the class of *Beurling ultradistributions* $\mathscr{D}'_{(\omega)}(\Omega)$ is defined to be the strong dual of

$$\mathscr{D}_{(\omega)}(\Omega) := \left\{ f \in \mathscr{D}(\Omega) : \ \forall k \in \mathbb{N} \ \sup_{x \in \Omega} \sup_{\alpha \in \mathbb{N}^d} |f^{(\alpha)}(x)| \exp\left(-k\varphi^*\left(\frac{|\alpha|}{k}\right)\right) < \infty \right\}$$

equipped with the inductive limit topology (steps are Fréchet spaces):

$$\mathscr{D}_{(\omega)}(\Omega) = \operatorname{ind}_{N \in \mathbb{N}} \mathscr{D}_{(\omega)}(K_N), \qquad \mathscr{D}_{(\omega)}(K_N) := \{f \in \mathscr{D}_{(\omega)}(\Omega) : \operatorname{supp} f \subseteq K_N\}.$$

Thus $\mathscr{D}'_{(\omega)}(\Omega)$ is a PLN-space. We consider the weight $\omega(t) = \log(2 + |t|)$ since it gives $\mathscr{D}'_{(\omega)}(\Omega)$ to be the classical space of distributions $\mathscr{D}'(\Omega)$.

The family of classes $\mathscr{D}'_{(\omega)}$ and $\mathscr{E}_{\{\omega\}}$ is well adapted to Fourier transform but it contains also most of the Denjoy-Carleman style classes defined via growth conditions of Taylor coefficients controlled by sequences (M_p), see [6] and [24]. The best known among them are Gevrey classes (see, for instance, [46]): $\Gamma_{\{p\}} = \mathscr{E}_{\{\omega\}}$ or $\gamma_{(p)} = \mathscr{E}_{(\omega)}$ for $\omega(t) = t^{1/p}$ for $p > 1$.

We will use only PLS-type Köthe sequence spaces with ℓ_∞-norms (and call them *Köthe type PLS-spaces of infinite order* $\Lambda^\infty(A)$) as defined, for instance, in [3], where the definition of PLS-type power series spaces $\Lambda_{r,s}(\alpha, \beta)$ is also given.

Let us recall the definition of the ε-product of Schwartz for complete X, E:

$$X\varepsilon E := L_e(X'_{co}, E)$$

the space of linear continuous operators from the dual X' equipped with the compact-open topology. The subscript e means that we equip $L(X'_{co}, E)$ with the topology of uniform convergence on equicontinuous sets. It is important that ε-product is commutative. Clearly $X \otimes E \subseteq X\varepsilon E$ and if one of the spaces is nuclear and both are complete then $X \otimes E$ is dense in $X\varepsilon E$ and the latter is the unique natural completion of $X \otimes E$! If X, E are PLS-spaces then $X\varepsilon E = L'_b(X', E)$ and if X is ultrabornological then by [15, Prop. 4.3] the space $X\varepsilon E$ is a PLS-space.

It is well known that for complete X spaces of vector valued functions have the following tensor representations (for $\mathscr{A}$ see [1, Th. 16]):

$$C^\infty(U, X) = C^\infty(U)\varepsilon X, \qquad H(U, X) = H(U)\varepsilon X, \qquad \mathscr{A}(U, X) = \mathscr{A}(U)\varepsilon X.$$

For vector valued distributions ε-product is just the definition:

$$\mathscr{D}'(U, X) := \mathscr{D}'(U)\varepsilon X, \quad \mathscr{D}'_{(\omega)}(U, X) := \mathscr{D}'_{(\omega)}(U)\varepsilon X, \quad \mathscr{E}_{\{\omega\}}(U, X) := \mathscr{E}_{\{\omega\}}(U)\varepsilon X.$$

If $T : X \to Y$ and $S : E \to F$ then $T \otimes S : X\varepsilon E \to Y\varepsilon F$ is uniquely defined on $X \otimes E$ as the tensor of T and S and generally on $X\varepsilon E$ as $(T \otimes S)(V) := S \circ V \circ T'$.

We will use so-called dual interpolation estimates and P-properties which are important linear topological invariant for PLS-spaces, generalizations of (Ω)-type and dual (DN)-type properties introduced by Vogt and Zahariuta and were defined for the first time in [3], [4], [5]. A lot of information below is taken from [4] and [13].

A PLS-space X has a *dual interpolation estimate* if

$$\begin{gathered} \forall N\ \exists M \geqslant N\ \forall K \geqslant M\ \exists n\ \forall m \geqslant n\ \forall \theta \in]0,1[\ \exists k \geqslant m, C\ \forall x \in X'_N \\ \|x \circ i_N^M\|^*_{M,m} \leqslant C \left(\|x \circ i_N^K\|^*_{K,k}\right)^{1-\theta} \left(\|x\|^*_{N,n}\right)^{\theta}. \end{gathered} \tag{1}$$

If we replace the quantifier $\forall \theta \in]0,1[$ by $\exists \theta_0 \in]0,1[\ \forall \theta \in]0,\theta_0[$ (or $\exists \theta_0 \in]0,1[\ \forall \theta \in]\theta_0,1[$) we get the definition of the *dual interpolation estimate for small (big) theta*.

Using the proof of [3, Lemma 5.1] we get the following reformulation of the dual interpolation estimate:

$$\forall N\ \exists M \geqslant N\ \forall K \geqslant M\ \exists n\ \forall m \geqslant n\ \forall \theta \in]0,1[\ \exists k \geqslant m, C\ \forall r > 0\ \forall x \in X'_N :$$
$$\|x \circ i_N^M\|^*_{M,m} \leqslant C \left(r^\theta \|x \circ i_N^K\|^*_{K,k} + \frac{1}{r^{1-\theta}} \|x\|^*_{N,n} \right).$$

Clearly taking θ close to 0 or 1 we get the dual interpolation estimate for small or big θ. If we consider only $r \geqslant 1$ then we get the condition $(P\overline{\overline{\Omega}})$ (for all θ or, equivalently, θ close to 0) or $(P\Omega)$ (for θ close to 1 or, equivalently, for some $\theta \in]0,1[$). Analogously, if we consider only $0 < r \leqslant 1$ then we get the condition (PA) (for all θ or, equivalently, θ close to 1) or $(P\underline{A})$ (for θ close to 0 or, equivalently, for some $\theta \in]0,1[$).

The dual interpolation estimate for small θ is equivalent to the combination of $(P\overline{\overline{\Omega}})$ and $(P\underline{A})$, for big θ is equivalent to $(P\Omega)$ and (PA). Moreover, $(P\overline{\overline{\Omega}})$ implies $(P\Omega)$ and (PA) implies $(P\underline{A})$. Thus the dual interpolation estimate for all θ is equivalent to the combination of $(P\overline{\overline{\Omega}})$ and (PA).

There are also reformulations of these P-conditions in the similar spirit as in the definition of the dual interpolation estimate. For instance, $(P\Omega)$ is equivalent to

$$\forall N \ \exists M \geqslant N \ \forall K \geqslant M \ \exists n \ \forall m \geqslant n \ \exists \theta \in]0,1[\ \exists k \geqslant m, C \ \forall x \in X'_N :$$
$$\|x \circ i_N^M\|^*_{M,m} \leqslant C \left(\|x \circ i_N^K\|^*_{K,k}\right)^{1-\theta} \cdot \max\left(\|x\|^*_{N,n}, \|x \circ i_N^K\|^*_{K,k}\right)^{\theta}.$$

If we take $\forall \theta \in]0,1[$ then we get $(P\overline{\overline{\Omega}})$. Analogously, $(P\underline{A})$ is equivalent to

$$\forall N \ \exists M \geqslant N \ \forall K \geqslant M \ \exists n \ \forall m \geqslant n \ \exists \theta \in]0,1[\ \exists k \geqslant m, C \ \forall x \in X'_N :$$
$$\|x \circ i_N^M\|^*_{M,m} \leqslant C \max\left(\|x \circ i_N^K\|^*_{K,k}, \|x\|^*_{N,n}\right)^{1-\theta} \left(\|x\|^*_{N,n}\right)^{\theta}$$

and if we take $\forall \theta \in]0,1[$ we get (PA). All these conditions are inherited by countable products and complete quotients. In the papers [3], [5] and [4] many examples of spaces with P-properties are collected — we summarize them for the sake of convenience repeating [13, Cor. 2.3].

Corollary 2.1.

(a) *A Fréchet space has a dual interpolation estimate for big θ iff it has (Ω). It has a dual interpolation estimate for small (all) θ iff it has $(\overline{\overline{\Omega}})$.*

(b) *An LS-space has a dual interpolation estimate for small θ iff its dual has $(\underline{DN})$. It has a dual interpolation estimate for big (all) θ iff its dual has (DN).*

(c) *The space of distributions $\mathscr{D}'(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ arbitrary open, or the space of Beurling ultradistributions $\mathscr{D}'_{(\omega)}(\Omega)$ has the dual interpolation estimate for all θ.*

(d) *The space of real analytic functions $\mathscr{A}(\Omega)$, Ω a real analytic manifold, or the space of Roumieu ultradifferentiable functions $\mathscr{E}_{\{\omega\}}(\Omega)$ (ω non-quasianalytic, $\Omega \subseteq \mathbb{R}^d$ arbitrary, or ω quasianalytic with property (α_1) and convex $\Omega \subseteq \mathbb{R}^d$) has the dual interpolation estimate for small θ.*

(e) *The PLS-type power series space $\Lambda_{r,s}(\alpha,\beta)$ has the dual interpolation estimate for big θ iff $s = \infty$ or it is a Fréchet space. It has the dual interpolation estimate for small θ (or $(P\overline{\overline{\Omega}})$) iff it is an LS-space.*

Remark. The condition (α_1) means that

$$\sup_{\lambda\geqslant 1}\limsup_{t\to\infty}\frac{\omega(\lambda t)}{\lambda\omega(t)}<\infty.$$

For non-explained notions from functional analysis see [40] or/and [22]. For the theory of tensor products see [22]. For applications of homological algebra and derived functors to partial differential equations and functional analysis see [53]. For the theory of plurisubharmonic functions (pluripotential theory) see [23]. For linear partial differential equations see [21], [42].

3. Parameter dependence of solutions and surjectivity of tensorized maps

Let us assume that

$$T:\mathscr{E}_{\{\omega\}}(\Omega)\to\mathscr{E}_{\{\omega\}}(\Omega)$$

is a surjective operator and U is a real analytic manifold. We say that the equation

$$Tu=f,$$

where u is an unknown function has the *real analytic parameter dependence (of solutions)* if for every vector valued real analytic function

$$U\ni\lambda\mapsto f_\lambda\in\mathscr{E}_{\{\omega\}}(\Omega)$$

there is a vector valued real analytic function

$$U\ni\lambda\mapsto u_\lambda\in\mathscr{E}_{\{\omega\}}(\Omega)$$

such that

$$Tu_\lambda=f_\lambda\qquad\text{for every }\lambda\in U.$$

If $\lambda\mapsto f_\lambda, u_\lambda$ are ultradifferentiable $\mathscr{E}_{\{\omega\}}$ then we can speak about Roumieu ultradifferentiable parameter dependence. Let us recall that a vector valued function $F:U\to X$ is analytic ($\mathscr{E}_{\{\omega\}}$ ultradifferentiable etc.) if and only if for any $\varphi\in X'$, $\varphi\circ F$ is analytic ($\mathscr{E}_{\{\omega\}}$ ultradifferentiable etc.).

Recall that for complete locally convex space X:

$$\mathscr{A}(U,X):=\{f:U\to X:\ \ \forall\, g\in X'\ \ g\circ f\in\mathscr{A}(U)\}$$

is naturally isomorphic to $\mathscr{A}(U)\varepsilon X$.

As shown in [25], $\mathscr{A}(U,X)$ contains functions for which vector valued Taylor series is not convergent. In [1], [2] a characterization is given for which Fréchet spaces X every function $F\in\mathscr{A}(U,X)$ has a Taylor series local representation.

Let $T:\mathscr{E}_{\{\omega\}}(\Omega)\to\mathscr{E}_{\{\omega\}}(\Omega)$ be an arbitrary operator. From the considerations above (see [13, Sec. 3]) it follows that the equation $Tu=f$ has the real analytic parameter dependence of solutions if and only if

$$T\otimes\mathrm{id}\,:\mathscr{E}_{\{\omega\}}(\Omega)\varepsilon\mathscr{A}(U)\to\mathscr{E}_{\{\omega\}}(\Omega)\varepsilon\mathscr{A}(U)$$

is surjective. An analogous conclusion holds for smooth, holomorphic, ultradifferentiable, (ultra)distributional dependence for operators on spaces of real analytic, smooth, ultradifferentiable, holomorphic functions or (ultra)distributions.

It is explained in [13, Sec. 4] that surjectivity of tensorized maps can be handled by the homological approach. For reader's convenience we shortly recall this approach.

Let $V = \mathrm{proj}_{N\in\mathbb{N}} V_N$ be a projective limit of linear spaces. We construct the following exact sequence

$$0 \longrightarrow V \xrightarrow{\ i\ } \prod_{N\in\mathbb{N}} V_N \xrightarrow{\ \sigma\ } \prod_{N\in\mathbb{N}} V_N,$$

where i is the canonical embedding and $\sigma((x_n)_{N\in\mathbb{N}}) := (i_N^{N+1} x_{N+1} - x_N)_{N\in\mathbb{N}}$ and $i_N^{N+1} : V_{N+1} \to V_N$ as usual denote linking maps. We define

$$\mathrm{Proj}^1 \left((V_N, i_N^{N+1})_{N\in\mathbb{N}}\right) := \prod_{N\in\mathbb{N}} V_N / \operatorname{im} \sigma.$$

That functor was introduced to functional analysis by Palamodov [43], [44], and analyzed in depth by Vogt [49], [50], Frerick [18], Frerick and Wengenroth [19] and others, see a detailed exposition in [53].

If we assume that (V_N) are complete and the spectrum is *strongly reduced* (i.e., canonical images of V in V_N are dense) then all such spectra are equivalent and we can write $\mathrm{Proj}^1\, V$ since it does not depend on such a spectrum representing V. If V is a PLS-space then it is a strongly reduced projective limit of LS-spaces.

Theorem 3.1 ([13, Cor. 4.6]). *Let $S : X \to Y$, $T : E \to E$ be continuous open surjective operators between PLS-spaces. Assume that either E or* $\ker T$ *and* $\ker S$ *are PLN-spaces or nuclear Fréchet spaces. Then*

$$S \otimes T : X\varepsilon E \to Y\varepsilon E$$

is surjective whenever

$$\mathrm{Proj}^1\, (\ker S\varepsilon E) = \mathrm{Proj}^1\, (Y\varepsilon \ker T) = 0.$$

If, additionally, $\mathrm{Proj}^1\, (X\varepsilon E) = \mathrm{Proj}^1\, (Y\varepsilon E) = 0$ *then this is also a necessary condition.*

Since by [45, Prop. 4], every PLS-space with the dual interpolation estimate is so-called deeply reduced then, by [13, Cor. 5.6], [5, Th. 4.1, Th. 3.4] we get the following conclusion:

Theorem 3.2. *Let X, E be PLS-spaces and let one of them be either a PLN-space or a nuclear Fréchet space or a Köthe type PLS-space of infinite order $\Lambda^\infty(A)$. If both E and X have the dual interpolation estimate for small (big) θ then $L_b(E'_b, X) = X\varepsilon E$ is an ultrabornological space and* $\mathrm{Proj}^1\, (X\varepsilon E) = 0$.

The results above imply the following characterization of surjectivity of tensorized maps.

Theorem 3.3. *Let* $T : \mathscr{E}_{\{\omega\}}(\Omega) \to \mathscr{E}_{\{\omega\}}(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ *open, be a surjective operator and let* ω *be either non-quasianalytic or* ω *have the property* (α_1) *and* Ω *be convex. Then:*

(a) *The operator*

$$T \otimes \mathrm{id}_{\mathscr{A}(U)} : \mathscr{E}_{\{\omega\}}(\Omega)\varepsilon\mathscr{A}(U) \to \mathscr{E}_{\{\omega\}}(\Omega)\varepsilon\mathscr{A}(U)$$

is surjective if and only if ker T *has the dual interpolation estimate for small* θ *(for* U *having a non-compact connected component) or if and only if* ker T *has* $(P\overline{\overline{\Omega}})$ *(for* U *having only compact connected components).*

(b) *The operator*

$$T \otimes \mathrm{id}_{\mathscr{E}_{\{\omega_1\}}(\Omega_1)} : \mathscr{E}_{\{\omega\}}(\Omega)\varepsilon\mathscr{E}_{\{\omega_1\}}(\Omega_1) \to \mathscr{E}_{\{\omega\}}(\Omega)\varepsilon\mathscr{E}_{\{\omega_1\}}(\Omega_1)$$

is surjective for ω_1 *non-quasianalytic,* $\Omega_1 \subseteq \mathbb{R}^n$ *open, if and only if* ker T *has* $(P\overline{\overline{\Omega}})$.

(c) *The operator*

$$T \otimes \mathrm{id}_{\mathscr{D}'_{(\omega_1)}(\Omega_1)} : \mathscr{E}_{\{\omega\}}(\Omega)\varepsilon\mathscr{D}'_{(\omega_1)}(\Omega_1) \to \mathscr{E}_{\{\omega\}}(\Omega)\varepsilon\mathscr{D}'_{(\omega_1)}(\Omega_1)$$

is surjective for ω_1 *non-quasianalytic,* $\Omega_1 \subseteq \mathbb{R}^n$ *open, if and only if* ker T *has* $(P\Omega)$.

Proof. (a): By Theorem 3.2 and Corollary 2.1,

$$\mathrm{Proj}^1 \left(\mathscr{E}_{\{\omega\}}(\Omega)\varepsilon\mathscr{A}(U)\right) = 0.$$

The conclusion follows from Theorem 3.1 and [13, Cor. 6.1].

(b) and (c): By Theorem 3.2 and Corollary 2.1,

$$\mathrm{Proj}^1 \left(\mathscr{E}_{\{\omega\}}(\Omega)\varepsilon\mathscr{E}_{\{\omega_1\}}(\Omega_1)\right) = \mathrm{Proj}^1 \left(\mathscr{E}_{\{\omega\}}(\Omega)\varepsilon\mathscr{D}'_{(\omega_1)}(\Omega_1)\right) = 0.$$

The conclusion follows by Theorem 3.1 and [13, Cor. 6.2]. ■

Remarks.

(a) Since for an arbitrary real analytic manifold Ω the space $\mathscr{A}(\Omega)$ has the dual interpolation estimate for small θ the whole result applies also to $\mathscr{A}(\Omega)$ instead of $\mathscr{E}_{\{\omega\}}(\Omega)$.

(b) Langenbruch has pointed out to the author that in [27, Cor. 6.8] a characterization of surjectivity of

$$P(D) : \mathscr{E}_{\{\omega\}}(\Omega)\varepsilon\mathscr{E}_{\{\omega\}}(\mathbb{R}) \to \mathscr{E}_{\{\omega\}}(\Omega)\varepsilon\mathscr{E}_{\{\omega\}}(\mathbb{R})$$

was given for some linear partial differential operators with constant coefficients

$$P(D) : \mathscr{E}_{\{\omega\}}(\Omega) \to \mathscr{E}_{\{\omega\}}(\Omega).$$

Using known descriptions of kernels of convolution operators we can prove the following characterizations parallel to [13, Cor. 6.7]:

Corollary 3.4 (comp. [13, Cor. 6.7]). *Let ω be a non-quasianalytic weight and $\mu \in \mathscr{D}'_{\{\omega\}}(\mathbb{R})$ be an ultradistribution with compact support. The convolution operator*

$$T_\mu : \mathscr{E}_{\{\omega\}}(\mathbb{R}, \mathscr{A}(U)) \to \mathscr{E}_{\{\omega\}}(\mathbb{R}, \mathscr{A}(U)) \tag{2}$$

is surjective if and only if

$$T_\mu : \mathscr{E}_{\{\omega\}}(\mathbb{R}) \to \mathscr{E}_{\{\omega\}}(\mathbb{R}) \tag{3}$$

has a linear continuous right inverse.

The same result holds for a convolution operators

$$T_\mu : \mathscr{A}(I) \to \mathscr{A}(I)$$

and $\mu \in \mathscr{A}(I)'$, where $I = \mathbb{R}$ or I is an open interval and $\operatorname{supp}\mu = \{0\}$.

Remark. A characterization of existence of right inverse for T_μ in terms of $\hat{\mu}$ is given in [39, Th. 3.9, 2.1, 4.4, Prop. 4.6] and [26].

Proof. Sufficiency is obvious.

Necessity for ultradifferentiable functions. A characterization of surjectivity of (3) is given in [10]. If (2) is surjective then (3) must be surjective. Then, by [31, Th. 3.6], $\ker T_\mu \subseteq \mathscr{E}_{\{\omega\}}(\mathbb{R})$, $\dim \ker T_\mu = \infty$, is isomorphic to $\Lambda_{\infty,0}(\alpha, \beta)$, where $\alpha_j = |\operatorname{Im} a_j|$, $\beta_j = \omega(a_j)$ and (a_j) is a sequence with multiplicities of zeros of the Fourier transform $\hat{\mu}$ of μ. By Theorem 3.3 and Corollary 2.1, $\ker T_\mu$ is an LS-space. This completes the proof by [39, Th. 4.4, Prop. 4.6].

Necessity for the real analytic case. By [41, 2.11], $\ker T_\mu \simeq \Lambda_{\infty,0}(\alpha, \beta)$ for $I = \mathbb{R}$, where α, β as above. By Theorem 3.3 and Cor. 2.1, $\ker T_\mu$ is an LS-space. By surjectivity of T_μ, $\hat{\mu}$ is "slowly decreasing" see [41, Theorem]. The two conditions together imply the existence of a continuous linear right inverse by [26]. The case of I instead of $\mathbb{R}$ can be proved similarly (here $\ker T_\mu \simeq \Lambda_{0,0}(\alpha, \beta)$). ■

Unfortunately, we cannot check interpolation estimates for general operators or even differential operators with variable coefficients. In case of constant coefficients differential operators we may apply the Fundamental Principle of Ehrenpreis and Palamodov in a way similar to [13]. This is the content of the next section.

4. Evaluation of the dual interpolation estimates via Phragmén-Lindelöf type conditions

First we fix some notation as in [13]. Assume from now on that V is an algebraic variety, $\Omega \subseteq \mathbb{R}^d$ is a convex open set and ω is a weight (in the sense of definition of the space $\mathscr{E}_{\{\omega\}}(\Omega)$). We fix a convex compact exhaustion (K_N) of Ω. We define $PSH(V)$ to be the set of plurisubharmonic functions $u : V \to [-\infty, \infty[$, i.e., locally bounded from above and plurisubharmonic at regular points of V. We assume that the values in singular points of V are defined by $u(z) := \limsup_{\xi \in V_{reg}, \xi \to z} u(\xi)$ for $z \in V_{sing}$. For a convex compact set $\emptyset \neq K \subseteq \mathbb{R}^d$ we define its support function

$$h_K : \mathbb{R}^d \to \mathbb{R}, \qquad h_K(\xi) := \sup\{\langle \xi, x\rangle : x \in K\},$$

moreover, for any $N \in \mathbb{N}$ we define $h_N := h_{K_N}$.

Now, we introduce a family of Phragmén-Lindelöf type conditions modifying the conditions in [13] by adding one more parameter. Let $R \in \mathbb{R} \cup \{\infty\}$, then

$$\mathrm{PSH}(V, N, R) := \{u \in \mathrm{PSH}(V) : \forall s < R\, \exists C_0\, \forall z \in V \;\; u(z) \leqslant h_N(\operatorname{Im} z) - s\omega(z) + C_0\}.$$

We say that the variety V satisfies $\mathrm{IPL}(\Omega, \omega, R)$ if and only if

$$\begin{aligned} &\forall N \;\exists M \geqslant N \;\forall K \geqslant M \;\exists n < R \;\forall m < R \\ &\qquad \forall \theta \in]0,1[\;\exists k < R, C \;\forall t \in \mathbb{R} \;\forall u \in \mathrm{PSH}(V, N, R) \qquad (a) + (b) \Rightarrow (c), \end{aligned}$$

where

(a) $\forall z \in V \quad u(z) \leqslant h_N(\operatorname{Im} z) - n\omega(z) + t;$
(b) $\forall z \in V \quad u(z) \leqslant h_K(\operatorname{Im} z) - k\omega(z);$
(c) $\forall z \in V \quad u(z) \leqslant h_M(\operatorname{Im} z) - m\omega(z) + \theta t + C.$

If all the conditions are considered only for $t \geqslant 0$ (or $t \leqslant 0$) we write $\mathrm{IPL}_+(\Omega, \omega, R)$ (or $\mathrm{IPL}_-(\Omega, \omega, R)$) and say that V satisfies the IPL condition for *positive* (*negative*) t. Similarly, if we replace the quantifier $\forall \theta \in]0,1[$ by $\exists \theta_0 \in]0,1[\;\forall \theta \in]0,\theta_0[$ we say that IPL is satisfied for *small* θ and denote it by $\mathrm{IPL}^0(\Omega, \omega, R)$. Analogously, if we put $\exists \theta_0 \in]0,1[\;\forall \theta \in]\theta_0, 1[$ then we call the condition for *big* θ and denote it by $\mathrm{IPL}^1(\Omega, \omega, R)$. Of course, all the possible combinations of subscripts and superscripts are possible so we have defined nine versions of IPL.

It is easy to observe that $\mathrm{IPL}_+(\Omega, \omega, R)$ is equivalent to $\mathrm{IPL}^0_+(\Omega, \omega, R)$. Similarly, $\mathrm{IPL}_-(\Omega, \omega, R)$ is equivalent to $\mathrm{IPL}^1_-(\Omega, \omega, R)$, hence $\mathrm{IPL}(\Omega, \omega, R)$ is equivalent to the combination of $\mathrm{IPL}^0_+(\Omega, \omega, R)$ and $\mathrm{IPL}^1_-(\Omega, \omega, R)$. Clearly we are interested only in $R = 0$ or ∞. For $R = 0$ we consider all weights ω such that $\log(2 + |z|) = o(\omega(z))$, $\omega(z) = O(z)$ (so including $\omega(z) = |z|$). For $R = \infty$ we consider all non-quasianalytic weights ω such that $\log(2 + |z|) = o(\omega(z))$ or $\omega(z) = \log(2 + |z|)$ corresponding to the space of classical distributions $\mathscr{D}'(\Omega)$. In all cases we may assume $\omega(0) \geqslant 1$. The versions with $R = \infty$ were studied in [13], where they were denoted by $IPL^\alpha_\beta(\Omega, \omega)$. Now, we concentrate on $R = 0$.

The inspiration of these conditions is explained in [13, Sec. 7], similarly as there we apply many ideas from [38]. We will show that in many cases everything true for the right inverse is true for the real analytic parameter dependence (comp. Problem 7.1).

Now, we formulate the main result of this section which is an analogue of [13, Th. 7.1] for the Roumieu classes. Let us recall that for $\omega(z) = |z|$ the space $\mathscr{E}_{\{\omega\}}$ is just the space of real analytic functions $\mathscr{A}$, so the results cover also the case $\mathscr{E}_{\{\omega\}}(\Omega) = \mathscr{A}(\Omega)$.

Theorem 4.1. *Let $\Omega \subseteq \mathbb{R}^d$ be open convex and let $V = \{z : P(-z) = 0\}$ be the zero variety of the polynomial P. Let $P(D) : \mathscr{E}_{\{\omega\}}(\Omega) \to \mathscr{E}_{\{\omega\}}(\Omega)$ be a linear partial differential operator with constant coefficients.*

(i) $\ker P(D)$ *has the dual interpolation estimate if and only if V satisfies* $\mathrm{IPL}(\Omega, \omega, 0)$.
(ii) $\ker P(D)$ *has the dual interpolation estimate for small θ (big θ) if and only if V has* $\mathrm{IPL}^0(\Omega, \omega, 0)$ *(* $\mathrm{IPL}^1(\Omega, \omega, 0)$ *).*
(iii) $\ker P(D)$ *has* $(P\overline{\overline{\Omega}})$ *if and only if V has* $\mathrm{IPL}_+(\Omega, \omega, 0)$ *or, equivalently,* $\mathrm{IPL}^0_+(\Omega, \omega, 0)$.
(iv) $\ker P(D)$ *has* $(P\Omega)$ *if and only if V has* $\mathrm{IPL}^1_+(\Omega, \omega, 0)$.
(v) $\ker P(D)$ *has* (PA) *if and only if V has* $\mathrm{IPL}_-(\Omega, \omega, 0)$ *or, equivalently,* $\mathrm{IPL}^1_-(\Omega, \omega, 0)$.
(vi) *if* $\ker P(D)$ *has* $(P\underline{A})$ *then V has* $\mathrm{IPL}^0_-(\Omega, \omega, 0)$.

First, exactly as in [13, Lemma 7.2], we reduce the result to irreducible varieties (we omit the proof).

Lemma 4.2. *An algebraic variety has one of the* IPL *properties if and only if each of its irreducible component has the same property.*

Again exactly as [13, Lemma 7.4] we prove:

Lemma 4.3. *Let P be a polynomial, $P = Q_1 \cdot \dots \cdot Q_n$ its decomposition into irreducible factors. The space* $\ker P(D) \subseteq \mathscr{E}_{\{\omega\}}(\Omega)$ *has one of the conditions* $(P\overline{\overline{\Omega}})$, $(P\Omega)$, (PA), *the dual interpolation estimate for all (big, small) θ if and only if every space* $\ker Q_j(D) \subseteq \mathscr{E}_{\{\omega\}}(\Omega)$ *has the same property for $j = 1, \dots, n$.*

Please observe that as in [13] necessity holds also for $(P\underline{A})$.

Finally, we need the following approximation result of Meise, Taylor and Vogt [33, Th. 5.1] and Franken [17, Th. 8]. In fact, their formulation is much more complicated and based on the stronger assumption $|u(z)| \leqslant L|z|$ — the result in the form below was given and proved in [13, Th. 7.5].

Theorem 4.4. *For any algebraic variety $V \subseteq \mathbb{C}^d$ there is a constant C_V such that for any $u \in \mathrm{PSH}(V)$ such that*

$$|u(z)| \leqslant L|z| + L \qquad \text{for } z \in V \tag{4}$$

there exists for every point $z \in V$ an entire function $f_z : \mathbb{C}^d \to \mathbb{C}$ such that

(a) $u(z) \leqslant \log|f_z(\zeta)| + L^2 + C_V \log(2 + |z|)$ *for some* $\zeta \in V$, $|\zeta - z| \leqslant 2$;
(b) $\forall \zeta \in V \ \exists \zeta' \in V \ \log|f_z(\zeta)| \leqslant u(\zeta') + C_V \log(2 + |\zeta|)$ *and* $|\zeta - \zeta'| \leqslant 1$.

Proof of Theorem 4.1. The proof is similar to that of [13, Th. 7.1]. We just prove the part (i) — the other parts are completely analogous. By Lemmas 4.2 and 4.3 we may restrict our attention to irreducible polynomials P. Without loss of generality we may assume that $0 \in K_1 \subset \Omega$.

By the Fundamental Principle for $P(D):\mathscr{E}_{\{\omega\}}(\Omega)\to\mathscr{E}_{\{\omega\}}(\Omega)$ (see [47, Satz 2.19]) we have

$$(\ker P(D))' = \{f\in H(V):\ \exists N\in\mathbb{N}\ \forall n\in\mathbb{N}:\quad \|f\|_{N,-\frac{1}{n}}<\infty\},$$

where

$$\|f\|_{N,-\frac{1}{n}} := \sup_{z\in V}|f(z)|\exp\left(-h_N(\operatorname{Im} z)-\frac{\omega(z)}{n}\right).$$

Moreover, $\ker P(D)=\operatorname{proj}_{N\in\mathbb{N}} X_N$ where X_N are LN-spaces and

$$X_N' = \{f\in H(V):\ \forall n\in\mathbb{N}\quad \|f\|_{N,-\frac{1}{n}}<\infty\}.$$

The dual interpolation estimate for $\ker P(D)\subseteq\mathscr{E}_{\{\omega\}}(\Omega)$ means that

$$\begin{gathered}\forall N'\ \exists M'\ \forall K'\ \exists n'\ \forall m',\theta\in]0,1[\ \exists k',C'\ \forall r>0\ \forall f\in X_N'\\ \frac{1}{r^\theta}\|f\|_{M',-\frac{1}{m'}}\leqslant C'\left(\|f\|_{K',-\frac{1}{k'}}+\frac{1}{r}\|f\|_{N',-\frac{1}{n'}}\right).\end{gathered}\tag{5}$$

Taking logarithms on both sides and $t=\log r$ we get $\mathrm{IPL}(\Omega,\omega,0)$ with the above preamble of quantifiers but only for u of the form $u(z)=\log|f(z)|\in\mathrm{PSH}(V,N,0)$, $f\in H(V)$. So it suffices to show that if $\mathrm{IPL}(\Omega,\omega,0)$ holds for such holomorphic functions then we have the same condition for $u\in\mathrm{PSH}(V)$.

Let us recall that there are constants A and D such that

$$\begin{aligned}\omega(x+1)&\leqslant A\omega(x)+A &&\text{for all } x\in\mathbb{R}_+,\\ \omega(z)&\leqslant D|z|+D &&\text{for suitable } D>0.\end{aligned}\tag{6}$$

and we may assume that $\log(2+|z|)\leqslant o(\omega(z))$ for any z, thus

$$C_V\log(2+|\zeta|)\leqslant B(\varepsilon)+\varepsilon\omega(|\zeta|).$$

We take arbitrary N, $N':=N$, find M' from (5), $M:=M'$, $K=K'$ arbitrary. We find n' from (5) and take $n=2n'A$, choose m arbitrary, $m':=2mA^2$. Finally we fix θ and find k', C' by (5), $k:=2k'A$.

For $t\geqslant 0$ we take $u\in\mathrm{PSH}(V,N,0)$ such that for $z\in V$ we have:

$$u(z)\leqslant h_N(\operatorname{Im} z)+\frac{\omega(z)}{n}+t,\tag{7}$$

$$u(z)\leqslant h_K(\operatorname{Im} z)+\frac{\omega(z)}{k}.\tag{8}$$

If

$$|h_N(\operatorname{Im} z)|\leqslant|h_K(\operatorname{Im} z)|+\omega(z)\leqslant L|z|+L$$

we define

$$\tilde u(z):=\max(0,u(z)).$$

Therefore, by (6),

$$0 \leqslant \tilde{u}(z) \leqslant h_K(\operatorname{Im} z) + \omega(z) \leqslant L|z| + L$$

for suitable $L > 0$.

Clearly, $\tilde{u} \in \mathrm{PSH}(V, N, 0)$ since $u \in \mathrm{PSH}(V, N, 0)$. For $z \in V$ and $t \geqslant 0$ we get by (7), (8):

$$\tilde{u}(z) \leqslant h_N(\operatorname{Im} z) + \frac{\omega(z)}{n} + t \tag{9}$$

and

$$\tilde{u}(z) \leqslant h_K(\operatorname{Im} z) + \frac{\omega(z)}{k}. \tag{10}$$

We apply to the function $\tilde{u}$ Theorem 4.4, then we find entire functions f_z. Let us take $\zeta \in V$, then, by (9) and (6) for $C_N := \max_{|x|\leqslant 1} h_N(x)$ and for suitable small ε:

$$\begin{aligned}\log|f_z(\zeta)| &\leqslant \max\{\tilde{u}(\zeta') : |\zeta' - \zeta| \leqslant 1\} + C_V \log(2 + |\zeta|) \\ &\leqslant h_N(\operatorname{Im}\zeta) + \max_{|x|\leqslant 1} h_N(x) + \frac{1}{n}\omega(|\zeta| + 1) + t + C_V \log(2 + |\zeta|) \\ &\leqslant h_N(\operatorname{Im}\zeta) + C_N + \frac{1}{n}\left(A\omega(\zeta) + A\right) + t + \varepsilon\omega(\zeta) + B(\varepsilon) \\ &\leqslant h_N(\operatorname{Im}\zeta) + \frac{1}{n'}\omega(\zeta) + t + C_N + B(\varepsilon) + \frac{A}{n}.\end{aligned}$$

On the other hand, (10) implies:

$$\log|f_z(\zeta)| \leqslant h_K(\operatorname{Im}\zeta) + \frac{1}{k'}\omega(\zeta) + C_K + B(\varepsilon) + \frac{A}{k}.$$

Analogously, since $\tilde{u} \in \mathrm{PSH}(V, N, 0)$ we get also $\log|f_z| \in \mathrm{PSH}(V, N, 0)$. We apply IPL for logarithms of holomorphic functions to get

$$\begin{aligned}\log|f_z(\zeta)| \leqslant{}& h_M(\operatorname{Im}\zeta) + \frac{1}{m'}\omega(\zeta) + \theta t \\ &+ \max\left(C_N + B(\varepsilon) + \frac{A}{n}, C_K + B(\varepsilon) + \frac{A}{k}\right) + C'.\end{aligned}$$

Let us take $z \in V$, then by (a) in Theorem 4.4, for

$$C'' := \max\left(C_N + B(\varepsilon) + \frac{A}{n}, C_K + B(\varepsilon) + \frac{A}{k}\right) + C' + L^2$$

and some $\zeta \in V$, $|\zeta - z| \leqslant 2$ we get

$$\begin{aligned} u(z) &\leqslant \log|f_z(\zeta)| + L^2 + C_V \log(2+|z|) \\ &\leqslant h_M(\operatorname{Im} z) + \max_{|x|\leqslant 2} h_M(x) + \frac{1}{m'}\omega(|z|+2) + \theta t + C'' + C_V \log(2+|z|) \\ &\leqslant h_M(\operatorname{Im} z) + 2C_M + \frac{1}{m'}\left(A^2\omega(z) + 2A\right) + \theta t + \varepsilon\omega(z) + B(\varepsilon) + C'' \\ &\leqslant h_M(\operatorname{Im} z) + \frac{\omega(z)}{m} + \theta t + 2C_M + C'' + B(\varepsilon) + \frac{2A}{m'}. \end{aligned}$$

We have finished the proof for $t \geqslant 0$ with constant $C = 2C_M + C'' + B(\varepsilon) + \frac{2A}{m'}$.

For the case $t \leqslant 0$ we use

$$\begin{aligned} u(z) &\leqslant h_N(\operatorname{Im} z) + \frac{\omega(z)}{n}, \\ u(z) &\leqslant h_K(\operatorname{Im} z) + \frac{\omega(z)}{k} - t. \end{aligned}$$

The proof is similar getting

$$u(z) \leqslant h_M(\operatorname{Im} z) + \frac{\omega(z)}{m} + (\theta - 1)t + \text{const}\,. \qquad \blacksquare$$

We will need the following auxiliary lemma (see [13, Lemma 7.6] where it was proved for $IPL(V,N,\infty)$-type conditions, comp. [38, Lemma 2.10]).

Lemma 4.5. *Let V be an algebraic variety, $A > 1$. Assume that for some fixed convex compact sets $N \Subset M \Subset \mathbb{R}^d$ the following condition holds:*

$$\exists\, n\ \ \forall\, m\ \ \exists\, \theta_0(m)\ \ \forall\, \theta \in]\theta_0(m), 1[\ \ \exists\, k(m,\theta), C\ \ \forall\, t \geqslant 0\ \ \forall\, u \in \mathrm{PSH}(V,N,0)$$
$$(a) + (b) \Rightarrow (c),$$

where

(a) $\forall z \in V\ \ u(z) \leqslant h_N(\operatorname{Im} z) + \frac{\omega(z)}{n} + t;$
(b) $\forall z \in V\ \ u(z) \leqslant A h_M(\operatorname{Im} z) + \frac{\omega(z)}{k(m,\theta)};$
(c) $\forall z \in V\ \ u(z) \leqslant h_M(\operatorname{Im} z) + \frac{\omega(z)}{m} + \theta t + C.$

Then

$$\forall\, p \geqslant 2\ \ \forall\, m\ \ \exists\, \theta_0(m,p)\ \ \forall\, \theta \in]\theta_0(m,p), 1[$$
$$\exists\, k(m,\theta,p), C\ \ \forall\, t \geqslant 0\ \ \forall\, u \in \mathrm{PSH}(V,N,0) \qquad (a_p) + (b_p) \Rightarrow (c_p),$$

where

(a_p) $\forall z \in V\ \ u(z) \leqslant h_N(\operatorname{Im} z) + \frac{\omega(z)}{n} + t;$
(b_p) $\forall z \in V\ \ u(z) \leqslant A^p h_M(\operatorname{Im} z) + \frac{\omega(z)}{k(m,\theta,p)};$
(c_p) $\forall z \in V\ \ u(z) \leqslant h_M(\operatorname{Im} z) + \frac{\omega(z)}{m} + \theta t + C.$

An analogous result holds for other IPL*-type conditions, i.e., for $t \leqslant 0$ or $\theta \in]0, \theta_0[$ with suitable modifications of conditions.*

Proof. We go by induction with respect to p. Choose $\theta_1 > \theta_0(m)$, inductively $\theta_p := \theta_{p-1}(1-\theta_1) + \theta_1$ and we take

$$k(m, \theta_p, p) \geqslant k(Ak(m, \theta_1), \theta_{p-1}, p-1).$$

Assume that $u \in \mathrm{PSH}(V, N, 0)$ satisfies for $z \in V$ both (a$_p$) and (b$_p$). We define $v(z) := u(z)/A$. Clearly $v \in \mathrm{PSH}(V, N, 0)$ and for $z \in V$

$$v(z) \leqslant h_N(\operatorname{Im} z) + \frac{\omega(z)}{An} + \frac{t}{A},$$
$$v(z) \leqslant A^{p-1} h_M(\operatorname{Im} z) + \frac{\omega(z)}{k(Ak(m, \theta_1), \theta_{p-1}, p-1)}.$$

We may apply the inductive hypothesis:

$$v(z) \leqslant h_M(\operatorname{Im} z) + \frac{\omega(z)}{Ak(m, \theta_1)} + \frac{\theta_{p-1} t}{A} + C.$$

Then, by the assumption applied to $u(z) - \theta_{p-1}t - CA$ (since $t \geqslant \theta_{p-1}t$), we get

$$u(z) - CA - \theta_{p-1}t \leqslant h_M(\operatorname{Im} z) + \frac{\omega(z)}{m} + \theta_1 t - \theta_1\theta_{p-1}t$$

and

$$u(z) \leqslant h_M(\operatorname{Im} z) + \frac{\omega(z)}{m} + \theta_p t + C + CA.$$

The proof in the case $t \leqslant 0$ is exactly the same since $t \leqslant \theta_{p-1}t$. In case θ is close to 0 we just take θ_1 so close to 0 that θ_p is suitable small. Other cases also have similar proofs – we omit details. ■

We show that we can manipulate the set Ω (similar as in [13, Prop. 7.7 and 7.8]):

Proposition 4.6. *Let $\Omega \subseteq \mathbb{R}^d$ be an open convex set, V an arbitrary algebraic variety satisfying one of the* $\mathrm{IPL}(\Omega, \omega, R)$ *conditions. Then for every $s > 0$ and $a \in \mathbb{R}^d$ the variety V satisfies the same* IPL *condition with the sets*

$$s\Omega = \{sx : x \in \Omega\}, \qquad \Omega + a = \{x + a : x \in \Omega\}, \qquad -\Omega = \{-x : x \in \Omega\}.$$

Proof. Let us observe that for any compact set K

$$h_{sK}(x) = sh_K(x), \quad h_{K+a}(x) = h_K(x) + \langle x, a\rangle, \quad h_{-K}(x) = h_K(-x).$$

Observe also that if $u(z)$ is plurisubharmonic then $u(z)/s$, $u(z) + \langle \operatorname{Im} z, a\rangle$, $u(-z)$ are also plurisubharmonic functions with respect to z. The conclusion follows easily. ■

Proposition 4.7. *Let $\Omega_1 \subset \Omega_2 \subset \cdots \subset \mathbb{R}^d$ be an increasing sequence of open convex sets such that an algebraic variety V satisfies* $\mathrm{IPL}(\Omega_j, \omega, 0)$ *for any $j \in \mathbb{N}$. Then V satisfies* $\mathrm{IPL}(\Omega, \omega, 0)$ *for $\Omega = \bigcup_j \Omega_j$. The same result holds for other IPL-conditions.*

Proof. By Proposition 4.6, without loss of generality we may assume that $0 \in \Omega_1$. Let $K_N \Subset \Omega$ arbitrary, then $K_N \Subset \Omega_j$ for some j. By $\mathrm{IPL}(\Omega_j, \omega, 0)$ there is $K_M \Subset \Omega_j$ and for every $K_K = AK_M$, $A > 1$, $K_M \Subset K_K \Subset \Omega_j$ such that $0 \in \operatorname{Int} K_M$ and

$$\exists\, n\ \forall\, m\ \exists\, \theta_0(m)\ \forall\, \theta \in]\theta_0(m), 1[\ \exists\, k(m,\theta), C\ \forall\, t\ \forall\, u \in \mathrm{PSH}(V, N, 0)$$
$$(a) + (b) \Rightarrow (c),$$

where

(a) $\forall z \in V\ \ u(z) \leqslant h_N(\operatorname{Im} z) + \frac{\omega(z)}{n} + t,$
(b) $\forall z \in V\ \ u(z) \leqslant h_K(\operatorname{Im} z) + \frac{\omega(z)}{k},$
(c) $\forall z \in V\ \ u(z) \leqslant h_M(\operatorname{Im} z) + \frac{\omega(z)}{m} + \theta t + C.$

Since $h_K = Ah_M$, the assumptions of Lemma 4.5 are satisfied. Let us take any $L \Subset \Omega$, clearly for some $p \in \mathbb{N}$ we have $A^p K_M \Supset L$. The conclusion follows from Lemma 4.5 for $\mathrm{IPL}(\Omega, \omega, 0)$. ■

Now, we give a reformulation of IPL conditions.

Corollary 4.8.

(i) *The condition* $\mathrm{IPL}(\Omega, \omega, 0)$ *is equivalent to*

$$\forall\, N\ \exists\, M\ \forall\, \rho > 0\ \exists\, n\ \forall\, m\ \forall\, \theta \in]0,1[$$
$$\exists\, k, C\ \forall\, t \in \mathbb{R}\ \forall\, u \in \mathrm{PSH}(V, N, 0) \qquad (a) + (b) \Rightarrow (c),$$

where

(a) $\forall z \in V\ \ u(z) \leqslant h_N(\operatorname{Im} z) + \frac{\omega(z)}{n} + t;$
(b) $\forall z \in V\ \ u(z) \leqslant \rho|\operatorname{Im} z| + \frac{\omega(z)}{k};$
(c) $\forall z \in V\ \ u(z) \leqslant h_M(\operatorname{Im} z) + \frac{\omega(z)}{m} + \theta t + C.$

(ii) *The condition* $\mathrm{IPL}(\mathbb{R}^d, \omega, 0)$ *is equivalent to*

$$\exists\, A > 1\ \forall\, \rho > 0\ \exists\, n\ \forall\, m\ \forall\, \theta \in]0,1[$$
$$\exists\, k, C\ \forall\, t \in \mathbb{R}\ \forall\, u \in \mathrm{PSH}(V, N, 0) \qquad (a) + (b) \Rightarrow (c),$$

where $K_N = \overline{B}(0,1)$

(a) $\forall z \in V\ \ u(z) \leqslant |\operatorname{Im} z| + \frac{\omega(z)}{n} + t;$
(b) $\forall z \in V\ \ u(z) \leqslant \rho|\operatorname{Im} z| + \frac{\omega(z)}{k};$

(c) $\forall z \in V \ \ u(z) \leqslant A|\operatorname{Im} z| + \frac{\omega(z)}{m} + \theta t + C.$

On the other hand IPL$(B(0,1),\omega,0)$ *is equivalent to*

$$\forall A>1 \ \forall \rho>0 \ \exists n \ \forall m \ \forall \theta \in]0,1[\quad \exists k, C \ \forall t \in \mathbb{R} \ \forall u \in \mathrm{PSH}(V,N,0) \qquad (a)+(b) \Rightarrow (c),$$

where (a), (b) and (c) are the same as above.

(iii) *The same results hold for other* IPL*-conditions.*

Proof. (i): The condition implies the original $\mathrm{IPL}^0(\Omega,\omega,0)$ since $h_K(\operatorname{Im} z) \leqslant \rho|\operatorname{Im} z|$ for some ρ. For the other direction, by Proposition 4.6 and its proof, we may assume without loss of generality that $0 \in K_N \Subset K_M \Subset \Omega$. The result follows from Lemma 4.5. The proof of other cases is similar. ■

5. Evaluation of Phragmén-Lindelöf type conditions

We consider now the differential operators

$$P(D) : \mathscr{E}_{\{\omega\}}(\Omega) \to \mathscr{E}_{\{\omega\}}(\Omega),$$

Ω open convex, ω a weight such that $\log(2+|z|) = o(\omega(z))$ and $\omega(z) = O(|z|)$. Here the situation is more complicated than in the distributional case although we solve partially [3, Problem 9.4] for non-quasianalytic weights (see Corollary 5.6, Theorem 5.8 and 5.9). We cannot prove that ker $P(D)$ always have some P-property (i.e., an analogue of [13, Cor. 8.2] is not known) — see Problem 7.6. Since every P-property implies $\mathrm{Proj}^1 = 0$, by [53, 3.4.6] or [47, 3.26], it follows that any P-property of ker $P(D)$ implies surjectivity of $P(D)$. Even under surjectivity assumption we have examples that ker $P(D) \notin (PA)$ since $\mathscr{E}_{\{\omega\}}(\Omega)$ itself does not have (PA).

Proposition 5.1. *If P is an elliptic polynomial then* ker $P(D) \subseteq \mathscr{E}_{\{\omega\}}(\Omega)$ *has* (PA) *and* $(P\Omega)$ *(i.e., it has the dual interpolation estimate for big θ) but for arbitrary open subset $\Omega \subseteq \mathbb{R}^d$ it has never $(P\overline{\overline{\Omega}})$ for $d>1$.*

Proof. In that case, ker $P(D)$ is topologically the same as the corresponding kernel in $C^\infty(\Omega)$ or $\mathscr{D}'(\Omega)$ which implies the conclusion by [13, Cor. 8.2, 8.3], [51, Th. 14, Th. 3]. ■

Our previous study gives the following result.

Corollary 5.2. *Let $P(D) : \mathscr{E}_{\{\omega\}}(\Omega) \to \mathscr{E}_{\{\omega\}}(\Omega)$ be a linear partial differential operator with constant coefficients, $\Omega \subseteq \mathbb{R}^d$ open convex, let ω be a non-quasianalytic weight or let ω have (α_1), let V be the zero variety of P.*

(i) *The following assertions are equivalent:*

(a) *V has* $\mathrm{IPL}^0_+(\Omega,\omega,0)$;

(b) $\ker P(D) \subseteq \mathscr{E}_{\{\omega\}}(\Omega)$ *has* $(P\overline{\overline{\Omega}})$*;*

(c) $P(D) : \mathscr{E}_{\{\omega\}}(\Omega, \Lambda_0'(\alpha)) \to \mathscr{E}_{\{\omega\}}(\Omega, \Lambda_0'(\alpha))$ *is surjective for some (all) stable* α*;*

(d) $P(D) : \mathscr{E}_{\{\omega\}}(\Omega, \mathscr{E}_{\{\omega_1\}}(\Omega_1)) \to \mathscr{E}_{\{\omega\}}(\Omega, \mathscr{E}_{\{\omega_1\}}(\Omega_1))$ *is surjective for some (all) non-quasianalytic weights* ω_1*,* $\Omega_1 \subseteq \mathbb{R}^n$*.*

(e) $P(D) : \mathscr{E}_{\{\omega\}}(\Omega, \mathscr{A}(U)) \to \mathscr{E}_{\{\omega\}}(\Omega, \mathscr{A}(U))$*,* U *a real analytic manifold with only compact connected components, is surjective.*

(ii) *The following assertions are equivalent:*

(a) V *has* $\mathrm{IPL}^0(\Omega, \omega, 0)$*;*

(b) $\ker P(D) \subseteq \mathscr{E}_{\{\omega\}}(\Omega)$ *has the dual interpolation estimate for small* θ*;*

(c) $P(D) : \mathscr{E}_{\{\omega\}}(\Omega, \mathscr{A}(U)) \to \mathscr{E}_{\{\omega\}}(\Omega, \mathscr{A}(U))$*,* U *a real analytic manifold with at least one non-compact connected component, is surjective.*

(iii) *The following assertions are equivalent:*

(a) V *has* $\mathrm{IPL}^1_+(\Omega, \omega, 0)$*;*

(b) $\ker P(D) \subseteq \mathscr{E}_{\{\omega\}}(\Omega)$ *has* $(P\Omega)$*;*

(c) $P(D) : \mathscr{E}_{\{\omega\}}(\Omega, \Lambda_\infty'(\alpha)) \to \mathscr{E}_{\{\omega\}}(\Omega, \Lambda_\infty'(\alpha))$ *is surjective for some (all) stable* α*;*

(d) $P(D) : \mathscr{E}_{\{\omega\}}(\Omega, \mathscr{D}'_{(\omega_1)}(U)) \to \mathscr{E}_{\{\omega\}}(\Omega, \mathscr{D}'_{(\omega_1)}(U))$ *is surjective for a non-quasianalytic weight* ω_1 *and arbitrary open set* U*.*

Proof. (i): (a)⇔(b) Theorem 4.1. (b)⇔(c) Theorem 3.1 and [3, Cor. 3.8, Th. 6.2, Cor. 7.2]. (b)⇔(d) Theorem 3.3. (b)⇔(e) Theorem 3.3. (ii): (a)⇔(b) Theorem 4.1. (b)⇔(c) Theorem 3.3. (iii) (a)⇔(b) Theorem 4.1. (b)⇔(c) Theorem 3.1 and [3, Cor. 3.8, Th. 6.2, Cor. 7.2]. (b)⇔(d) Use (b)⇔(c) and the fact that $\mathscr{D}'_{(\omega_1)}(U) \simeq \prod \Lambda_\infty'(\alpha)$ for some stable α (see [48]). ■

Corollary 5.3. *For a fixed linear partial differential operator with constant coefficients* $P(D) : \mathscr{E}_{\{\omega\}}(\Omega) \to \mathscr{E}_{\{\omega\}}(\Omega)$ *where* $\Omega \subseteq \mathbb{R}^d$ *is fixed open convex subset and* ω *is a fixed non-quasianalytic weight or a weight having* (α_1)*, the real analytic parameter dependence of solutions of the equation* $P(D)u = f$ *depends only if manifold* U *of parameters has a non-compact connected component or not.*

By Proposition 5.1 and Theorem 3.3, we have:

Corollary 5.4. *Let* $d > 1$*,* $\Omega \subseteq \mathbb{R}^d$ *arbitrary open,* P *elliptic. The operator*

$$P(D) : \mathscr{E}_{\{\omega\}}(\Omega, \mathscr{A}(U)) \to \mathscr{E}_{\{\omega\}}(\Omega, \mathscr{A}(U))$$

is never surjective.

An analogous result as [13, Cor. 8.4] holds with the same proof for $\bar\partial$ on $\mathscr{E}^{(0,0)}_{\{\omega\}}$, ω arbitrary.

The next result corresponds to [13, Th. 8.6].

Theorem 5.5. *Let $P \in \mathbb{C}[z_1, \dots, z_d]$ be an arbitrary polynomial and P_p its principal part, $\Omega \subseteq \mathbb{R}^d$ an open convex set, let ω be a non-quasinanalytic weight or let ω satisfy (α_1). If*

$$P(D) : \mathscr{E}_{\{\omega\}}(\Omega, \mathscr{A}(U)) \to \mathscr{E}_{\{\omega\}}(\Omega, \mathscr{A}(U)) \tag{11}$$

is surjective then the principal parts

$$P_p(D) : \mathscr{E}_{\{\omega_1\}}(\Omega) \to \mathscr{E}_{\{\omega_1\}}(\Omega) \qquad \text{and} \qquad P_p(D) : C^\infty(\Omega) \to C^\infty(\Omega)$$

have continuous linear right inverses for any non-quasianalytic weight ω_1.

For homogeneous polynomials we get immediately:

Corollary 5.6. *Let $P \in \mathbb{C}[z_1, \dots, z_d]$ be a homogeneous polynomial, $\Omega \subseteq \mathbb{R}^d$ an open convex set, ω a non-quasianalytic weight. The operator*

$$P(D) : \mathscr{E}_{\{\omega\}}(\Omega, \mathscr{A}(U)) \to \mathscr{E}_{\{\omega\}}(\Omega, \mathscr{A}(U))$$

is surjective if and only if

$$P(D) : \mathscr{E}_{\{\omega\}}(\Omega) \to \mathscr{E}_{\{\omega\}}(\Omega) \quad (\text{or, equivalently, } P(D) : C^\infty(\Omega) \to C^\infty(\Omega))$$

has a continuous linear right inverse.

Remarks.

(1) Note that the existence of a continuous linear right inverse for $P(D) : \mathscr{E}_{\{\omega\}}(\Omega) \to \mathscr{E}_{\{\omega\}}(\Omega)$ for non-quasianalytic weights ω is characterized in [35].

(2) Braun [7, 5.5.3] showed that for homogenenous P, non-quasianalytic ω,

$$P(D) : \mathscr{E}_{\{\omega\}}(\mathbb{R}^d) \to \mathscr{E}_{\{\omega\}}(\mathbb{R}^d)$$

has a continuous linear right inverse if and only if

$$P(D) : \mathscr{E}_{\{\omega\}}(\mathbb{R}^d, \mathscr{E}_{\{\omega\}}(\mathbb{R})) \to \mathscr{E}_{\{\omega\}}(\mathbb{R}^d, \mathscr{E}_{\{\omega\}}(\mathbb{R}))$$

is surjective.

In the proof of the Theorem 5.5 we will need the following definition. A homogeneous algebraic variety V in $\mathbb{C}^d$ satisfies HPL(Ω, loc) at $\xi \in V \cap \mathbb{R}^d$ for an open convex set $\Omega \subseteq \mathbb{R}^d$ if there are open sets $U_1 \subset U_2 \subset U_3 \Subset \mathbb{C}^d$ with $\xi \in U_1$ such that for each compact convex set $K \Subset \Omega$ there exists a compact convex set $L \Subset \Omega$ and $\delta > 0$ such that each plurisubharmonic function u on $U_3 \cap V$ satisfies $(a) + (b) \Rightarrow (c)$, where

(a) $u(z) \leqslant h_K(\operatorname{Im} z) + \delta$ for $z \in U_3 \cap V$,

(b) $u(z) \leqslant 0$ for $z \in U_2 \cap \mathbb{R}^d \cap V$,

(c) $u(z) \leqslant h_L(\operatorname{Im} z)$ for $z \in U_1 \cap V$.

Proof of Theorem 5.5. First, we observe that by [35, Th. 5.6], for homogeneous P the existence of a right linear continuous inverse on $\mathscr{E}_{\{\omega_1\}}(\Omega)$, ω_1 non-quasianalytic, does not depend on ω_1 and it is equivalent to the existence of a right inverse on $C^\infty(\Omega)$.

The remaining proof is quite analogous to the proof of [13, Th. 8.6] but since it plays a fundamental role in the next section we give its details. In fact, following closely [38, Th. 4.1] we prove that if V has $\mathrm{IPL}_+^0(\Omega,\omega,0)$ then V_p has $\mathrm{HPL}(\Omega, loc)$ at zero (comp. [38]). This proves the result by Corollary 5.2 and [35, Th. 5.5], [38, Th. 3.3].

Let us take

$$\begin{aligned} U_E(z) &:= U_E(z; h, V, D) \\ &:= \sup\{u(z) : u \in PSH(V\cap D),\ u \leqslant h \text{ on } D\cap V,\ u(z)\leqslant 0 \text{ for } z \in E\cap V\}, \end{aligned}$$

where D is a domain in $\mathbb{C}^d$, h is a function on D, $E \subseteq D$. For $0<\varepsilon<1$ we define

$$K_\varepsilon := \{z \in \mathbb{C}^d : |z| \leqslant 2 \text{ and } |\operatorname{Im} z| \leqslant \varepsilon|z| \text{ or } |z|\leqslant\varepsilon\}.$$

For $j \in \mathbb{N}$ we define

$$V_j := \{z/j : z \in V\}, \qquad D = \{z \in \mathbb{C}^d : |z| < 3\}, \qquad E = \{z \in \mathbb{R}^d : |z| \leqslant 2\}.$$

Let us take any N, without loss of generality we may assume that $K_N \supset B(0,1)$. We find N_1 such that

$$h_{N_1}(\operatorname{Im} z) \geqslant h_N(\operatorname{Im} z) + 2\eta|\operatorname{Im} z| > |\operatorname{Im} z|$$

for suitable η, $0<\eta<1$, small enough. Using Corollary 4.8 we find M such that for each $\rho>0$ if a plurisubharmonic function φ satisfies $\varphi(z) = h_{N_1}(\operatorname{Im} z) + o(\omega(|z|))$ and

$$\varphi(z) \leqslant h_{N_1}(\operatorname{Im} z) + \frac{\omega(z)}{n} + t, \qquad \varphi(z) \leqslant \rho|\operatorname{Im} z|$$

then

$$\varphi(z) \leqslant h_M(\operatorname{Im} z) + \frac{\omega(z)}{m} + \theta t + C_{\theta,m,\rho}.$$

Choose $\delta>0$ so small that $\delta < \frac{\eta}{2}$ and $0<\varepsilon<1$, $\rho_1>0$ so large that

$$h_{N_1}(\operatorname{Im} z) \leqslant \rho_1|\operatorname{Im} z|.$$

Let

$$\rho := \rho(\varepsilon) := 1 + \rho_1 + \delta\varepsilon^{-2}.$$

Take a plurisubharmonic function u on $V_j \cap D$ such that

$$\begin{aligned} u(z) &\leqslant h_N(\operatorname{Im} z) + \delta, && z \in V_j\cap D,\\ u(z) &\leqslant 0 && z \in K_\varepsilon \cap V_j. \end{aligned}$$

Define a plurisubharmonic function

$$H(z) := \frac{1}{2}\left(|\operatorname{Im} z|^2 - |\operatorname{Re} z|^2\right),$$

see its properties in [34, Lemma 2.9].

Take $z_0 \in V_j$ with $|z_0| < 1$. Define

$$\psi : V \cap \{z \in \mathbb{C}^d : |z - \operatorname{Re}(jz_0)| < j\} \to [-1, \infty[$$

by

$$\psi(z) := \max\left\{ju\left(\frac{z}{j}\right) + \eta j H\left(\frac{z - \operatorname{Re}(jz_0)}{j}\right); h_{N_1}(\operatorname{Im} z)\right\}.$$

Observe that for $|z - \operatorname{Re}(jz_0)| = j$ we have

$$\begin{aligned} ju\left(\frac{z}{j}\right) + \eta j H\left(\frac{z - \operatorname{Re}(jz_0)}{j}\right) &\leqslant j h_N\left(\frac{\operatorname{Im} z}{j}\right) + j\delta + \eta j\left(j^{-1}|\operatorname{Im} z| - \frac{1}{2}\right) \\ &\leqslant h_N(\operatorname{Im} z) + \eta|\operatorname{Im} z| + \left(\delta - \frac{1}{2}\eta\right) j \\ &\leqslant h_{N_1}(\operatorname{Im} z). \end{aligned}$$

Therefore we can extend ψ to a global plurisubharmonic function on V in $PSH(V, N_1, 0)$ taking $\psi(z) = h_{N_1}(\operatorname{Im} z)$ for $|z - \operatorname{Re}(jz_0)| > j$.

Clearly, for $|z - \operatorname{Re}(jz_0)| \leqslant j$ we have

$$\psi(z) \leqslant h_{N_1}(\operatorname{Im} z) + j\delta + \eta j\left|\frac{\operatorname{Im} z}{j}\right| \leqslant h_{N_1}(\operatorname{Im} z) + j\delta + \eta j.$$

Therefore for all z we have

$$\psi(z) \leqslant h_{N_1}(\operatorname{Im} z) + j(\delta + \eta).$$

Moreover, we observe that:

(1) for $|z - \operatorname{Re}(jz_0)| \geqslant j$ we have:

$$\psi(z) \leqslant h_{N_1}(\operatorname{Im} z) \leqslant \rho_1|\operatorname{Im} z| \leqslant \rho|\operatorname{Im} z|;$$

(2) for $|z - \operatorname{Re}(jz_0)| \leqslant j$, $\frac{z}{j} \in K_\varepsilon$ we have

$$\begin{aligned} \psi(z) &\leqslant \max\left(\eta j H\left(\frac{z - \operatorname{Re}(jz_0)}{j}\right), h_{N_1}(\operatorname{Im} z)\right) \\ &\leqslant \max\left(\eta|\operatorname{Im} z|, h_{N_1}(\operatorname{Im} z)\right) \leqslant \rho|\operatorname{Im} z|, \end{aligned}$$

since in this case $u\left(\frac{z}{j}\right) \leqslant 0$;

(3) for $|z - \mathrm{Re}\,(jz_0)| \leqslant j$, $\frac{z}{j} \neq K_\varepsilon$ we have:

$$\begin{aligned}\psi(z) &\leqslant h_N(\mathrm{Im}\, z) + j\delta + \eta|\,\mathrm{Im}\, z| \\ &\leqslant h_{N_1}(\mathrm{Im}\, z) + \frac{|z|\delta}{\varepsilon} \leqslant h_{N_1}(\mathrm{Im}\, z) + \frac{\delta}{\varepsilon^2}|\,\mathrm{Im}\, z| \\ &\leqslant \rho|\,\mathrm{Im}\, z|,\end{aligned}$$

since in this case $j \leqslant \frac{|z|}{\varepsilon}$, $|z| \leqslant \frac{|\,\mathrm{Im}\, z|}{\varepsilon}$.

Thus we have

$$\psi(z) \leqslant h_M(\mathrm{Im}\, z) + \frac{\omega(z)}{m} + \theta j(\delta + \eta) + C_{\rho,m,\theta}.$$

Clearly at $z = jz_0 \in V$ using $H(iy) \geqslant 0$ for $y \in [-1, 1]$ we obtain

$$ju(z_0) \leqslant \psi(jz_0) \leqslant h_M(j\,\mathrm{Im}\, z_0) + \frac{\omega(jz_0)}{m} + \theta j(\delta + \eta) + C_{\rho,m,\theta}$$

and

$$u(z_0) \leqslant h_M(\mathrm{Im}\, z_0) + \frac{\omega(jz_0)}{jm} + \theta(\delta + \eta) + \frac{C_{\rho,m,\theta}}{j}.$$

This is true for all $z_0 \in V_j$, $|z_0| < 1$.

We have proved that

$$U_{K_\varepsilon}(z_0, h_N \circ \mathrm{Im} + \delta, V_j, B(0, 3)) \leqslant h_M(\mathrm{Im}\, z_0) + \frac{\omega(jz_0)}{mj} + \theta(\delta + \eta) + \frac{C_{\theta,m,\rho}}{j}$$

for every m and j, $z_0 \in V_j$, $|z_0| < 1$. If we let $j \to \infty$ and use the facts that $\omega(z) \leqslant Q|z| + Q$ for some constant Q and that V_j converge to V_p the zero variety of the principal part of P in the sense of [34, 4.4] we get:

$$U_{K_\varepsilon}(z; h_N \circ \mathrm{Im} + \delta, V, D) \leqslant h_M(\mathrm{Im}\, z) + \theta(\delta + \eta) + \frac{Q|z|}{m}.$$

Since this is true for every m and $\theta \in]0, \theta_0[$ we have:

$$U_{K_\varepsilon}(z; h_N \circ \mathrm{Im} + \delta, V, D) \leqslant h_M(\mathrm{Im}\, z).$$

Now, $E = \bigcap_{\varepsilon \to 0} K_\varepsilon$ thus, by [34, Prop. 4.2], for $|z| < 1$,

$$U_E(z; h_N \circ \mathrm{Im} + \delta, V, D) \leqslant h_M(\mathrm{Im}\, z).$$

This implies $\mathrm{HPL}(\Omega, loc)$ at zero. ■

Theorem 5.7. *Let $P \in \mathbb{C}[z_1, \ldots, z_d]$ be an arbitrary polynomial, $\Omega \subseteq \mathbb{R}^d$ an open convex set, let ω be a non-quasianalytic weight or let ω satisfy (α_1).*

If

$$P(D) : \mathscr{E}_{\{\omega\}}(\Omega, \mathscr{A}(U)) \to \mathscr{E}_{\{\omega\}}(\Omega, \mathscr{A}(U))$$

is surjective then

$$P(D): \mathscr{E}_{\{\omega\}}(\mathbb{R}^d, \mathscr{A}(U)) \to \mathscr{E}_{\{\omega\}}(\mathbb{R}^d, \mathscr{A}(U))$$

is surjective as well.

The same holds for $\mathscr{A}(U)$ substituted by $\Lambda_0'(\alpha)$, $\Lambda_\infty'(\alpha)$. In fact, if $\ker P(D) \subseteq \mathscr{E}_{\{\omega\}}(\Omega)$ *has $(P\overline{\overline{\Omega}})$, $(P\Omega)$, (PA) or the dual interpolation estimate (for all θ, for small θ, for big θ) then* $\ker P(D) \subseteq \mathscr{E}_{\{\omega\}}(\mathbb{R}^d)$ *has the same property.*

Proof. To prove the last statement by Prop. 4.6 we can shift Ω so that $0 \in \Omega$ and then apply Prop. 4.6 and 4.7 for $\mathbb{R}^d = \bigcup_{n=1}^\infty n\Omega$. The rest follows from Cor. 5.2. ■

Theorem 5.8. *If the weight ω satisfies $\omega(t) = o(t^{1/2})$ then for a polynomial P of order $\leqslant 2$ the following conditions are equivalent:*

(a) *The operator $P(D): \mathscr{E}_{\{\omega\}}(\mathbb{R}^d, \mathscr{A}(U)) \to \mathscr{E}_{\{\omega\}}(\mathbb{R}^d, \mathscr{A}(U))$ is surjective.*
(b) *The operator $P(D): \mathscr{E}_{\{\omega\}}(\mathbb{R}^d) \to \mathscr{E}_{\{\omega\}}(\mathbb{R}^d)$ has a continuous linear right inverse.*
(c) *The operator $P(D): C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ has a continuous linear right inverse.*

In particular, the above result holds for Gevrey classes $\Gamma_{\{p\}}$ for $p > 2$.

Remark. The description of polynomials which satisfy the above conditions is given in [36], comp. [13, Theorem A].

Proof. (c)⇒(b): [35, Cor. 3.12]. (b)⇒(a): Obvious.

(a)⇒(c): By Theorem 5.5, the operator

$$P_p(D): C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$$

has a continuous linear right inverse. Using in the proof of [13, Lemma 8.9] the function u instead of v we get the same conclusion under the assumption $IPL_+^0(\mathbb{R}^d, \omega, 0)$. This allows to prove [13, Prop. 8.10] again under the assumption $IPL_+^0(\mathbb{R}^{d+2}, \omega, 0)$ instead of $IPL_+^0(\mathbb{R}^{d+2}, \omega, \infty)$. Then we apply the proof of [13, Theorem 8.11] for ω with $\omega(t) = o(t^{1/2})$ instead of $\omega(t) = \log(2 + |t|)$. ■

For polynomials of two variables we can repeat the method of [13, Thm. 8.14]. A polynomial P is $\{\omega\}$-*hyperbolic with respect to a non-characteristic vector* v if $P(D)$ admits a fundamental solution $E \in \mathscr{D}'_{\{\omega\}}(\mathbb{R}^d)$ with $\operatorname{supp} E \subseteq \overline{H_+(v)}$. It is $\{\omega\}$-*hyperbolic* if it is $\{\omega\}$-hyperbolic with respect to at least one non-characteristic vector. By [37, Lemma 2.14], P is $\{\omega\}$-hyperbolic with respect to v if and only if it is (σ)-hyperbolic with respect to v for some weight $\sigma = o(\omega)$.

Theorem 5.9. *Let $P \in \mathbb{C}[z_1, z_2]$ be an arbitrary polynomial. The following assertions are equivalent for any non-quasianalytic weight ω:*

(a) *$P(D): \mathscr{E}_{\{\omega\}}(\Omega, \mathscr{A}(U)) \to \mathscr{E}_{\{\omega\}}(\Omega, \mathscr{A}(U))$ is surjective for some open convex $\Omega \subseteq \mathbb{R}^2$;*

(b) $P(D) : \mathscr{E}_{\{\omega\}}(\Omega) \to \mathscr{E}_{\{\omega\}}(\Omega)$ *has a continuous linear right inverse for all open convex sets* $\Omega \subseteq \mathbb{R}^2$*;*
(c) $P(D)$ *is* $\{\omega\}$*-hyperbolic or, equivalently,* $\{\omega\}$*-hyperbolic with respect to all non-characteristic directions.*

Proof. (b)⇒(a) obvious. (c)⇒(b) follows from [35, Th. 5.14]. (a)⇒(c): We repeat the proof of [13, Thm. 8.13] (a)⇒(c) taking, $v = \tilde{u}$ (so σ is superfluous). Thus we get for every m:

$$ER^\alpha \leqslant 2^{\alpha+1}\theta DR^\alpha + \omega(a)/m + C.$$

Therefore taking $\theta = E/(2^{\alpha+2}D)$ we get $R^\alpha \leqslant 2C/E + (C_2/m)\omega(R)$ for any m, C depending on m. Then $t^\alpha = o(\omega(t))$. Clearly then

$$|\operatorname{Im} z_1| \leqslant A + B|z_2|^\alpha$$

which means that $V(P)$ is (t^α)-hyperbolic and thus $\{\omega\}$-hyperbolic. ∎

6. The real analytic case

Now, we consider

$$P(D) : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega),$$

where $\Omega \subseteq \mathbb{R}^d$ is convex open. Clearly $\mathscr{A}(\Omega, \mathscr{A}(U)) \simeq \mathscr{A}(\Omega \times U)$ thus surjectivity of $P(D)$ on $\mathscr{A}(\Omega, \mathscr{A}(U))$ can be checked by the criteria of surjectivity of differential operators on spaces of scalar valued real analytic functions. For open convex sets Ω surjectivity of $P(D)$ on $\mathscr{A}(\Omega)$ was characterized by Hörmander [20]. For non convex open sets Ω this was done Langenbruch [28], [29].

Since in general U is a manifold so the above criteria are useless. Using our method we can prove the following main theorem.

Theorem 6.1. *Let* $\Omega \subseteq \mathbb{R}^d$ *be open convex, let* U *be an arbitrary real analytic manifold and let* $P(D) : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ *be an arbitrary linear partial differential operator with constant coefficients. The following conditions are equivalent:*

(a) *The equation* $P(D)u = f$ *has a real analytic parameter dependence of solutions.*
(b) $P(D) : \mathscr{A}(\Omega, \mathscr{A}(U)) \to \mathscr{A}(\Omega, \mathscr{A}(U))$ *is surjective.*
(c) $P(D) : \mathscr{A}(\Omega, \mathscr{A}(\mathbb{R})) \to \mathscr{A}(\Omega, \mathscr{A}(\mathbb{R}))$ *is surjective.*
(d) $P(D) : \mathscr{A}(\Omega, \mathscr{A}(\mathbb{T})) \to \mathscr{A}(\Omega, \mathscr{A}(\mathbb{T}))$ *is surjective.*
(e) *The principal part operator* $P_p(D) : \mathscr{A}(\Omega, \mathscr{A}(U)) \to \mathscr{A}(\Omega, \mathscr{A}(U))$ *is surjective.*
(f) *The principal part operator* $P_p(D) : \mathscr{A}(\Omega, \mathscr{A}(\mathbb{R})) \to \mathscr{A}(\Omega, \mathscr{A}(\mathbb{R}))$ *is surjective.*
(g) *The principal part operator* $P_p(D) : \mathscr{A}(\Omega, \mathscr{A}(\mathbb{T})) \to \mathscr{A}(\Omega, \mathscr{A}(\mathbb{T}))$ *is surjective.*
(h) $P_p(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ *or, equivalently,* $P_p(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ *has a linear continuous right inverse.*

For more characterizations of condition (h) in Theorem 6.1 see [9] and references therein, comp. also [13, Th. A].

We first collect consequences of the result above. By Corollary 5.2, we get immediately:

Corollary 6.2. *Under the assumptions of the theorem above the following assertions are equivalent.*

(a) $\ker P(D) \subseteq \mathscr{A}(\Omega)$ *has* $(P\overline{\overline{\Omega}})$.
(b) $\ker P(D) \subseteq \mathscr{A}(\Omega)$ *has the dual interpolation estimate for small* θ.
(c) $\ker P_p(D) \subseteq \mathscr{A}(\Omega)$ *has* $(P\overline{\overline{\Omega}})$.
(d) $\ker P_p(D) \subseteq \mathscr{A}(\Omega)$ *has the dual interpolation estimate for small* θ.

By Cor. 5.2, Cor. 5.6, [38, Th. 3.3] and [13, Th. 8.8, Cor. 8.2], we also get (for more equivalent conditions see [38, Th. 3.3]):

Corollary 6.3. *For every homogeneous algebraic manifold* V *the following conditions are equivalent:*

(a) V *has* $\mathrm{HPL}(\Omega, loc)$ *at zero.*
(b) V *has* $\mathrm{IPL}^0_+(\Omega, |z|, 0)$ *or, equivalently,* $\mathrm{IPL}^0(\Omega, |z|, 0)$.
(c) V *has* $\mathrm{IPL}(\Omega, \omega, \infty)$ *for some/every non-quasianalytic weight* ω.
(d) V *has* $\mathrm{IPL}^0_+(\Omega, \omega, 0)$ *or, equivalently,* $\mathrm{IPL}^0(\Omega, \omega, 0)$ *for some/every non-quasianalytic* ω.

Using [36] we get:

Corollary 6.4. *An operator of order two on* $\mathscr{A}(\mathbb{R}^d)$ *satisfies the real analytic parameter dependence of solutions if and only if after some linear invertible change of variables its principal part is of the form*

$$\mu\partial_1^2 \qquad or \qquad \mu\left[\sum_{j=1}^{r}\partial_j^2 - \sum_{j=r+1}^{s}\partial_j^2\right]$$

for $\mu \in \mathbb{C}$, $1 \leqslant r < s$.

Using [32] or [13, Th. 8.13] we get:

Corollary 6.5. *An operator of two variables on* $\mathscr{A}(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ *convex open, satisfies the real analytic parameter dependence of solutions if and only if its principal part is hyperbolic.*

Proof of Theorem 6.1. Theorem 5.5 shows that (b), (c), (d), (e), (f) or (g) all imply (h) (for equivalence of C^∞ and $\mathscr{D}'$ cases see [32]).

By [52, Prop. 3.2], (h)⇒(f) which means that

$$P_p(D) : \mathscr{A}(\Omega \times \mathbb{R}) \to \mathscr{A}(\Omega \times \mathbb{R})$$

is surjective. Since $\Omega \times \mathbb{R}$ is convex, by [20] this implies surjectivity of

$$P(D) : \mathscr{A}(\Omega \times \mathbb{R}) \to \mathscr{A}(\Omega \times \mathbb{R}),$$

i.e., the condition (c). By Cor. 5.2, (c)⇒(b), (d), (f)⇒(g), (e). We have proved that (b) does not depend on the choice of U and (a) is just (b) for all U. ■

Using quite different methods Vogt [52] proved for $\Omega \subseteq \mathbb{R}^d$, $d > 1$, that if

$$P(D) : \mathscr{A}(\Omega, \mathscr{A}(\mathbb{R}))) \to \mathscr{A}(\Omega, \mathscr{A}(\mathbb{R}))$$

is surjective then

$$P(D) : \mathscr{A}(\mathbb{R}^d, \mathscr{A}(\mathbb{R})) \to \mathscr{A}(\mathbb{R}^d, \mathscr{A}(\mathbb{R})) \tag{12}$$

is surjective [52, Prop. 3.1, Th. 3.4] and [29, Th. 4.4] (surjectivity of $P_p(D)$ on $\mathscr{A}(\Omega)$ implies surjectivity on $\mathscr{A}(\mathbb{R}^d)$). Then surjectivity of (12) is equivalent to surjectivity of

$$P_p(D) : \mathscr{A}(\mathbb{R}^d, \mathscr{A}(\mathbb{R})) \to \mathscr{A}(\mathbb{R}^d, \mathscr{A}(\mathbb{R}))$$

[52, Th. 3.4] and also to the existence of a continuous linear right inverse for

$$P_p(D) : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$$

[52, Th. 3.4, Prop. 3.3]. Vogt [52, Th. 3.7] also characterized polynomials of order two on bounded convex sets $\Omega \subseteq \mathbb{R}^d$ with C^1-boundary for which

$$P(D) : \mathscr{A}(\Omega, \mathscr{A}(U)) \to \mathscr{A}(\Omega, \mathscr{A}(U))$$

is surjective (those for which its principal part is proportional to a product of two real linear forms).

We get an immediate consequence of Theorem 5.7:

Corollary 6.6. *If $\Omega \subseteq \mathbb{R}^d$ is an open convex set and $P(D) : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ satisfies the real analytic parameter dependence of solutions then the same holds for $P(D) : \mathscr{A}(\mathbb{R}^d) \to \mathscr{A}(\mathbb{R}^d)$.*

7. Open problems

We collect some open problems suggested by the theory described in this paper (comp. Section 9 in [13]).

Problem 7.1. *Is it true that for arbitrary polynomial P the positive solution of the real analytic parameter dependence problem for $P(D) : \mathscr{E}_{\{\omega\}}(\Omega) \to \mathscr{E}_{\{\omega\}}(\Omega)$ is equivalent to the existence of continuous linear right inverse for the same operator?*

This problem is exactly equivalent to the following question (comp. [3, Problem 9.5]):

Problem 7.2. *Can we characterize existence of a continuous linear right inverse for $P(D) : \mathscr{E}_{\{\omega\}}(\Omega) \to \mathscr{E}_{\{\omega\}}(\Omega)$ by the dual interpolation estimate for the kernel of $P(D)$?*

There are two tempting particular cases of Problem 7.1: $\Omega = H_+$ a half space and Ω an open bounded set with C^1-boundary. Especially interesting is the following problem.

Problem 7.3. *Is it true that the following conditions are equivalent:*

(a) $P(D) : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ *has a continuous linear right inverse;*
(b) $P(D)$ *has the real analytic parameter dependence of solutions.*

Clearly, (a)⇒(b). So far Vogt proved [52, Th. 3.6] that for homogeneous operators of order two (b) implies that P is proportional to the product of two linear forms. That means P is a composition of two directional derivatives. Such an operator has a linear continuous right inverse for many sets Ω.

Of course our approach only gives a chance to solve Problems 7.1, 7.2 and 7.3 for convex Ω. For non-convex Ω we know anything only for elliptic operators.

Problem 7.4. *Prove an analogue of Theorem 6.1 for quasianalytic classes* $\mathscr{E}_{\{\omega\}}(\Omega)$.

As Rudnicki (Katowice) pointed out to the author the following problem is very natural.

Problem 7.5. *Characterize the analytic parameter dependence of solutions for* $P(D)$ *in terms of the existence of special fundamental solutions of* $P(D)$.

The same problem for existence of a linear continuous right inverse is solved in [32] — so the problem is related to Problem 7.1.

As we have shown in [13, Prop. 8.1], ker $P(D) \subseteq \mathscr{D}'(\Omega)$ always has some dual interpolation estimate. Moreover, by Cor. 6.2, if $\ker P(D) \subseteq \mathscr{A}(\Omega)$ has $(P\overline{\overline{\Omega}})$ then it has $(P\underline{A})$.

Problem 7.6. *Is it true that* ker $P(D) \subseteq \mathscr{E}_{\{\omega\}}(\Omega)$ *for surjective* $P(D) : \mathscr{E}_{\{\omega\}}(\Omega) \to \mathscr{E}_{\{\omega\}}(\Omega)$ *has always* $(P\underline{A})$ *or* $(P\Omega)$*? The same question for* $P(D) : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$.

This is true in the elliptic case. Might be this the only case?

In order to improve Th. 4.1 (vi) one should solve the following problem:

Problem 7.7. *Find a nuclear Fréchet space* X *such that for every PLS-space* Y *the condition* $\mathrm{Proj}^1\,(X\varepsilon Y) = 0$ *is equivalent to* $Y \in (P\underline{A})$.

Clearly such a space X should have the property $(\overline{\overline{\Omega}})$. Moreover, it would allow to prove that $(P\underline{A})$ is the three space property.

It seems that the most interesting would be the problem to characterize in terms of the symbol kernels of linear partial differential operators with *variable* coefficients on $\mathscr{A}(\Omega)$ which have the dual interpolation estimate for small θ since this would solve the real analytic parameter dependence problem for partial differential operators with variable coefficients via Theorem 3.3. Creating a suitable theory would require a nice description of the kernel and this requires a substitute of the fundamental principle.

References

[1] J. Bonet, P. Domański, *Real analytic curves in Fréchet spaces and their duals*, Mh. Math. **126** (1998), 13–36.

[2] J. Bonet, P. Domański, *Parameter dependence of solutions of partial differential equations in spaces of real analytic functions*, Proc. Amer. Math. Soc. **129** (2000), 495–503.

[3] J. Bonet, P. Domański, *Parameter dependence of solutions of differential equations on spaces of distributions and the splitting of short exact sequences*, J. Funct. Anal., **230** (2006), 329–381.

[4] J. Bonet, P. Domański, *The structure of spaces of quasianalytic functions of Roumieu type*, Arch. Math. **89** (2007), 430–441.

[5] J. Bonet, P. Domański, *The splitting of exact sequences of PLS-spaces and smooth dependence of solutions of linear partial differential equations*, Advances Math. **217** (2008), 561–585.

[6] J. Bonet, R. Meise, S. N. Melikhov, *A comparison of two different ways to define classes of ultradifferentiable functions*, Bull. Belg. Math. Simon Stevin **14** (2007), 425–444.

[7] R. W. Braun, *Surjektivität partieller Differentialoperatoren auf Roumieu-Klassen*, Habilitationsschrift, Düsseldorf 1993.

[8] R. W. Braun, R. Meise, B. A. Taylor, *Ultradifferentiable functions and Fourier analysis*, Results Math. **17** (1990), 207–237.

[9] R. W. Braun, R. Meise, B. A. Taylor, *Characterization of the linear partial differential equations that admit solution operators on Gevrey classes*, J. reine angew. Math. **588** (2005), 169–220.

[10] R. W. Braun, R. Meise, D. Vogt, *Existence of fundamental solutions and surjectivity of convolution operators on classes of ultradifferentiable functions*, Proc. London Math. Soc. **61** (1990), 344–370

[11] F. E. Browder, *Analyticity and partial differential equations I*, Amer. J. Math. **84** (1962), 660–710

[12] P. Domański, *Classical PLS-spaces: spaces of distributions, real analytic functions and their relatives*, in: Orlicz Centenary Volume, Banach Center Publications, **64**, Z. Ciesielski, A. Pełczyński and L. Skrzypczak (eds.), Institute of Mathematics, Warszawa 2004, pp. 51–70.

[13] P. Domański, *Real analytic parameter dependence of solutions of differential equations*, Rev. Mat. Iberoamericana **26** (2010), 175–238.

[14] P. Domański, *Notes on real analytic functions and classical operators*, preprint 2010.

[15] P. Domański and M. Langenbruch, *Vector valued hyperfunctions and boundary values of vector valued harmonic and holomorphic functions*, Publ. RIMS Kyoto Univ. **44**(4) (2008), 1097–1142.

[16] P. Domański, D. Vogt, *The space of real analytic functions has no basis*, Studia Math. **142** (2000), 187–200.

[17] U. Franken, *On the equivalence of holomorphic and plurisubharmonic Phragmén-Lindelöf principles*, Michigan Math. J. **42** (1995), no. 1, 163–173.

[18] L. Frerick, *A splitting theorem for nuclear Fréchet spaces*, in: Functional Analysis, Proc. of the First International Workshop held at Trier University, S. Dierolf, P. Domański, S. Dineen (eds.), Walter de Gruyter, Berlin 1996, pp. 165–167.

[19] L. Frerick, J. Wengenroth, *A sufficient condition for the vanishing of the derived projective limit functor*, Arch. Math., **67** (1996), 296–301.
[20] L. Hörmander, *On the existence of real analytic solutions of partial differential equations with constant coefficients*, Inventiones Math. **21** (1973), 151–182.
[21] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Springer, Berlin 1983.
[22] H. Jarchow, *Locally Convex Spaces*, B. G. Teubner, Stuttgart 1981.
[23] M. Klimek, *Pluripotential Theory*, London Mathematical Society Monographs, New Series, **6**, Oxford University Press, New York, 1991.
[24] H. Komatsu, *Ultradistributions I. Structure theorems and characterization*, J. Fac. Sci. Univ. Tokyo, Sec. 1 A. Math. **20** (1973), 25–105.
[25] A. Kriegl, P. W. Michor, *The convenient setting for real analytic mappings*, Acta Math. **165** (1990), 105–159.
[26] M. Langenbruch, *Continuous linear right inverses for convolution operators in spaces of real analytic functions*, Studia Math. **110** (1994), 65–82.
[27] M. Langenbruch, *Surjectivity of partial differential operators on Gevrey classs and extension of regularity*, Math. Nachr. **196** (1998), 103–140.
[28] M. Langenbruch, *Characterization of surjective partial differential operators on spaces of real analytic functions*, Studia Math. **162** (2004) 53–96.
[29] M. Langenbruch, *Inheritance of surjectivity for partial differential operators on spaces of real analytic functions*, J. Math. Anal. Appl. **297** (2004), 696–719.
[30] A. Martineau, *Sur la topologie des espaces de fonctions holomorphes*, Math. Ann. **163** (1966), 62–88.
[31] R. Meise, *Sequence space representations for zero-solutions of convolution equations on ultradifferentiable functions of Roumieu type*, Studia Math. **92** (1989), 211–230.
[32] R. Meise, B. A. Taylor, D. Vogt, *Characterization of the linear partial differential operators with constant coefficients that admit a continuous linear right inverse*, Ann. Inst. Fourier, Grenoble **40** (1990), 619–655.
[33] R. Meise, B. A. Taylor, D. Vogt, *Equivalence of analytic and plurisubharmonic Phragmén-Lindelöf conditions*, Proc. Symposia Pure Math. **52** (1991), part 3, 287–308.
[34] R. Meise, B. A. Taylor, D. Vogt, *Extremal plurisubharmonic functions of linear growth on algebraic varieties*, Math. Z. **219** (1995), 515–537.
[35] R. Meise, B. A. Taylor, D. Vogt, *Continuous linear right inverses for partial differential operators on non-quasianalytic classes and on ultradistributions*, Math. Nachr. **180** (1996), 213–242.
[36] R. Meise, B. A. Taylor, D. Vogt, *Continuous linear right inverses for partial differential operators of order 2 and fundamental solutions in half spaces*, Manuscr. Math. **90** (1996), 449–464.
[37] R. Meise, B. A. Taylor, D. Vogt, *ω-Hyperbolicity of linear partial differential operators with constant coefficients*, Complex Analysis, Harmonic Analysis and Applications, R. Deville, J. Esterle, V. Petkov, A Sebbar, A Yger, (eds.), Pitman Research Notes, Math. Ser. **347** (1996), 157–182.

[38] R. Meise, B. A. Taylor, D. Vogt, *Phragmén-Lindelöf principles on algebraic varieties*, J. Amer. Math. Soc. **11** (1998), 1–39.

[39] R. Meise, D. Vogt, *Characterization of convolution operators on spaces of C^∞-functions admitting a continuous linear right inverse*, Math. Ann. **279** (1987), 141–155.

[40] R. Meise, D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford 1997.

[41] T. Meyer, *Surjectivity of convolution operators on spaces of ultradifferentiable functions of Roumieu type*, Studia Math. **125** (1997), 101–129.

[42] V. P. Palamodov, *Linear Differential Operators with Constant Coefficients*, Nauka, Moscow 1967 (Russian), English transl., Springer, Berlin 1971.

[43] V. P. Palamodov, *Functor of projective limit in the category of topological vector spaces*, Mat. Sb. **75** (1968), 567-603 (in Russian); English transl., Math. USSR Sbornik **17** (1972), 289–315.

[44] V. P. Palamodov, *Homological methods in the theory of locally convex spaces*, Uspekhi Mat. Nauk **26** (1) (1971), 3–66 (in Russian); English transl., Russian Math. Surveys **26** (1) (1971), 1–64.

[45] K. Piszczek, *On a property of PLS-spaces inherited by their tensor products*, Bull. Belg. Math. Soc. Simon Stevin **17** (2010), 155–170.

[46] L. Rodino, *Linear Partial Differential Operators in Gevrey Classes*, World Scientific, Singapore 1993.

[47] T. Rösner, *Surjektivität partieller Differentialoperatoren auf quasianalytischen Roumieu-Klassen*, Dissertation, Düsseldorf 1997.

[48] D. Vogt, *Sequence space representations of spaces of test functions and distributions*, in: Functional Analysis, Holomorphy and Approximation Theory, G. L. Zapata (ed.), Lecture Notes Pure Appl. Math. **83**, Marcel Dekker, New York 1983, pp. 405–443.

[49] D. Vogt, *Lectures on projective spectra of DF-spaces*, Seminar Lectures, AG Funktionalanalysis, Düsseldorf/Wuppertal 1987.

[50] D. Vogt, *Topics on projective spectra of LB-spaces*, in: Advances in the Theory of Fréchet Spaces, T. Terzioğlu (ed.), Kluwer, Dordrecht 1989, pp. 11–27.

[51] D. Vogt, *Invariants and spaces of zero solutions of linear partial differential operators*, Arch. Math. **87** (2006), 163–171.

[52] D. Vogt, *Real analytic zero solutions of linear partial differential operators with constant coefficients*, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), 577–586.

[53] J. Wengenroth, *Derived Functors in Functional Analysis*, Lecture Notes Math. **1810**, Springer, Berlin 2003.

Address: Paweł Domański: Faculty of Mathematics and Comp. Sci., A. Mickiewicz University Poznań, Umultowska 87, 61-614 Poznań, Poland.

E-mail: domanski@amu.edu.pl

Received: 28 July 2010; **revised:** 27 September 2010