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GLOBAL STRUCTURE OF SOME ULTRADISTRIBUTIONS

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To the memory of our dear friend
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Abstract: Given $p \in \mathbb{N}$, a non empty open subset Ω of $\mathbb{R}^k$ and a semi-regular matrix $\mathfrak{M}$, we characterize the elements of the duals of the Beurling classes $\mathcal{D}^{(\mathfrak{M})}(\Omega)$ and $\mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega)$ of ultradifferentiable functions. We provide a global representation of these ultradistributions with and without compact support by means of series involving measures in the first case and elements of $L^q_{\mathrm{loc}}(\Omega)$ in the second.

Keywords: countable intersection, non quasi-analytic class, ultradifferentiable function, ultradistribution, global representation.

1. Introduction

For the notations, we refer to the Paragraphs 2 and 7.

In this paper, we continue the study of the locally convex properties of the countable intersections of non quasi-analytic classes of ultradifferentiable functions, initiated in [6]. After the study of the mixed intersections in [7] and their tensor product characterization in [8], we obtain a global structure of the elements of the dual of the space $\mathcal{D}^{(\mathfrak{M})}(\Omega)$ in the first part and of the space $\mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega)$ in a second part, $\mathfrak{M}$ being a semi-regular matrix.

We adopt the method used by Valdivia to obtain global representations of the ultradistributions $u \in \mathcal{D}^{(\boldsymbol{M})}(\Omega)'$ in [12] and $u \in \mathcal{D}_{L^p}^{(\boldsymbol{M})}(\Omega)'$ in [13], where $\boldsymbol{M}$ is an increasing, normalized and non quasi-analytic sequence of positive numbers. This leads to a global representation of the continuous linear functionals on $\mathcal{E}_0^{(\mathfrak{M})}(\Omega)$ and of the ultradistributions (i.e. the elements of $\mathcal{D}^{(\mathfrak{M})}(\Omega)'$) with and without compact support (cf. Theorems 5.1 and 6.2) by means of Borel and Radon measures on Ω.

Starting with Paragraph 7, we follow the introduction by Schwartz ([11], p. 199) of the space $\mathcal{D}_{L^p}(\mathbb{R}^k)$ and introduce the space $\mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega)$. Here also the method is fruitful: it leads to a global representation of the ultradistributions $S \in \mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega)'$ with and without compact support (cf. Theorems 9.1 and 9.3). If the matrix $\mathfrak{M}$

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is regular, this leads to a global representation of the elements of $\mathcal{D}^{(\mathfrak{M})}(\Omega)'$ by means of a series of the type $\sum_{\alpha\in\mathbb{N}_0^k}\int_\Omega g_\alpha \mathrm{D}^\alpha\varphi\,dx$ where the functions g_α belong to $L^q_{\mathrm{loc}}(\Omega)$.

There is a huge literature on the non quasi-analytic classes of ultradifferentiable functions of Beurling type $\mathcal{E}^{(\boldsymbol{M})}(\Omega)$ and $\mathcal{D}^{(\boldsymbol{M})}(\Omega)$; a basic reference is given by [5]. Very similar spaces can also be introduced by means of a weight; in this case, a basic reference is given by [3]. In these two papers, one finds local representations of the ultradistributions.

Intersections of non quasi-analytic classes of ultradifferentiable functions have first been investigated by Chaumat and Chollet in [4] in the case when the matrix $\mathfrak{M}$ is defined by $M_{j,p} = M_p^{a_j}$ where $(M_p)_{p\in\mathbb{N}_0}$ is a sequence with moderate growth and $(a_j)_{j\in\mathbb{N}}$ a sequence of positive numbers strictly decreasing to 0. They obtained a Whitney extension theorem, a Łojasiewicz theorem on regular situation, some theorems of division and preparation and a Whitney spectral theorem.

Later on Beaugendre studied extensively such intersections in [1] and [2] when the numbers $M_{j,p}$ are defined by means of a convex and increasing function Φ on $[0,+\infty[$ such that $\lim_{t\to\infty}\Phi(t)/t=\infty$. In particular he obtained extension results for Whitney jets and an explicit continuous linear extension map for Whitney jets.

The introduction of semi-regular matrices $\mathfrak{M}$ appeared in [9] where analytic and holomorphic extensions of Whitney jets are obtained and has been used in [10] to describe an explicit continuous linear extension map for Whitney jets.

2. Notations

Let us first introduce the matrices $\mathfrak{m}$ and $\mathfrak{M}$ used to define the countable intersections of non quasi-analytic Beurling classes of ultradifferentiable functions considered in this paper.

Whenever $\boldsymbol{m}$ is a sequence $(m_p)_{p\in\mathbb{N}_0}$ of real numbers, the notation $\boldsymbol{M}$ designates the sequence $(M_p)_{p\in\mathbb{N}_0}$ where $M_p = m_0\ldots m_p$ for every $p\in\mathbb{N}_0$. Such a sequence $\boldsymbol{m}$ is

(a) *normalized* if $m_0 = 1$ and $m_p \geqslant 1$ for every $p\in\mathbb{N}$;
(b) *non quasi-analytic* if $\sum_{p=0}^\infty 1/m_p < \infty$.

From now on $\mathfrak{m} = (m_{j,p})_{j\in\mathbb{N},p\in\mathbb{N}_0}$ designates a *semi-regular* matrix, i.e. a matrix of real numbers such that, for every $j\in\mathbb{N}$, the sequence $\boldsymbol{m}_j = (m_{j,p})_{p\in\mathbb{N}_0}$ is normalized, increasing, non quasi-analytic and such that

(a) $m_{j,p}\geqslant m_{j+1,p}$ for every $p\in\mathbb{N}_0$;
(b) $\lim_{p\to\infty} m_{j+1,p}/m_{j,p} = 0$.

Of course, $\boldsymbol{M}_j$ designates the sequence $(M_{j,p})_{p\in\mathbb{N}_0}$ for every $j\in\mathbb{N}$ and $\mathfrak{M}$ the matrix $(M_{j,p})_{j\in\mathbb{N},p\in\mathbb{N}_0}$.

The matrix $\mathfrak{m}$ or equivalently $\mathfrak{M}$ is *regular* if it is semi-regular and if, for every $j\in\mathbb{N}$, there are constants $A_j>1$ and $H_j>1$ such that $M_{j+1,p+1}\leqslant A_jH_j^pM_{j,p}$ for every $p\in\mathbb{N}_0$.

Let us say once for all that the functions we consider are complex valued and that all vector spaces are $\mathbb{C}$-vector spaces. Moreover, throughout the paper,

(a) k is a positive integer;
(b) if f is a function on $A \subset \mathbb{R}^k$, we set $\|f\|_A := \sup_{x\in A} |f(x)|$;
(c) Ω is a non void open subset of $\mathbb{R}^k$.

Now all is set up to introduce the spaces we deal with in the first part of this paper.

The Banach space $\mathcal{C}_0(\Omega)$. Its elements are the continuous functions f on Ω "tending to 0 at infinity" (i.e. for every $\varepsilon > 0$, there is a compact subset K of Ω such that $\|f\|_{\Omega\setminus K} \leqslant \varepsilon$) and its norm is $\|.\|_\Omega$. By the Riesz representation theorem, for every continuous linear functional u on $\mathcal{C}_0(\Omega)$, there is a Borel measure μ on Ω such that $\langle u, .\rangle = \int_\Omega . \, d\mu$ on $\mathcal{C}_0(\Omega)$ and $\|u\| = |\mu|(\Omega)$.

The Banach space $\mathcal{K}(K)$ *and the* (LB)*-space* $\mathcal{K}(\Omega)$. Given a non void compact subset K of $\mathbb{R}^k$, $\mathcal{K}(K)$ is the space of the continuous fonctions on $\mathbb{R}^k$ having their support contained in K; its norm is $\|.\|_K$. The space $\mathcal{K}(\Omega)$ is the inductive limit of the spaces $\mathcal{K}(H)$ where H runs through the family of the non void compact subsets of Ω. The elements of the topological dual of $\mathcal{K}(\Omega)$ are the Radon measures on Ω. Given a Radon measure u on Ω and a non void compact subset H of Ω, $\|u\|(H)$ designates the norm of the restriction of u to $\mathcal{K}(H)$.

The Fréchet space $\mathcal{E}_0^{(\mathfrak{M})}(\Omega)$ is the projective limit of the spaces $\mathcal{E}_0^{(\boldsymbol{M}_j)}(\Omega)$. For every $j \in \mathbb{N}$, $\mathcal{E}_0^{(\boldsymbol{M}_j)}(\Omega)$ is the projective limit of the spaces $\mathcal{E}_0^{(\boldsymbol{M}_j),1/m}(\Omega)$ where $\mathcal{E}_0^{(\boldsymbol{M}_j),h}(\Omega)$ is the following Banach space: its elements are the functions $f \in \mathcal{C}^\infty(\Omega)$ such that $\mathrm{D}^\alpha f \in \mathcal{C}_0(\Omega)$ for every $\alpha \in \mathbb{N}_0^k$ and

$$\|f\|_{j,h} := \sup_{\alpha\in\mathbb{N}_0^k} \frac{\|\mathrm{D}^\alpha f\|_\Omega}{h^{|\alpha|} M_{j,|\alpha|}} < \infty;$$

its norm is $\|.\|_{j,h}$.

The (FS)*-space* $\mathcal{E}^{(\mathfrak{M})}(\Omega)$ is the projective limit of the (FS)-spaces $\mathcal{E}^{(\boldsymbol{M}_j)}(\Omega)$. For every $j \in \mathbb{N}$, $\mathcal{E}^{(\boldsymbol{M}_j)}(\Omega)$ is the usual Beurling class of the elements f in $\mathcal{C}^\infty(\Omega)$ such that, for every non void compact subset H of Ω and every $h > 0$,

$$|f|_{j,H,h} := \sup_{\alpha\in\mathbb{N}_0^k} \frac{\|\mathrm{D}^\alpha f\|_H}{h^{|\alpha|} M_{j,|\alpha|}} < \infty$$

and it is endowed with the system of semi-norms $\{|.|_{j,H,h} : H \Subset \Omega, h > 0\}$.

The (LFS)*-space* $\mathcal{D}^{(\mathfrak{M})}(\Omega)$ is the inductive limit of the spaces $\mathcal{D}^{(\mathfrak{M})}(H) = \mathcal{E}_0^{(\mathfrak{M})}(H^\circ)$ where H runs through the non void compact subsets of Ω.

Let us recall ([6], Theorem 8.2) that, if $\mathfrak{M}$ is regular, then the spaces $\mathcal{E}^{(\mathfrak{M})}(\Omega)$, $\mathcal{D}^{(\mathfrak{M})}(K)$ and $\mathcal{D}^{(\mathfrak{M})}(\Omega)$ are nuclear.

3. Intermediate step

Given a Banach space $X = (X, \|.\|)$, it is possible to construct a Fréchet space Z "similar" for instance to $\mathcal{E}^{(\mathfrak{M})}(\Omega)$.

Its elements are the elements $\varkappa = (x_\alpha)_{\alpha\in\mathbb{N}_0^k}$ of $X^{\mathbb{N}_0^k}$ such that

$$\|\varkappa\|_j := \sup_{\alpha\in\mathbb{N}_0^k} \frac{j^{|\alpha|}\,\|x_\alpha\|}{M_{j,|\alpha|}} < \infty$$

for every $j \in \mathbb{N}$, its system of continuous semi-norms being $\{\|.\|_j : j \in \mathbb{N}\}$. It is a vector-valued Köthe space.

We are interested in the use of its dual. For this purpose, given $u \in Z'$ and $\alpha \in \mathbb{N}_0^k$, we denote by u_α the functional

$$u_\alpha \colon X \to \mathbb{C}; \qquad \langle x, u_\alpha\rangle := \langle \varkappa, u\rangle$$

where $\varkappa$ is defined by $x_\alpha = x$ and $x_\beta = 0$ if $\beta \neq \alpha$. It is clear that u_α belongs to X'.

For the sake of completeness, let us state and prove the following two known results that will be of systematic use later on.

Proposition 3.1. *For every $u \in Z'$, there is $j \in \mathbb{N}$ such that*

$$\sup_{\alpha\in\mathbb{N}_0^k} j^{-|\alpha|} M_{j,|\alpha|}\,\|u_\alpha\| \leqslant \|u\|_{(j)} := \sup_{\|\varkappa\|_j\leqslant 1} |\langle \varkappa, u\rangle| < \infty \tag{1}$$

and we have

$$\langle \varkappa, u\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \langle x_\alpha, u_\alpha\rangle, \qquad \forall \varkappa \in Z, \tag{2}$$

these series converging absolutely and uniformly on the bounded subsets of Z.

Proof. As u belongs to Z', there is $j \in \mathbb{N}$ such that $\|u\|_{(j)} < \infty$. For every $\alpha \in \mathbb{N}_0^k$ and $x \in X$, $\varkappa \in Z$ defined by $x_\alpha = x$ and $x_\beta = 0$ if $\beta \neq \alpha$ verifies $\|\varkappa\|_j = j^{|\alpha|} M_{j,|\alpha|}^{-1}\,\|x\|$ hence $\|u\|_{(j)} \geqslant j^{-|\alpha|} M_{j,|\alpha|} \|u_\alpha\|$ and the inequality (1).

Given $\varkappa \in Z$ and $\beta \in \mathbb{N}_0^k$, let us define $\varkappa^\beta$ by $x_\beta^\beta = x_\beta$ and $x_\alpha^\beta = 0$ if $\alpha \neq \beta$. Then the family $(\varkappa^\beta \colon \beta \in \mathbb{N}_0^k)$ is summable in Z, with limit $\varkappa$. Indeed for every j, $q \in \mathbb{N}$, we successively have

$$\|\varkappa - \sum_{|\beta|\leqslant q} \varkappa^\beta\|_j = \sup_{|\alpha|>q} \frac{j^{|\alpha|}\,\|x_\alpha\|}{M_{j,|\alpha|}} \leqslant \sup_{|\alpha|>q} \frac{(2j)^{|\alpha|}\,\|x_\alpha\|}{2^{|\alpha|} M_{2j,|\alpha|}} \leqslant 2^{-q}\,\|\varkappa\|_{2j}$$

hence the equality (2).

Now let B be any bounded subset of Z. Setting $b := \sup_{\varkappa\in B} \|\varkappa\|_{2kj} < \infty$, for every $\varkappa \in B$ and $\beta \in \mathbb{N}_0^k$, we successively have

$$\begin{aligned} |\langle x_\beta, u_\beta\rangle| \leqslant \|x_\beta\|\,\|u_\beta\| &= \frac{(2kj)^{|\beta|}\,\|x_\beta\|}{M_{j,|\beta|}} \frac{M_{j,|\beta|}\,\|u_\beta\|}{(2kj)^{|\beta|}} \\ &\leqslant \frac{1}{(2k)^{|\beta|}} \frac{(2kj)^{|\beta|}\,\|x_\beta\|}{M_{2kj,|\beta|}} \sup_{\alpha\in\mathbb{N}_0^k} \frac{M_{j,|\alpha|}\,\|u_\alpha\|}{j^{|\alpha|}} \leqslant \frac{b}{(2k)^{|\beta|}}\,\|u\|_{(j)}\,. \end{aligned}$$

Hence the conclusion since the series $\sum_{\beta\in\mathbb{N}_0^k} (2k)^{-|\beta|}$ converges. ■

Proposition 3.2. *Let* $(v_\alpha \colon \alpha \in \mathbb{N}_0^k)$ *be a family of elements of* X'. *If there is* $j \in \mathbb{N}$ *such that* $\sup_{\alpha\in\mathbb{N}_0^k} j^{-|\alpha|} M_{j,|\alpha|} \|v_\alpha\| < \infty$, *there is a unique element* u *of* Z' *such that* $u_\alpha = v_\alpha$ *for every* $\alpha \in \mathbb{N}_0^k$.

Proof. Let us note that $u \colon Z \to \mathbb{C}$ defined by $\langle \varkappa, u\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \langle x_\alpha, v_\alpha\rangle$ for every $\varkappa \in Z$ is a well defined continuous linear functional on Z since, for every $\varkappa \in Z$ and $\beta \in \mathbb{N}_0^k$, we successively have

$$|\langle x_\beta, v_\beta\rangle| \leqslant \|x_\beta\| \, \|v_\beta\| \leqslant \frac{1}{(2k)^{|\beta|}} \|\varkappa\|_{2kj} \sup_{\alpha\in\mathbb{N}_0^k} j^{-|\alpha|} M_{j,|\alpha|} \|v_\alpha\| \,.$$

Now the previous Proposition leads to $\langle \varkappa, u\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \langle x_\alpha, u_\alpha\rangle$ for every $\varkappa \in Z$. In fact, we have $u_\alpha = v_\alpha$ for every $\alpha \in \mathbb{N}_0^k$ since, for every $\beta \in \mathbb{N}_0^k$ and $x \in X$, we have $\langle x, u_\beta\rangle = \langle \varkappa^\beta, u\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \langle x_\alpha^\beta, v_\alpha\rangle = \langle x, v_\beta\rangle$.

To conclude, we note that the uniqueness of u comes from the fact that, in the previous proof, we obtained as a by-result that $\{\varkappa^\beta \colon x \in X, \beta \in \mathbb{N}_0^k\}$ is total in Z. ■

4. Structure of the elements of $\mathcal{E}_0^{(\mathfrak{M})}(\Omega)'$

In this paragraph we are going to apply the results of the preceding one with $X = \mathcal{C}_0(\Omega)$.

Let us consider

$$V := \{(\mathrm{D}^\alpha f)_{\alpha\in\mathbb{N}_0^k} \colon f \in \mathcal{E}_0^{(\mathfrak{M})}(\Omega)\}$$

as a topological vector subspace of Z and introduce the map

$$\Phi \colon \mathcal{E}_0^{(\mathfrak{M})}(\Omega) \to V; \qquad f \mapsto (\mathrm{D}^\alpha f)_{\alpha\in\mathbb{N}_0^k};$$

it is clear that Φ is a topological isomorphism.

Theorem 4.1. *Let* $(\mu_\alpha \colon \alpha \in \mathbb{N}_0^k)$ *be a family of Borel measures on* Ω. *If there is* $j \in \mathbb{N}$ *such that* $\sup_{\alpha\in\mathbb{N}_0^k} j^{-|\alpha|} M_{j,|\alpha|} |\mu_\alpha|(\Omega) < \infty$, *then*

$$S \colon \mathcal{E}_0^{(\mathfrak{M})}(\Omega) \to \mathbb{C}; \qquad f \mapsto \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha f \, d\mu_\alpha$$

is a well defined continuous linear functional, these series converging absolutely and uniformly on the bounded subsets of $\mathcal{E}_0^{(\mathfrak{M})}(\Omega)$.

Proof. For every $\alpha \in \mathbb{N}_0^k$, $\langle ., \mu_\alpha\rangle = \int_\Omega . \, d\mu_\alpha$ is a continuous linear functional on $\mathcal{C}_0(\Omega)$, of norm $\|\mu_\alpha\| = |\mu_\alpha|(\Omega)$. Therefore, by the Propositions 3.2 and 3.1 successively, there is $u \in Z'$ such that

$$\langle (f_\alpha)_{\alpha\in\mathbb{N}_0^k}, u\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \langle f_\alpha, \mu_\alpha\rangle, \qquad \forall (f_\alpha)_{\alpha\in\mathbb{N}_0^k} \in Z,$$

and these series converge absolutely and uniformly on the bounded subsets of Z.

Now we consider the restriction of u to V, that we still denote by u to simplify the notation. For every $f \in \mathcal{E}_0^{(\mathfrak{M})}(\Omega)$, we get

$$\langle (\mathrm{D}^\alpha f)_{\alpha \in \mathbb{N}_0^k}, u \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \langle \mathrm{D}^\alpha f, \mu_\alpha \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega \mathrm{D}^\alpha f \, d\mu_\alpha.$$

Let finally ${}^t\Phi \colon V' \to \mathcal{E}_0^{(\mathfrak{M})}(\Omega)'$ denote the transpose of Φ and set $S := {}^t\Phi u$. For every $f \in \mathcal{E}_0^{(\mathfrak{M})}(\Omega)$, we then get

$$\langle (\mathrm{D}^\alpha f)_{\alpha \in \mathbb{N}_0^k}, u \rangle = \langle \Phi f, u \rangle = \langle f, {}^t\Phi u \rangle = \langle f, S \rangle$$

hence the conclusion since it is clear that these series converge absolutely and uniformly on the bounded subsets of $\mathcal{E}_0^{(\mathfrak{M})}(\Omega)$. ■

Theorem 4.2. *For every $S \in \mathcal{E}_0^{(\mathfrak{M})}(\Omega)'$, there are $j \in \mathbb{N}$ and a family $(\mu_\alpha \colon \alpha \in \mathbb{N}_0^k)$ of Borel measures on Ω such that*

$$\sup_{\alpha \in \mathbb{N}_0^k} j^{-|\alpha|} M_{j,|\alpha|} |\mu_\alpha|(\Omega) < \infty$$

and

$$\langle f, S \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha f \, d\mu_\alpha, \qquad \forall f \in \mathcal{E}_0^{(\mathfrak{M})}(\Omega),$$

these series converging absolutely and uniformly on the bounded subsets of $\mathcal{E}_0^{(\mathfrak{M})}(\Omega)$.

Proof. Let us denote by Ψ the map Φ considered as a map from $\mathcal{E}_0^{(\mathfrak{M})}(\Omega)$ into Z. As its transpose ${}^t\Psi$ is surjective, there is $u \in Z'$ such that ${}^t\Psi u = S$. The Proposition 3.1 provides then $j \in \mathbb{N}$ and a family $(u_\alpha \colon \alpha \in \mathbb{N}_0^k)$ of elements of $\mathcal{C}_0(\Omega)'$ such that

$$\sup_{\alpha \in \mathbb{N}_0^k} j^{-|\alpha|} M_{j,|\alpha|} \, \|u_\alpha\| < \infty$$

and

$$\langle (\mathrm{D}^\alpha f)_{\alpha \in \mathbb{N}_0^k}, u \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \langle \mathrm{D}^\alpha f, u_\alpha \rangle, \qquad \forall f \in \mathcal{E}_0^{(\mathfrak{M})}(\Omega),$$

these series converging absolutely and uniformly on the bounded subsets of $\mathcal{E}_0^{(\mathfrak{M})}(\Omega)$. For every $\alpha \in \mathbb{N}_0^k$, the Riesz representation theorem provides then a Borel measure μ_α on Ω such that $\|u_\alpha\| = |\mu_\alpha|(\Omega)$ and $\langle g, u_\alpha \rangle = \int_\Omega g \, d\mu_\alpha$ for every $g \in \mathcal{C}_0(\Omega)$. Hence the conclusion since we have

$$\langle (\mathrm{D}^\alpha f)_{\alpha \in \mathbb{N}_0^k}, u \rangle = \langle \Psi f, u \rangle = \langle f, {}^t\Psi u \rangle = \langle f, S \rangle.$$ ■

5. Structure of the elements of $\mathcal{D}^{(\mathfrak{M})}(\Omega)'$ with compact support

Theorem 5.1. *Let $S \in \mathcal{D}^{(\mathfrak{M})}(\Omega)'$ have a compact support H and let K be a compact subset of Ω such that $H \subset K^\circ$.*

Then there are an integer $s \in \mathbb{N}$ and a family $(\nu_\alpha\colon \alpha \in \mathbb{N}_0^k)$ of Borel measures on Ω such that

$$\sup_{\alpha \in \mathbb{N}_0^k} s^{-|\alpha|} M_{s,|\alpha|} |\nu_\alpha|(\Omega) < \infty,$$

$$\operatorname{supp}(\nu_\alpha) \subset K, \qquad \forall \alpha \in \mathbb{N}_0^k,$$

$$\langle \varphi, S\rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \, d\nu_\alpha, \qquad \forall \varphi \in \mathcal{D}^{(\mathfrak{M})}(\Omega),$$

these series converging absolutely and uniformly on the bounded subsets of $\mathcal{E}_0^{(\mathfrak{M})}(\Omega)$ hence of $\mathcal{D}^{(\mathfrak{M})}(\Omega)$.

Proof. Let ψ be an element of $\mathcal{D}^{(\mathfrak{M})}(\Omega)$, identically 1 on a neighbourhood of H and with support contained in K°. Proceeding as in [5], S is the restriction to $\mathcal{D}^{(\mathfrak{M})}(\Omega)$ of the element $\langle .\psi, S\rangle$ of $\mathcal{E}^{(\mathfrak{M})}(\Omega)'$. Therefore, as the canonical injection from $\mathcal{E}_0^{(\mathfrak{M})}(\Omega)$ into $\mathcal{E}^{(\mathfrak{M})}(\Omega)$ is continuous, the restriction T of $\langle .\psi, S\rangle$ to $\mathcal{E}_0^{(\mathfrak{M})}(\Omega)$ is a continuous linear extension of S. Moreover Theorem 4.2 provides an integer $j \in \mathbb{N}$ and a family $(\mu_\alpha\colon \alpha \in \mathbb{N}_0^k)$ of Borel measures on Ω such that

$$\sup_{\alpha \in \mathbb{N}_0^k} j^{-|\alpha|} M_{j,|\alpha|} |\mu_\alpha|(\Omega) < \infty,$$

$$\langle f, T\rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega \mathrm{D}^\alpha f \, d\mu_\alpha, \quad \forall f \in \mathcal{E}_0^{(\mathfrak{M})}(\Omega),$$

these series converging absolutely and uniformly on the bounded subsets of $\mathcal{E}_0^{(\mathfrak{M})}(\Omega)$.

Let us fix the integer s by the condition $s > 4kj$.

For every $f \in \mathcal{E}_0^{(\mathfrak{M})}(\Omega)$, we have $\langle f\psi, S\rangle = \langle f\psi^2, S\rangle$ hence

$$\langle f, T\rangle = \langle f\psi, T\rangle = \sum_{\alpha \in \mathbb{N}_0^k} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \int_\Omega \mathrm{D}^\beta \psi \, \mathrm{D}^{\alpha-\beta} f \, d\mu_\alpha.$$

Let us prove that this series converges absolutely so that, setting $\gamma = \alpha - \beta$, we also have

$$\langle f, T\rangle = \sum_{\gamma \in \mathbb{N}_0^k} \sum_{\beta \in \mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\gamma!} \int_\Omega \mathrm{D}^\beta \psi \, \mathrm{D}^\gamma f \, d\mu_{\beta+\gamma}.$$

For this purpose, it suffices to note that

$$\sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \left\| \mathrm{D}^\beta \psi \right\|_\Omega \left\| \mathrm{D}^{\alpha-\beta} f \right\|_\Omega \leqslant (2k)^{-|\alpha|} \left\|\psi\right\|_{s,1/s} \left\|f\right\|_{s,1/s} j^{-|\alpha|} M_{s,|\alpha|}$$

leads to

$$\sum_{\alpha\in\mathbb{N}_0^k}\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\int_\Omega |\mathrm{D}^\beta\psi|\,|\mathrm{D}^{\alpha-\beta}f|\,d|\mu_\alpha|$$
$$\leqslant \|f\|_{s,1/s}\,\|\psi\|_{s,1/s}\sum_{\alpha\in\mathbb{N}_0^k}(2k)^{-|\alpha|}\sup_{\gamma\in\mathbb{N}_0^k} j^{-|\gamma|}M_{j,|\gamma|}|\mu_\gamma|(\Omega)<\infty.$$

Next we prove that, for every $\gamma\in\mathbb{N}_0^k$,

$$V_\gamma\colon \mathcal{C}_0(\Omega)\to\mathbb{C};\qquad g\mapsto\sum_{\beta\in\mathbb{N}_0^k}\frac{(\beta+\gamma)!}{\beta!\gamma!}\int_\Omega g\mathrm{D}^\beta\psi\,d\mu_{\beta+\gamma}$$

is a well defined continuous linear functional. In fact, it suffices to note that we successively have

$$|\langle g,V_\gamma\rangle|\leqslant\|\psi\|_{s,1/s}\,\|g\|_\Omega\sum_{\beta\in\mathbb{N}_0^k}2^{|\beta+\gamma|}s^{-|\beta|}M_{s,|\beta|}|\mu_{\beta+\gamma}|(\Omega)$$
$$\leqslant\|\psi\|_{s,1/s}\,\|g\|_\Omega\,\frac{s^{|\gamma|}}{M_{s,|\gamma|}}\sum_{\beta\in\mathbb{N}_0^k}(2k)^{-|\beta|}\sup_{\alpha\in\mathbb{N}_0^k}j^{-|\alpha|}M_{j,|\alpha|}|\mu_\alpha|(\Omega).$$

So, for every $\gamma\in\mathbb{N}_0^k$, the Riesz representation theorem provides a Borel measure ν_γ on Ω such that $\langle g,V_\gamma\rangle=\int_\Omega g\,d\nu_\gamma$ for every $g\in\mathcal{C}_0(\Omega)$, in particular for $g=\mathrm{D}^\gamma f$. Moreover it is clear that ν_γ has its support contained in K.

Let us now prove that we have

$$\sup_{\gamma\in\mathbb{N}_0^k}s^{-|\gamma|}M_{s,|\gamma|}|\nu_\gamma|(\Omega)<\infty.$$

Indeed, for every $\gamma\in\mathbb{N}_0^k$, we can choose $g\in\mathcal{C}_0(\Omega)$ such that $\|g\|_\Omega\leqslant 2$ and $\langle g,V_\gamma\rangle=|\nu|(\Omega)$. Hence the conclusion since this leads to

$$s^{-|\gamma|}M_{s,|\gamma|}|\nu_\gamma|(\Omega)=s^{-|\gamma|}M_{s,|\gamma|}|\langle g,V_\gamma\rangle|$$
$$\leqslant 2\,\|\psi\|_{s,1/s}\sum_{\beta\in\mathbb{N}_0^k}(2k)^{-|\beta|}\sup_{\alpha\in\mathbb{N}_0^k}j^{-|\alpha|}M_{j,|\alpha|}|\mu_\alpha|(\Omega).$$

Therefore we may apply Theorem 4.1 to the family $(\nu_\alpha\colon\alpha\in\mathbb{N}_0^k)$ of Borel measures on Ω and obtain a continuous linear functional

$$R\colon\mathcal{E}_0^{(\mathfrak{M})}(\Omega)\to\mathbb{C};\qquad f\mapsto\langle f,R\rangle=\sum_{\alpha\in\mathbb{N}_0^k}\int_\Omega\mathrm{D}^\alpha f\,d\nu_\alpha,$$

the series converging absolutely and uniformly on the bounded subsets of $\mathcal{E}_0^{(\mathfrak{M})}(\Omega)$.

Hence the conclusion since we have

$$\langle f,R\rangle=\sum_{\alpha\in\mathbb{N}_0^k}\int_\Omega\mathrm{D}^\alpha f\,d\nu_\alpha=\sum_{\alpha\in\mathbb{N}_0^k}\langle\mathrm{D}^\alpha f,V_\alpha\rangle=\langle f,T\rangle$$

for every $f\in\mathcal{E}_0^{(\mathfrak{M})}(\Omega)$. ■

6. Structure of the elements of $\mathcal{D}^{(\mathfrak{M})}(\Omega)'$

Theorem 6.1. *Let $(u_\alpha : \alpha \in \mathbb{N}_0^k)$ be a family of Radon measures on Ω. If, for every non void compact subset K of Ω, there is $j \in \mathbb{N}$ such that*

$$\sup_{\alpha \in \mathbb{N}_0^k} j^{-|\alpha|} M_{j,|\alpha|} \|u_\alpha\| (K) < \infty,$$

then

$$S : \mathcal{D}^{(\mathfrak{M})}(\Omega) \to \mathbb{C}; \qquad \varphi \mapsto \sum_{\alpha \in \mathbb{N}_0^k} \langle D^\alpha \varphi, u_\alpha \rangle$$

is a well defined continuous linear functional, these series converging absolutely and uniformly on the bounded subsets of $\mathcal{D}^{(\mathfrak{M})}(\Omega)$.

Proof. Let $(K_m)_{m\in\mathbb{N}}$ be a compact exhaustion of Ω such that $K_1^\circ \neq \emptyset$ and $K_m \subset K_{m+1}^\circ$ for every $m \in \mathbb{N}$. For every $m \in \mathbb{N}$, let us identify the spaces $\mathcal{K}(K_m)$ and $\mathcal{C}_0(K_m^\circ)$ and designate by u_α^m the restriction of u_α to $\mathcal{K}(K_m)$. The Riesz representation theorem provides then, for every $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^k$, a Borel measure μ_α^m on K_m° such that

$$\langle f, u_\alpha^m \rangle = \int_{K_m^\circ} f \, d\mu_\alpha^m, \qquad \forall f \in \mathcal{C}_0(K_m^\circ),$$
$$||u_\alpha^m|| = |\mu_\alpha^m|(K_m^\circ), \qquad \forall m \in \mathbb{N}, \alpha \in \mathbb{N}_0^k.$$

So, for every $m \in \mathbb{N}$, there is an integer $j_m \in \mathbb{N}$ such that

$$\sup_{\alpha \in \mathbb{N}_0^k} j_m^{-|\alpha|} M_{j_m,|\alpha|} |\mu_\alpha^m|(K_m^\circ) < \infty$$

and the Theorem 4.1 asserts that

$$S_m : \mathcal{E}_0^{(\mathfrak{M})}(K_m^\circ) \to \mathbb{C}; \qquad f \mapsto \sum_{\alpha \in \mathbb{N}_0^k} \int_{K_m^\circ} \mathrm{D}^\alpha f \, d\mu_\alpha^m$$

is a well defined continuous linear functional and that these series converge absolutely and uniformly on the bounded subsets of $\mathcal{E}_0^{(\mathfrak{M})}(K_m^\circ)$.

The conclusion is now a standard matter. ■

Theorem 6.2. *For every $S \in \mathcal{D}^{(\mathfrak{M})}(\Omega)'$, there is a family $(u_\alpha : \alpha \in \mathbb{N}_0^k)$ of Radon measures on Ω such that*

$$\langle \varphi, S \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \langle D^\alpha \varphi, u_\alpha \rangle, \qquad \forall \varphi \in \mathcal{D}^{(\mathfrak{M})}(\Omega),$$

these series converging absolutely and uniformly on the bounded subsets of $\mathcal{D}^{(\mathfrak{M})}(\Omega)$.

Moreover, for every non empty compact subset K of Ω, there is $j \in \mathbb{N}$ such that

$$\sup_{\alpha \in \mathbb{N}_0^k} j^{-|\alpha|} M_{j,|\alpha|} ||u_\alpha||(K) < \infty.$$

Proof. Let $\{\Omega_m : m \in \mathbb{N}\}$ be an open, locally finite and relatively compact cover of Ω. Let moreover $\{\psi_m : m \in \mathbb{N}\}$ be a partition of unity subordinate to this cover and such that $\psi_m \in \mathcal{D}^{(\mathfrak{M})}(\Omega_m)$ for every $m \in \mathbb{N}$.

For every $m \in \mathbb{N}$, $\psi_m S$ belongs to $\mathcal{D}^{(\mathfrak{M})}(\Omega)'$ and has a compact support contained in Ω_m. Therefore, by Theorem 5.1, there are an integer $s_m \in \mathbb{N}$ and a family $(\nu_\alpha^m : \alpha \in \mathbb{N}_0^k)$ of Borel measures on Ω such that

$$\sup_{\alpha \in \mathbb{N}_0^k} s_m^{-|\alpha|} M_{s_m,|\alpha|} |\nu_\alpha^m|(\Omega) < \infty;$$

$$\operatorname{supp}(\nu_\alpha^m) \subset \Omega_m, \qquad \forall \alpha \in \mathbb{N}_0^k,$$

$$\langle f, \psi_m S\rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega \mathrm{D}^\alpha f \, d\nu_\alpha^m, \qquad \forall f \in \mathcal{E}_0^{(\mathfrak{M})}(\Omega),$$

these series converging absolutely and uniformly on the bounded subsets of $\mathcal{E}_0^{(\mathfrak{M})}(\Omega)$.

For every $f \in \mathcal{K}(\Omega)$, there is only a finite number of integers $m \in \mathbb{N}$ such that $\Omega_m \cap \operatorname{supp}(f) \neq \emptyset$. Therefore, for every $\alpha \in \mathbb{N}_0^k$,

$$u_\alpha : \mathcal{K}(\Omega) \to \mathbb{C}; \qquad f \mapsto \sum_{m \in \mathbb{N}} \int_\Omega f \, d\nu_\alpha^m$$

is a well defined linear functional.

Moreover, for every non void compact subset K of Ω, there is an integer $m_0 \in \mathbb{N}$ such that $K \cap \Omega_m = \emptyset$ for every $m > m_0$. Therefore, if $f \in \mathcal{K}(\Omega)$ has its support contained in K, we get

$$|\langle f, u_\alpha\rangle| \leqslant \|f\|_\Omega \sum_{m=1}^{m_0} |\nu_\alpha^m|(\Omega)$$

which implies that u_α is a Radon measure on Ω such that $\|u_\alpha\|(K) \leqslant \sum_{m=1}^{m_0} |\nu_\alpha^m|(\Omega)$. Moreover if we set $j := \sup\{s_m : m = 1, \ldots, m_0\}$, we obtain

$$\sup_{\alpha \in \mathbb{N}_0^k} j^{-|\alpha|} M_{j,|\alpha|} ||u_\alpha||(K) \leqslant \sum_{m=1}^{m_0} s_m^{-|\alpha|} M_{s_m,|\alpha|} |\nu_\alpha^m|(\Omega) < \infty.$$

Therefore, by Theorem 6.1,

$$R : \mathcal{D}^{(\mathfrak{M})}(\Omega) \to \mathbb{C}; \qquad \varphi \mapsto \sum_{\alpha \in \mathbb{N}_0^k} \langle \mathrm{D}^\alpha \varphi, u_\alpha\rangle$$

is a continuous linear functional, these series converging absolutely and uniformly on the bounded subsets of $\mathcal{D}^{(\mathfrak{M})}(\Omega)$.

To conclude, we just have to note that, if $\varphi \in \mathcal{D}^{(\mathfrak{M})}(\Omega)$ has its support contained in the non void compact subset K of Ω, we successively have

$$\begin{aligned}\langle \varphi, S\rangle &= \langle \varphi, \sum_{m=1}^{m_0} \psi_m S\rangle = \sum_{m=1}^{m_0} \langle \varphi, \psi_m S\rangle = \sum_{m=1}^{m_0} \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega \mathrm{D}^\alpha \varphi \, d\nu_\alpha^m \\ &= \sum_{\alpha\in\mathbb{N}_0^k} \sum_{m=1}^{m_0} \int_\Omega \mathrm{D}^\alpha \varphi \, d\nu_\alpha^m = \sum_{\alpha\in\mathbb{N}_0^k} \langle \mathrm{D}^\alpha \varphi, u_\alpha\rangle. \qquad \blacksquare\end{aligned}$$

7. Notations for the L^p-case

Given p such that $1 \leqslant p \leqslant \infty$, $L^p(\Omega)$ and $\mathcal{L}^p(\Omega)$ designate the classical Lebesgue spaces and for $f \in \tilde{f} \in \mathcal{L}^p(\Omega)$, we set

$$\|f\|_p = ||\tilde{f}||_p = (\int_\Omega |f|^p \; dx)^{1/p} \qquad \text{if } 1 \leqslant p < \infty$$

and

$$\|f\|_\infty = ||\tilde{f}||_\infty = \sup\operatorname{ess}\{|f(x)| : x \in \Omega\}.$$

Given $p \in [1, \infty[$, we now adapt the introduction by Schwartz of the space $\mathcal{D}_{L^p}(\mathbb{R}^k)$ (cf. [11], p. 199) to our setting.

The Fréchet spaces $\mathcal{B}_{L^p}^{(\boldsymbol{M}_j)}(\Omega)$ *and* $\mathcal{B}_{L^p}^{(\mathfrak{M})}(\Omega)$.

For every $j \in \mathbb{N}$, one first considers the Fréchet space $\mathcal{B}_{L^p}^{(\boldsymbol{M}_j)}(\Omega)$ introduced in [13]. Its elements are the $\mathcal{C}^\infty$-functions f on Ω such that $\mathrm{D}^\alpha f \in \mathcal{L}^p(\Omega)$ for every $\alpha \in \mathbb{N}_0^k$ and

$$|f|_{p,j,r} := \sup_{\alpha\in\mathbb{N}_0^k} \frac{||\mathrm{D}^\alpha f||_p}{r^{|\alpha|} M_{j,|\alpha|}} < \infty, \qquad \forall r > 0,$$

and $\{|\cdot|_{p,j,r} : r > 0\}$ is its fundamental system of semi-norms. We then introduce the Fréchet space $\mathcal{B}_{L^p}^{(\mathfrak{M})}(\Omega)$ as the projective limit of the spaces $\mathcal{B}_{L^p}^{(\boldsymbol{M}_j)}(\Omega)$.

The spaces $\mathcal{D}_{L^p}^{(\boldsymbol{M}_j)}(K)$, $\mathcal{D}_{L^p}^{(\boldsymbol{M}_j)}(\Omega)$, $\mathcal{D}_{L^p}^{(\mathfrak{M})}(K)$ *and* $\mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega)$.

Given a non void compact subset K of $\mathbb{R}^k$ and $j \in \mathbb{N}$, one can introduce as in [13] the Fréchet space $\mathcal{D}_{L^p}^{(\boldsymbol{M}_j)}(K)$ as the topological vector subspace of $\mathcal{B}_{L^p}^{(\boldsymbol{M}_j)}(\mathbb{R}^k)$ the elements of which have their support contained in K and the space $\mathcal{D}_{L^p}^{(\boldsymbol{M}_j)}(\Omega)$ as the inductive limit of the spaces $\mathcal{D}_{L^p}^{(\boldsymbol{M}_j)}(H)$ where H runs through the family of the non void compact subsets of Ω.

In the same way, *the Fréchet space* $\mathcal{D}_{L^p}^{(\mathfrak{M})}(K)$ is the topological vector subspace of $\mathcal{B}_{L^p}^{(\mathfrak{M})}(\mathbb{R}^k)$, the elements of which have their support contained in K and *the* (LF)-*space* $\mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega)$ is the inductive limit of the spaces $\mathcal{D}_{L^p}^{(\mathfrak{M})}(H)$ where H runs through the family of the non void compact subsets of Ω.

It is a direct matter to prove the following property.

Proposition 7.1. *The multiplication map*

$$\mathcal{B}\colon \mathcal{B}_{L^p}^{(\mathfrak{M})}(\Omega) \times \mathcal{E}_0^{(\mathfrak{M})}(\Omega) \to \mathcal{B}_{L^p}^{(\mathfrak{M})}(\Omega); \qquad (f, g) \mapsto fg$$

is a well defined continuous bilinear map.

Proposition 7.2. *The space $\mathcal{D}^{(\mathfrak{M})}(\Omega)$ is a dense vector subspace of the space $\mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega)$ and the canonical injection from $\mathcal{D}^{(\mathfrak{M})}(\Omega)$ into $\mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega)$ is a continuous linear map.*

Proof. From the paragraph 5 of [6], we know that, for every $r > 0$, there is $\varphi \in \mathcal{D}^{(\mathfrak{M})}(\mathbb{R}^k)$ identically 1 on a neighbourhood of 0 and having its support contained in $\{x\colon |x| \leqslant r\}$.

Therefore a direct adaptation of the proof of the proposition 5 of [13] with $\psi_i \in \mathcal{D}^{(\mathfrak{M})}(\mathbb{R}^k)$ for every $i \in \mathbb{N}$ leads to the conclusion. ■

The previous result justifies the fact that the elements of $\mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega)'$ may be considered as ultradistributions. The next one says that the spaces $\mathcal{D}^{(\mathfrak{M})}(\Omega)$ and $\mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega)$ coincide under the nuclearity condition that $\mathfrak{M}$ is regular.

Proposition 7.3. *If the matrix $\mathfrak{M}$ is regular, then, for every $p \in [1, \infty[$, the canonical injection I from $\mathcal{D}^{(\mathfrak{M})}(\Omega)$ into $\mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega)$ is a topological isomorphism.*

Proof. As a consequence of the previous result, we just need to prove that, if $\mathfrak{M}$ is regular, I is onto and has a continuous inverse. For this purpose, let us remark that the regularity of $\mathfrak{M}$ implies that, for every $j \in J$, there are $A'_j > 1$ and $H'_j > 1$ such that $M_{j+k,r+k} \leqslant A'_j H'^{rk}_j M_{j,r}$ for every $r \in \mathbb{N}_0$.

Now let K be any non void compact subset of Ω, φ be any element of $\mathcal{D}_{L^p}^{(\mathfrak{M})}(K)$ and $\|.\|_{j,h}$ be any continuous semi-norm on $\mathcal{D}^{(\mathfrak{M})}(K)$. Let us set $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}_0^k$ and let $C > 0$ be such that $K \subset [-C, C]^k$. For every $\alpha \in \mathbb{N}_0^k$, we then have

$$\mathrm{D}^\alpha \varphi(x) = \int_{\mathbb{R}^k} \chi_{[-C,x_1]\times\cdots[-C,x_k]}(t) \mathrm{D}^{\alpha+\mathbf{1}} \varphi(t)\, dt, \qquad \forall x \in \Omega,$$

hence $||\mathrm{D}^\alpha \varphi||_K \leqslant B||\mathrm{D}^{\alpha+\mathbf{1}}\varphi||_p$ with $B = (2C)^{k/q}$ if $q \in]1, \infty[$ and $B = 1$ if $q = \infty$. So, for $s \in]0, h/H'^k_j[$, we get

$$\|\varphi\|_{j,h} \leqslant B\, |\varphi|_{p,j+k,s} \sup_{\alpha \in \mathbb{N}_0^k} \frac{s^{|\alpha|+k} M_{j+k,|\alpha|+k}}{h^{|\alpha|} M_{j,|\alpha|}} \leqslant B A'_j h^k\, |\varphi|_{p,j+k,s}$$

and we conclude at once. ■

Open question. Are the spaces $\mathcal{D}^{(\mathfrak{M})}(\Omega)$ and $\mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega)$ different in general?

8. Structure of the elements of $\mathcal{B}_{L^p}^{(\mathfrak{M})}(\Omega)'$

In the following two proofs, we apply the intermediate step with $X = L^p(\Omega)$.

Let us consider
$$V := \{(\mathrm{D}^\alpha f)_{\alpha \in \mathbb{N}_0^k} \colon f \in \mathcal{B}_{L^p}^{(\mathfrak{M})}(\Omega)\}$$
as a topological vector subspace of Z and introduce the map

$$\Phi \colon \mathcal{B}_{L^p}^{(\mathfrak{M})}(\Omega) \to V; \qquad f \mapsto (\mathrm{D}^\alpha f)_{\alpha \in \mathbb{N}_0^k};$$

it is clear that Φ is a topological isomorphism.

Let also q designate the conjugate number of p, i.e. $q = \infty$ if $p = 1$ and $1/q = 1 - 1/p$ if $p \in]1, \infty[$.

Proceeding as in the proof of the Theorems 4.1 and 4.2 provides directly the following two results.

Theorem 8.1. *Let $(g_\alpha : \alpha \in \mathbb{N}_0^k)$ be a family of elements of $L^q(\Omega)$.*

If there is $j \in \mathbb{N}$ such that $\sup_{\alpha\in\mathbb{N}_0^k} j^{-|\alpha|} M_{j,|\alpha|} ||g_\alpha||_q < \infty$, then

$$S : \mathcal{B}_{L^p}^{(\mathfrak{M})}(\Omega) \to \mathbb{C}; \qquad f \mapsto \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega g_\alpha D^\alpha f \, dx$$

is a well defined continuous linear functional, these series converging absolutely and uniformly on the bounded subsets of $\mathcal{B}_{L^p}^{(\mathfrak{M})}(\Omega)$.

Theorem 8.2. *For every $S \in \mathcal{B}_{L^p}^{(\mathfrak{M})}(\Omega)'$, there are $j \in \mathbb{N}$ and a family $(g_\alpha : \alpha \in \mathbb{N}_0^k)$ of elements of $L^q(\Omega)$ such that*

$$\sup_{\alpha\in\mathbb{N}_0^k} j^{-|\alpha|} M_{j,|\alpha|} ||g_\alpha||_q < \infty$$

$$\langle f, S\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega g_\alpha D^\alpha f \, dx, \qquad \forall f \in \mathcal{B}_{L^p}^{(\mathfrak{M})}(\Omega),$$

these series converging absolutely and uniformly on the bounded subsets of $\mathcal{B}_{L^p}^{(\mathfrak{M})}(\Omega)$.

9. Structure of the elements of $\mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega)'$

Theorem 9.1. *Let $S \in \mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega)'$ have a compact support H and let K be a compact subset of Ω such that $H \subset K^\circ$.*

Then there are an integer $s \in \mathbb{N}$ and a family $(g_\alpha : \alpha \in \mathbb{N}_0^k)$ of elements of $L^q(\Omega)$ such that

$$\sup_{\alpha\in\mathbb{N}_0^k} s^{-|\alpha|} M_{s,|\alpha|} ||g_\alpha||_q < \infty,$$

$$\operatorname{supp}(g_\alpha) \subset K, \qquad \forall \alpha \in \mathbb{N}_0^k,$$

$$\langle \varphi, S\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega g_\alpha D^\alpha \varphi \, dx, \qquad \forall \varphi \in \mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega),$$

these series converging absolutely and uniformly on the bounded subsets of $\mathcal{B}_{L^p}^{(\mathfrak{M})}(\Omega)$ hence on those of $\mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega)$.

Proof. Let ψ be an element of $\mathcal{D}^{(\mathfrak{M})}(\Omega)$, identically 1 on a neighbourhood of H and with support contained in K°. By Proposition 7.1, we know that

$$T : \mathcal{B}_{L^p}^{(\mathfrak{M})}(\Omega) \to \mathbb{C}; \qquad f \mapsto \langle f\psi, S\rangle$$

is a continuous linear functional. Therefore, Theorem 8.2 provides an integer $j \in \mathbb{N}$ and a family $(g_\alpha \colon \alpha \in \mathbb{N}_0^k)$ of elements of $L^q(\Omega)$ such that

$$\sup_{\alpha \in \mathbb{N}_0^k} j^{-|\alpha|} M_{j,|\alpha|} ||g_\alpha||_q < \infty,$$

$$\langle f, T \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega g_\alpha \mathrm{D}^\alpha f \, dx, \qquad \forall f \in \mathcal{B}_{L^p}^{(\mathfrak{M})}(\Omega),$$

these series converging absolutely and uniformly on the bounded subsets of $\mathcal{B}_{L^p}^{(\mathfrak{M})}(\Omega)$.

To conclude, one has just then to follow the argument of the proof of the Theorem 5.1. ■

Theorem 9.2. *Let $(g_\alpha \colon \alpha \in \mathbb{N}_0^k)$ be a family of elements of $L^q_{\mathrm{loc}}(\Omega)$. If, for every non void compact subset K of Ω, there is $j \in \mathbb{N}$ such that*

$$\sup_{\alpha \in \mathbb{N}_0^k} j^{-|\alpha|} M_{j,|\alpha|} ||g_\alpha||_{q,K} < \infty,$$

then

$$S \colon \mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega) \to \mathbb{C}; \qquad \varphi \mapsto \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega g_\alpha D^\alpha \varphi \, dx$$

is a well defined continuous linear functional and these series converge absolutely and uniformly on the bounded subsets of $\mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega)$.

Proof. Let $(K_m)_{m \in \mathbb{N}}$ be a compact exhaustion of Ω with negligeable borders such that $K_1^\circ \neq \emptyset$ and $K_m \subset K_{m+1}^\circ$ for every $m \in \mathbb{N}$. For every $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^k$, $g_\alpha^m = g_\alpha|_{K_m^\circ}$ belongs to $L^q(K_m^\circ)$ with $\|g_\alpha^m\|_q = \|g_\alpha\|_{q,K_m}$. Therefore, by Theorem 8.1,

$$S_m \colon \mathcal{B}_{L^p}^{(\mathfrak{M})}(K_m^\circ) \to \mathbb{C}; \qquad f \mapsto \sum_{\alpha \in \mathbb{N}_0^k} \int_{K_m^\circ} g_m^\alpha \mathrm{D}^\alpha f \, dx$$

is a well defined continuous linear functional on $\mathcal{B}_{L^p}^{(\mathfrak{M})}(K_m^\circ)$, these series converging absolutely and uniformly on the bounded subsets of $\mathcal{B}_{L^p}^{(\mathfrak{M})}(K_m^\circ)$. As $\mathcal{D}_{L^p}^{(\mathfrak{M})}(K_m)$ can be considered as a topological vector subspace of $\mathcal{B}_{L^p}^{(\mathfrak{M})}(K_m^\circ)$, the conclusion is now a standard matter. ■

Theorem 9.3. *For every $S \in \mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega)'$, there is a family $(g_\alpha \colon \alpha \in \mathbb{N}_0^k)$ of elements of $L^q_{\mathrm{loc}}(\Omega)$ such that*

$$\langle \varphi, S \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega g_\alpha D^\alpha \varphi \, dx, \qquad \forall \varphi \in \mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega),$$

these series converging absolutely and uniformly on the bounded subsets of $\mathcal{D}_{L^p}^{(\mathfrak{M})}(\Omega)$.

Moreover, for every non void compact subset K of Ω, there is $j \in \mathbb{N}$ such that

$$\sup_{\alpha \in \mathbb{N}_0^k} j^{-|\alpha|} M_{j,|\alpha|} ||g_\alpha||_{q,K} < \infty.$$

Proof. One has just to proceed as in the proof of the Theorem 6.2, replacing $\mathcal{K}(\Omega)$ by $L^p_{\text{comp}}(\Omega)$. ∎

The Proposition 7.3 leads then directly to the following result.

Corollary 9.4. *If $\mathfrak{M}$ is regular and q belongs to $]1,\infty[\cup\{\infty\}$, then, for every $S \in \mathcal{D}^{(\mathfrak{M})}(\Omega)'$, there is a family $(g_\alpha : \alpha \in \mathbb{N}_0^k)$ of elements of $L^q_{\text{loc}}(\Omega)$ such that*

$$\langle \varphi, S \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega g_\alpha D^\alpha \varphi \, dx, \qquad \forall \varphi \in \mathcal{D}^{(\mathfrak{M})}(\Omega),$$

these series converging absolutely and uniformly on the bounded subsets of $\mathcal{D}^{(\mathfrak{M})}(\Omega)$.

Moreover, for every non void compact subset K of Ω, there is $j \in \mathbb{N}$ such that

$$\sup_{\alpha \in \mathbb{N}_0^k} j^{-|\alpha|} M_{j,|\alpha|} ||g_\alpha||_{q,K} < \infty.$$

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