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FRÉCHET SPACES INVARIANT UNDER DIFFERENTIAL OPERATORS

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Dedicated to the memory of Susanne Dierolf

Abstract: A Fréchet space of distributions which is stable under differential operators is continuously included in $\mathcal{C}^\infty$. We give extensions of this result to the ultradifferentiable setting and show their connection with the problem of iterates of differential operators.

Keywords: ultradistributions, elliptic and hypoelliptic polynomials, iterates of an operator

1. Introduction

M. Langenbruch and J. Voigt proved in [14] that a Fréchet space of distributions which is stable under differential operators is continuously included in $\mathcal{C}^\infty$. As they assert in [14], this result was more or less folklore. They also showed that to guarantee this continuous inclusion it is enough to assume that the Fréchet space is stable under a single hypoelliptic differential operator $P(D)$ and that this property in fact characterizes the hypoellipticity of the operator. Our aim is to provide extensions of these results to the ultradifferentiable setting and show their connection with the problem of iterates of differential operators.

The study of several classes of ultradifferentiable functions and ultra- distributions has been a very active area of research during the last two decades. These are intermediate classes between real analytic functions and the class of all $\mathcal{C}^\infty$-functions. There are essentially two ways to introduce them, the theory of Komatsu [11], in which one looks at the growth of the derivatives on compact sets, and the theory developed by Björck [1] in 1966, following the ideas previously announced by Beurling, in which one pays attention at the growth of the Fourier transforms. We will work with ultradifferentiable functions as defined by Braun, Meise and Taylor [5]. Their point of view permits a unified treatment of both theories, contains the most relevant cases of Komatsu's theory and it is strictly

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larger than Beurling-Björck 's one. The relation between both theories has been clarified in [3].

Observe that the class of ultradistributions of Roumieu type $E := \mathcal{D}'_{\{\omega\}}(\mathbb{R}^d)$ is a Fréchet space invariant under differential operators which is not included in $\mathcal{C}^\infty(\mathbb{R}^d)$. Nevertheless, a similar result to that of Langenbruch and Voigt can be obtained after imposing the extra assumption that E is stable under suitable differential operators of infinite order, available in any non quasianalytic class other than $\mathcal{C}^\infty(\mathbb{R}^d)$. This is the content of Theorem 5.

Each hypoelliptic linear partial differential operator with constant coefficients is also Gevrey hypoelliptic for some Gevrey class depending on the operator, hence it makes sense to study whether Fréchet spaces of distributions invariant under a single hypoelliptic operator and satisfying some extra assumptions should be contained not only in the space of all smooth functions but in a smaller class of ultradifferentiable functions. This question is related to the problem of iterates, which roughly speaking consists in characterizing the functions in a given non-quasianalytic class in terms of the behaviour of the iterates of a fixed differential operator when acting on these functions.

Let us recall that in 1960, H. Komatsu [10], using tools introduced by L. Hörmander [8], proved that given a elliptic differential operator $P(D)$ of order m, a function $f \in \mathcal{C}^\infty(\Omega)$ is real analytic if and only if for each compact subset $K \subset\subset \Omega$ there exists a constant $C > 0$ such that for each $j \in \mathbb{N}_0$

$$\|P^j(D)f\|_{2,K} \leqslant C^{j+1}(j!)^m,$$

where $P^j(D)$ is the j-th iterate of $P(D)$, i.e., $P^j(D) = P(D)\underbrace{\circ \cdots \circ}_{j} P(D)$. In 1973, E. Newberger and Z. Zielezny [16] considered extensions of this result to the setting of Gevrey classes: let $\mathcal{G}^s(\Omega)$ be the Gevrey class of exponent $s > 1$ and let $\mathcal{G}^s_P(\Omega)$ be the class of smooth functions in Ω such that for each $K \subset\subset \Omega$ there exists a constant $C > 0$ such that for any $j \in \mathbb{N}_0$,

$$\|P^j(D)f\|_{2,K} \leqslant C^{j+1}(j!)^s,$$

then

$$\mathcal{G}^s(\Omega) = \mathcal{G}^{ms}_P(\Omega)$$

whenever P is an elliptic polynomial with degree m.

The problem of the iterates consists in giving conditions on P in order to guarantee the equality $\mathcal{G}^s(\Omega) = \mathcal{G}^{ms}_P(\Omega)$.

In a recent paper, the third author [12] has considered the problem of iterates for non-quasianalytic classes in the sense of Braun, Meise and Taylor. He has shown that for a given elliptic polynomial P of degree m we always have $\mathcal{E}_{P,*(t^{\frac{1}{m}})}(\Omega) = \mathcal{E}_{*(t)}(\Omega)$. Moreover, for weight functions ω verifying a growth condition considered by J. Bonet, R. Meise and S.N. Melikhov in [3] the ellipticity of the polynomial is necessary in order to achieve the equality above.

In the last part of the paper we address the question whether a Fréchet space $E \subset \mathcal{D}'_{(\omega)}(\mathbb{R}^d)$ which is invariant under the action of a single one elliptic operator $P(D)$ that satisfies some extra assumptions on equicontinuity is necessarily contained in $\mathcal{E}_{(\omega)}(\mathbb{R}^d)$. The case that ellipticity is replaced by the weaker assumption of $(\omega)-$hypoellipticity is also considered. As a consequence we obtain several results related to the problem of iterates for non-quasianalytic classes.

2. Notation and Preliminaries

First we introduce the spaces of functions and ultradistributions and most of the notation that will be used in the sequel.

Definition 1 ([5]). *A non-quasianalytic weight function is an increasing continuous function* $\omega : [0,\infty[\to [0,\infty[$ *with the following properties:*

(α) *there exists* $L \geqslant 0$ *with* $\omega(et) \leqslant L(\omega(t)+1)$ *for all* $t \geqslant 0$*,*
(β) $\int_1^\infty \frac{\omega(t)}{t^2}dt < \infty$*,*
(γ) $\log(t) = o(\omega(t))$ *as* t *tends to* ∞*,*
(δ) $\varphi : t \to \omega(e^t)$ *is convex.*

The condition (β) is called non-quasianalytic condition and it implies $\omega(t) = o(t)$ as t tends to ∞. Moreover, it implies the existence of functions with compact support in the class of ultradifferentiable functions.

The *Young conjugate* $\varphi^* : [0,\infty[\to \mathbb{R}$ of φ is given by

$$\varphi^*(s) := \sup\{st - \varphi(t),\, t \geqslant 0\}.$$

There is no loss of generality to assume that ω vanishes on $[0,1]$. Then φ^* has only non-negative values, it is convex and $\varphi^*(t)/t$ is increasing and tends to ∞ as $t \to \infty$ and $\varphi^{**} = \varphi$.

Example 2. The following functions are, after a change in some interval $[0,M]$, examples of weight functions:

(i) $\omega(t) = t^d$ for $0 < d < 1$.
(ii) $\omega(t) = (\log(1+t))^s$, $s > 1$.
(iii) $\omega(t) = t(\log(e+t))^{-\beta}$, $\beta > 1$.
(iv) $\omega(t) = \exp(\beta(\log(1+t))^\alpha)$, $0 < \alpha < 1$.

Definition 3. ([5]) Let ω be a weight function. For an open set $\Omega \subset \mathbb{R}^d$ we set

$$\mathcal{E}_{(\omega)}(\Omega) := \{f \in \mathcal{C}^\infty(\Omega) : |f|_{K,\lambda} < \infty \text{ for every } \lambda > 0, \text{ and every } K \subset \Omega \text{ compact}\},$$

where

$$|f|_{K,\lambda} := \sup_{x\in K} \sup_{\alpha\in \mathbf{N}_0^N} |f^{(\alpha)}(x)| \exp(-\lambda\varphi^*(\frac{|\alpha|}{\lambda})).$$

$\mathcal{E}_{(\omega)}(\Omega)$ is a Fréchet space, that is, a complete and metrizable locally convex space and its topology can be described in terms of $\mathcal{L}^2$-norms, i.e., we can replace $|\cdot|_{K,\lambda}$ by the seminorms

$$q_{K,\lambda}(f) := \sup_{\alpha\in\mathbb{N}_0^N} \left\| f^{(\alpha)} \right\|_{2,K} \exp\left(-\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)\right),$$

where

$$\|f\|_{2,K} = \Big(\int_K |f|^2 \Big)^{\frac{1}{2}}.$$

By $\mathcal{D}_{(\omega)}(K)$, $K \subset \Omega$ compact, we denote the collection of all those $f \in \mathcal{E}_{(\omega)}(\Omega)$ with support contained in K. Then $\mathcal{D}_{(\omega)}(\Omega) = \mathrm{ind}_{n\to} \mathcal{D}_{(\omega)}(K_n)$, where (K_n) is any compact exhaustion of Ω. The elements of $\mathcal{D}'_{(\omega)}(\Omega)$ are called ultradistributions of Beurling type.

Definition 4 ([12]). *Let ω be a weight function. Given a polynomial P, an open set Ω of $\mathbb{R}^d$, a compact subset $K \subset\subset \Omega$ and $\lambda > 0$, we define*

$$\mathcal{E}^{\lambda}_{P,\omega}(K) = \{f \in \mathcal{C}^\infty(K) : \|f\|_{K,\lambda} := \sup_{j\in\mathbb{N}_0} \|P^j(D)f\|_{2,K} \exp\left(-\lambda\varphi^*(\frac{j}{\lambda})\right) < +\infty\}.$$

The spaces of ultradifferentiable functions of Beurling type with respect to the successive iterates of P are defined as follows:

$$\mathcal{E}_{P,(\omega)}(\Omega) = \{f \in \mathcal{C}^\infty(\Omega) : \ \|f\|_{K_n,n} < +\infty \ \textit{for each}\ n \in \mathbb{N}\}$$

where $\{K_n\}_{n\in\mathbb{N}}$ is a compact exhaustion of Ω.

This is a metrizable locally convex space whose topology is defined by the fundamental system of seminorms $\{\|\cdot\|_{K_n,n}\}_{n\in\mathbb{N}}$. $\mathcal{E}_{P,(\omega)}(\Omega)$ is complete if and only if P is a hypoelliptic polynomial (see [12]).

Let $G \in \mathcal{H}(\mathbb{C}^d)$ be an entire function such that $\log|G(z)| = \mathcal{O}(\omega(|z|))$ as $|z|$ tends to infinity. Then

$$T_G(\varphi) := \sum_{\alpha\in\mathbb{N}_0^d} (-i)^{|\alpha|} \frac{G^{(\alpha)}(0)}{\alpha!} \varphi^{(\alpha)}(0)$$

defines an ultradistribution $T_G \in \mathcal{E}'_{(\omega)}(\mathbb{R}^d)$ whose support reduces to $\{0\}$ and with Fourier transform $\widehat{T_G}(z) = G(-z)$. The operator

$$G(D) : \mathcal{D}'_{(\omega)}(\mathbb{R}^d) \to \mathcal{D}'_{(\omega)}(\mathbb{R}^d), \qquad G(D)\nu := T_G * \nu$$

is called an *ultradifferential operator* of class (ω). It has the important property that its restriction to $\mathcal{E}_{(\omega)}(\mathbb{R}^d)$ can be interpreted as a differential operator of infinite order

$$G(D) : \mathcal{E}_{(\omega)}(\mathbb{R}^d) \to \mathcal{E}_{(\omega)}(\mathbb{R}^d).$$

More precisely, for every $f \in \mathcal{E}_{(\omega)}(\mathbb{R}^d)$ one has,

$$G(D)f(x) = \sum_{\alpha \in \mathbb{N}_0^d} i^{|\alpha|} \frac{G^{(\alpha)}(0)}{\alpha!} f^{(\alpha)}(x).$$

The ultradifferential operator $G(D)$ is called *elliptic* if $G(D)^{-1}\big(\mathcal{A}(\mathbb{R}^d)\big) \subset \mathcal{A}(\mathbb{R}^d)$, where $\mathcal{A}(\mathbb{R}^d)$ denotes the class of real analytic functions on $\mathbb{R}^d$. It is said to be (ω)-*hypoelliptic* if $G(D)^{-1}\big(\mathcal{E}_{(\omega)}(\mathbb{R}^d)\big) \subset \mathcal{E}_{(\omega)}(\mathbb{R}^d)$. The ellipticity and the (ω)-hypoellipticity of an ultradifferential operator are characterized in terms of the distribution of zeros and the growth of the entire function $G(z)$ (see [6] and [2]). We will say that the (ω)-hypoelliptic ultradifferential operator $G(D)$ is *strongly* (ω)*-hypoelliptic* if there is a constant $C > 0$ such that

$$C\omega(|x|) \leqslant \log |G(x)|\,, \qquad x \in \mathbb{R}^d.$$

The existence of strongly (ω)-hypoelliptic ultradifferential operators follows from [4, 13].

3. Results

Our first result is an extension of [14, Theorem 1] to the ultradifferentiable setting.

Theorem 5. *Let E be a Fréchet space which is continuously included in $\mathcal{D}'_{(\omega)}(\mathbb{R}^d)$ and such that $G(D)E \subset E$ for some strongly (ω)-hypoelliptic ultradifferential operator $G(D)$ of class (ω). Then $E \subset \mathcal{E}_{(\omega)}(\mathbb{R}^d)$ with continuous inclusion.*

Proof. We need some preparation. Let (K_j) denote the closed ball centered at the origin and with radius j and

$$X_j := \{\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^d);\ \text{supp}\ \mu \subset K_j,\ ||\mu||_j^* := \sup_{z \in \mathbb{C}^N} |\widehat{\mu}(z)|\, e^{-j\omega(|z|)-j|\text{Im}\ z|} < \infty\}$$

where $\widehat{\mu}$ denotes the Fourier-Laplace transform of μ (see [5]). Then X_j is a Banach space and

$$\mathcal{E}'_{(\omega)}(\mathbb{R}^d) = \text{ind}_{j\to} X_j.$$

For every $j \in \mathbb{N}$, the bilinear form

$$B : E \times \mathcal{D}_{(\omega)}(K_{j+1}) \to \mathbb{C}, \qquad B(h, \varphi) := \langle h, \varphi \rangle\,,$$

is separately continuous, hence it is continuous. Consequently, if we fix a fundamental system of seminorms (p_m) of E then there are constants $C_j > 0$ and $m_j \in \mathbb{N}$, such that

$$|\langle h, \varphi\rangle| \leqslant C_j p_{m_j}(h)\, |\varphi|_{m_j} \qquad \forall h \in E, \varphi \in \mathcal{D}_{(\omega)}(K_{j+1}),$$

where

$$|\varphi|_m := \sup_{x\in\mathbb{R}^d} \sup_{\alpha \in \mathbb{N}_0^d} \Big|\varphi^{(\alpha)}(x)\Big| \exp\big(-m\varphi^*(\frac{|\alpha|}{m})\big).$$

According to property (α) in Definition 1, there is a constant $L \in \mathbb{N}$ such that

$$\omega(et) \leqslant L(1+\omega(t)) \qquad \forall t \geqslant 0. \tag{3.1}$$

By assumption there is a constant $C > 0$ with $C\omega(|x|) \leqslant \log|G(x)|$ for every $x \in \mathbb{R}^d$. Now, for each j we define $G_j(z)$ as a suitable power of $G(z)$ such that

$$(1+j+2m_jL)\ \omega(|x|) \leqslant \log|G_j(x)|\,, \qquad x \in \mathbb{R}^d. \tag{3.2}$$

Then $G_j(D)$ is an ultradifferential operator of (ω)-class which is strongly (ω)-hypoelliptic. Let now $\psi_j \in \mathcal{D}_{(\omega)}(\overset{\circ}{K}_{j+1})$ be a test function which is constant $\psi_j = 1$ on a neighborhood of K_j. For any $\mu \in X_j$ we define

$$f(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{\widehat{\mu}(t)}{G_j(t)} e^{i<x,t>}\ dt.$$

Then we can decompose

$$\mu = G_j(-D)f = G_j(-D)\big(\psi_j f\big) + G_j(-D)\big((1-\psi_j)f\big).$$

Moreover, we can apply (3.2) and

$$|t^\alpha| e^{-m\omega(t)} \leqslant \exp\big(m\varphi^*(\frac{|\alpha|}{m})\big) \qquad \forall\ m \in \mathbb{N}, \alpha \in \mathbb{N}_0^d,$$

to conclude that $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and

$$|f|_{2Lm_j} := \sup_{\alpha\in\mathbb{N}_0^d}\sup_{x\in\mathbb{R}^d}\Big|f^{(\alpha)}(x)\Big| \exp\big(-2m_jL\varphi^*(\frac{|\alpha|}{2m_jL})\big) \leqslant D_j||\mu||_j^*$$

for some constant $D_j > 0$ which does not depend on μ. Our aim is to prove that each ultradistribution $h \in E$ can be extended to a continuous and linear map

$$T_h : \mathcal{E}'_{(\omega)}(\mathbb{R}^d) \to \mathbb{C}.$$

First *we claim* that the linear map

$$\Phi : X_j \to \mathcal{E}_{(\omega)}(\mathbb{R}^d), \qquad \mu \mapsto \big(1-\psi_j(x)\big)\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\widehat{\mu}(t)}{G_j(t)}e^{i<x,t>}\ dt$$

is well-defined and continuous. In fact, since $G_j(-D)$ is an (ω)-hypoelliptic operator and $G_j(-D)f = \mu$ we have, for every $\mu \in X_j$,

$$\mathrm{sing}_{(\omega)}\mathrm{supp}\ f \subset \mathrm{sing}_{(\omega)}\mathrm{supp}\ \mu \subset K_j,$$

hence

$$(1-\psi_j)f \in \mathcal{E}_{(\omega)}(\mathbb{R}^d).$$

On the other hand, from the convexity of φ^* and (3.1) we get, for every $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}_0^d$,

$$\begin{aligned}\left|(\psi_j f)^{(\alpha)}(x)\right| &\leqslant \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \left|\psi_j^{(\beta)}(x)\right| \left|f^{(\alpha-\beta)}(x)\right| \\ &\leqslant |f|_{2Lm_j} |\psi_j|_{2Lm_j} \exp\big(\alpha + 2Lm_j\varphi^*(\frac{|\alpha|}{2Lm_j})\big) \\ &\leqslant |f|_{2Lm_j} |\psi_j|_{2Lm_j} \exp\big(2m_j\varphi^*(\frac{|\alpha|}{2m_j}) + m_j\big)\end{aligned}$$

and, consequently

$$|\psi_j f|_{2m_j} \leqslant e^{m_j} |f|_{2Lm_j} |\psi_j|_{2Lm_j} \leqslant D_j e^{m_j} |\psi_j|_{2Lm_j} ||\mu||_j^*. \tag{3.3}$$

Moreover, it follows from the previous estimates that $\Phi : X_j \to \mathcal{D}'_{(\omega)}(\mathbb{R}^d)$ is weakly continuous, hence $\Phi : X_j \to \mathcal{E}_{(\omega)}(\mathbb{R}^d)$ is continuous by the closed graph theorem and the claim is proved.

We now fix $h \in E$ and consider a regularizing family $(\eta_\epsilon)_{\epsilon\downarrow 0}$, $\eta_\epsilon \in \mathcal{D}_{(\omega)}(\mathbb{R}^d)$. Define

$$T_j : X_j \to \mathbb{C}$$

by

$$T_j(\mu) = \lim_{\epsilon\to 0} \langle G_j(D)h, (\psi_j f) * \eta_\epsilon\rangle + \big\langle h, G_j(-D)\big((1-\psi_j)f\big)\big\rangle,$$

where

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\widehat{\mu}(t)}{G_j(t)} e^{i<x,t>}\, dt.$$

Let us prove that T_j is a well-defined linear map. Since $G_j(D)h \in E$ and $(\psi_j f) * \eta_\epsilon \in \mathcal{D}_{(\omega)}(K_{j+1})$ we get

$$|\langle G_j(D)h, (\psi_j f) * (\eta_{\epsilon_1} - \eta_{\epsilon_2})\rangle| \leqslant C_j p_{m_j}\big(G_j(D)h\big) \left|(\psi_j f) * (\eta_{\epsilon_1} - \eta_{\epsilon_2})\right|_{m_j}.$$

On the other hand, using (3.3), we have

$$\lim_{\epsilon\to 0} |(\psi_j f) * \eta_\epsilon - (\psi_j f)|_{m_j} = 0,$$

from where it follows that

$$\big(\langle G_j(D)h, (\psi_j f) * \eta_\epsilon\rangle\big)_{\epsilon\downarrow 0}$$

is a Cauchy net. On the other hand,

$$G_j(-D)\big((1-\psi_j)f\big) = \mu - G_j(-D)(\psi_j f)$$

is compactly supported, hence

$$G_j(-D)\big((1-\psi_j)f\big) \in \mathcal{D}_{(\omega)}(\mathbb{R}^d).$$

Consequently T_j is a well-defined linear map. From (3.3) and the continuity of

$$G_j(-D)\circ\Phi : X_j \to \mathcal{D}_{(\omega)}(K_{j+1})$$

we conclude that there is a positive constant M_j such that

$$|T_j(\mu)| \leqslant C_j p_{m_j}\big(G_j(D)h\big)\,|\psi_j f|_{m_j} + |\langle h, (G_j(D)\circ\Phi)\mu\rangle| \leqslant M_j ||\mu||_j^*,$$

which proves that

$$T_j : X_j \to \mathbb{C}$$

is a continuous and linear form. Moreover, for $\mu \in X_j \cap \mathcal{D}_{(\omega)}(\mathbb{R}^d)$ we have

$$\begin{aligned} T_j(\mu) &= \lim_{\epsilon\to 0}\big(\langle h, G_j(-D)(\psi_j f) * \eta_\epsilon\rangle \\ &\quad + \langle h, G_j(-D)((1-\psi_j)f) * \eta_\epsilon\rangle\big) \\ &= \lim_{\epsilon\to 0}\langle h, G_j(-D)f * \eta_\epsilon\rangle \\ &= \lim_{\epsilon\to 0}\langle h, \mu * \eta_\epsilon\rangle = \langle h, \mu\rangle\,. \end{aligned}$$

Since the restriction of T_{j+1} to X_j coincides with T_j, we finally conclude that there is a continuous and linear form $T : \mathcal{E}'_{(\omega)}(\mathbb{R}^d) \to \mathbb{C}$ with the property that $T(\mu) = \langle h, \mu\rangle$ for all $\mu \in \mathcal{D}_{(\omega)}(\mathbb{R}^d)$. That is, as ultradistributions,

$$h = T \in \mathcal{E}_{(\omega)}(\mathbb{R}^d). \qquad \blacksquare$$

As we already mentioned in the introduction, the space of ultradistributions of Roumieu type $E := \mathcal{D}'_{\{\omega\}}(\mathbb{R}^d)$ is a Fréchet space which is stable under differential operators but which is not contained in $\mathcal{E}_{(\omega)}(\mathbb{R}^d)$.

We refer to [5] for the definition and properties of the spaces of Roumieu type.

Next, given a weight ω and a natural number m we put $\omega_m(t) = \omega(t^{\frac{1}{m}})$. We denote $\varphi^* = \varphi^*_\omega$ and observe that $\varphi^*_{\omega_m}(x) = \varphi^*(mx)$.

Definition 6. Let E be a Fréchet space such that $E \subset \mathcal{D}'_{(\omega)}(\mathbb{R}^d)$ with continuous inclusion and let $P(D)$ be a differential operator of degree m. Then E is said to be $(\omega, P(D))-$stable if $P(D)E \subset E$ and, moreover, for every $k \in \mathbb{N}$, the sequence of operators

$$P^j(D)e^{-k\varphi^*(m\frac{j}{k})} : E \to E$$

is equicontinuous, that is, for every $k \in \mathbb{N}$ and every continuous seminorm r on E there is a continuous seminorm s on E with

$$r\big(P^j(D)f\big) \leqslant e^{k\varphi^*(m\frac{j}{k})}s(f) \qquad \forall j \in \mathbb{N},\ f \in E. \tag{3.4}$$

Example 7. For every hypoelliptic polynomial P and for every $m \in \mathbb{N}$ the Fréchet space $\mathcal{E}_{P,(\omega_m)}(\mathbb{R}^d)$ is $(\omega, P(D))-$stable.

In the limit case $\omega(t) = \log(1+t)$, $(\omega, P(D))-$stability simply means that $P(D)E \subset E$.

We now consider the following question. Given a differential operator $P(D)$ and an $(\omega, P(D))-$stable Fréchet space $E \subset \mathcal{D}'_{(\omega)}(\mathbb{R}^d)$, we want to analyze whether E consists of smooth functions or even $E \subset \mathcal{E}_{(\lambda)}(\mathbb{R}^d)$ for some weight λ related to ω. We observe that, for an arbitrary polynomial $P(D)$, the space

$$E := \{S \in \mathcal{D}'_{\{\omega\}}(\mathbb{R}^d) \, : P(D)S = 0\},$$

consisting of the ultradistributions of Roumieu type in the Kernel of $P(D)$, is an $(\omega, P(D))-$stable Fréchet space. Hence in order to have a positive answer to the previous question the polynomial P has to be hypoelliptic.

As proved in [12], for any elliptic polynomial P of degree m the space $\mathcal{E}_{P,(\omega_m)}(\mathbb{R}^d)$ is contained in $\mathcal{E}_{(\omega)}(\mathbb{R}^d)$. This fact and [14] permit us to prove a similar result for arbitrary $(\omega, P(D))-$stable Fréchet spaces of distributions.

Proposition 8. *Let $P(D)$ be an elliptic differential operator of degree m. If the Fréchet space $E \subset \mathcal{D}'(\mathbb{R}^d)$ is $(\omega, P(D))-$stable then $E \subset \mathcal{E}_{(\omega)}(\mathbb{R}^d)$ with continuous inclusion.*

Proof. According to Langenbruch, Voigt [14, Theorem 1], E is continuously included in $\mathcal{C}^\infty(\mathbb{R}^d)$. Hence, for any $f \in E$, $k \in \mathbb{N}$ and $K \subset \mathbb{R}^d$,

$$\sup_{j \in \mathbb{N}_0} \| \, P^j(D)f \, \|_{2,K} \, e^{-k\varphi^*(m\frac{j}{k})} < \infty,$$

which means that

$$f \in \mathcal{E}_{P,(\omega_m)}(\mathbb{R}^d).$$

Since P is an elliptic operator, we can apply [12, Corollary 4.10] to conclude that $f \in \mathcal{E}_\omega(\mathbb{R}^d)$. ■

This result can be partially extended to Fréchet spaces of ultradistributions.

Theorem 9. *Let $P(D)$ be an elliptic differential operator of degree m such that its principal part has real coefficients. If the Fréchet space $E \subset \mathcal{D}'_{(\omega)}(\mathbb{R}^d)$ is $(\omega, P(D))$-stable then $E \subset \mathcal{E}_{(\omega)}(\mathbb{R}^d)$ with continuous inclusion.*

Proof. We will see that E is invariant under the action of a strongly (ω)-hypoelliptic operator and then the conclusion follows applying Theorem 5. Throughout the proof, in order to simplify the notation, we will put $\sigma := \omega_m$.

According to [13, Corollary 1.4] there are an entire function $g \in H(\mathbb{C})$ without zeros on the real line and a conic neighborhood Γ of $\mathbb{R} \setminus \{0\}$, defined by $|\mathrm{Im}\, z| < \epsilon |\mathrm{Re}\, z|$, such that

$$|g(z)| \leqslant Ae^{B\sigma(|z|)} \qquad \forall z \in \mathbb{C} \text{ and } |g(z)| \geqslant ae^{b\sigma(|z|)} \qquad \forall z \in \Gamma. \tag{3.5}$$

We now put

$$P = Q + P_m,$$

where P_m is the principal part of P and Q is a polynomial of degree at most $m-1$. For each $\xi \in \mathbb{R}^n$ we have, for some $d > 0$ and for $|\xi|$ large enough,

$$\left|\frac{\text{Im } P(\xi)}{\text{Re } P(\xi)}\right| \leqslant \frac{|Q(\xi)|}{|P_m(\xi)| - |Q(\xi)|} \leqslant \frac{|Q(\xi)|}{d|\xi|^m - |Q(\xi)|}.$$

Hence, there is $R > 0$ such that $P(\xi) \in \Gamma$ whenever $\xi \in \mathbb{R}^d$ and $|\xi| \geqslant R$. We now define $G \in H(\mathbb{C}^d)$ by

$$G(z) := g\big(\epsilon_0 P(z)\big),$$

where $\epsilon_0 > 0$ is such that $G(\xi) \neq 0$ for every $\xi \in \mathbb{R}^d$ with $|\xi| \leqslant R$. Hence $G(\xi) \neq 0$ for every $\xi \in \mathbb{R}^d$. Then, for some constants $C, D > 0$, we have

$$|G(z)| \leqslant Ae^{B\sigma(\epsilon_0|P(z)|)} \leqslant Ce^{D\sigma(|z|^m)} = Ce^{D\omega(|z|)}$$

for every $z \in \mathbb{C}^d$. On the other hand, there is $\delta > 0$ with $|P(\xi)| \geqslant \delta|\xi|^m$ for $|\xi|$ large enough. Consequently, there is $q \in \mathbb{N}$ such that

$$|G(\xi)| \geqslant ae^{b\sigma(\epsilon_0|P(\xi)|)} \geqslant ae^{b\omega(2^{-q}|\xi|)}$$

for each $\xi \in \mathbb{R}^d$ with $|\xi|$ large enough. Since G does not vanish on $\mathbb{R}^d$, we finally deduce

$$|G(\xi)| \geqslant a'e^{b'\omega(|\xi|)}$$

for $\xi \in \mathbb{R}^d$. To conclude that $G(D)$ is a strongly (ω)-hypoelliptic ultradifferential operator of (ω)-class it is enough to show that (see [2, Theorem 2.1])

$$\lim_{\substack{|z|\to\infty \\ G(z)=0}} \frac{|\text{Im } z|}{\omega(|z|)} = \infty.$$

We are going to prove that each $z \in \mathbb{C}^d$ with $|\text{Im}z| \leqslant A\omega(|z|)$ also satisfies $|\text{Im}P(z)| < \varepsilon|\text{Re}P(z)|$ provided that $|z|$ is big enough, and consequently $G(z) = g(P(z)) \neq 0$. Since $\omega(t) = o(t)$ we may assume $|\text{Im}z| \leqslant A\omega(|\text{Re}z|)$ for some different constant A. Clearly,

$$|\text{Im}Q(z)| \leqslant |Q(z)| \leqslant C|z|^{m-1} \leqslant C(1+A)^m|\text{Re}z|^{m-1}.$$

On the other hand, by Taylor formula

$$P_m(z) = P_m(\text{Re}z + i\text{Im}z) = P_m(\text{Re}z) + \sum_{\alpha\neq 0} \frac{P_m^{(\alpha)}(\text{Re}z)}{\alpha!}(i\text{Im}z)^\alpha.$$

Each term in the sum above is estimated as follows

$$|P_m^{(\alpha)}(\text{Re}z)(i\text{Im}z)^\alpha| \leqslant |\text{Re}z|^{m-|\alpha|}|\text{Im}z|^{|\alpha|}.$$

Therefore, for $|z|$ big enough, the whole sum is not bigger than $D|\text{Re}z|^{m-1}|\text{Im}z|$. Since the principal part of P has real coefficients we finally have

$$|\text{Im}P(z)| \leqslant D'|\text{Re}z|^{m-1}|\text{Im}z| \leqslant D'A|\text{Re}z|^{m-1}\omega(|\text{Re}z|)$$

whereas

$$\begin{aligned}|\mathrm{Re}P(z)| &\geqslant |P_m(\mathrm{Re}z)| - D'A|\mathrm{Re}z|^{m-1}\omega(|\mathrm{Re}z|)\\ &\geqslant d|\mathrm{Re}z|^m - D'A|\mathrm{Re}z|^{m-1}\omega(|\mathrm{Re}z|)\\ &\geqslant L|\mathrm{Re}z|^m.\end{aligned}$$

Hence, it is clear that

$$|\mathrm{Im}P(z)| < \varepsilon|\mathrm{Re}P(z)|$$

provided that $|z|$ is big enough.

We now check that $G(D)E \subset E$. Since the entire function

$$g(z) = \sum_{j\in\mathbb{N}_0} a_j z^j$$

satisfies (3.5), we can use the convexity of φ^* and Cauchy inequalities to find $k \in \mathbb{N}$ and $C > 0$ such that

$$|a_j| \leqslant Ce^{-k\varphi^*(m\frac{j}{k})} \qquad \forall j \in \mathbb{N}_0.$$

Moreover,

$$G(D) = \sum_{j\in\mathbb{N}_0} a_j P^j(D).$$

If r is a continuous seminorm on E then there is another continuous seminorm s on E such that, for every $f \in E$,

$$\sum_{j\in\mathbb{N}_0} |a_j| r\big(P^j(D)f\big) \leqslant CD \sup_{j\in\mathbb{N}_0} e^{-2k\varphi^*(m\frac{j}{2k})} r\big(P^j(D)f\big) \leqslant CDs(f),$$

where

$$D = \sum_{j\in\mathbb{N}_0} \frac{e^{2k\varphi^*(m\frac{j}{2k})}}{e^{k\varphi^*(m\frac{j}{k})}} < +\infty.$$

This proves that the series $\sum_{j\in\mathbb{N}_0} a_j P^j(D)f$ converges in the Fréchet space E and $G(D)f \in E$. ■

Remark 10. If E is invariant under complex conjugation, the assumption that the principal part of P should be real is redundant. Indeed, $(\omega, P(D))$-stability then implies $(\omega, Q(D))$-stability where $Q(D) = P(D)\overline{P(D)} = P(D)\overline{P}(-D)$ and the principal part of Q has real coefficients.

According to Hörmander [9, 11.1.3], for every hypoelliptic differential operator $P(D)$ of degree m, there are $c > 0$ and $0 < r < 1$ such that $|P(\xi)| \geqslant c|\xi|^{rm}$ and $|P^{(\alpha)}(\xi)| \leqslant c^{-1}|P(\xi)||\xi|^{-r|\alpha|}$ if $\xi \in \mathbb{R}^d$ and $|\xi|$ is large enough. Let us take a weight function ω such that $\lim_{t\to\infty} \frac{\omega(t)}{t^r} = 0$. In the sequel we will consider $\lambda(t) := \omega(t^r)$.

Proposition 11. *We assume that the hypoelliptic differential operator $P(D)$ of degree m satisfies*

$$\lim_{x\in\mathbb{R}^d, |x|\to\infty} \frac{\mathrm{Im}P(x)}{\mathrm{Re}P(x)} = 0.$$

If the Fréchet space $E \subset \mathcal{D}'_{(\lambda)}(\mathbb{R}^d)$ is $(\omega, P(D))-$stable then $E \subset \mathcal{E}_{(\lambda)}(\mathbb{R}^d)$ with continuous inclusion.

Proof. Let (K_j) denote the closed ball centered at the origin and with radius j and

$$X_j := \{\mu \in \mathcal{E}'_{(\lambda)}(\mathbb{R}^d);\ \mathrm{supp}\ \mu \subset K_j,\ ||\mu||_j^* := \sup_{z\in\mathbb{C}^d} |\widehat{\mu}(z)|\, e^{-j\lambda(|z|)-j|\mathrm{Im}\ z|} < \infty\}.$$

Then X_j is a Banach space and

$$\mathcal{E}'_{(\lambda)}(\mathbb{R}^d) = \mathrm{ind}_{j\to} X_j.$$

For every $j \in \mathbb{N}$, the bilinear form

$$B : E \times \mathcal{D}_{(\lambda)}(K_{j+1}) \to \mathbb{C},\ B(h,\varphi) := \langle h, \varphi\rangle\,,$$

is separately continuous, hence it is continuous. Consequently, if we fix a fundamental system of seminorms (p_m) of E there are constants $C_j > 0$ and $m_j \in \mathbb{N}$, such that

$$|\langle h,\varphi\rangle| \leqslant C_j p_{m_j}(h)\, |\varphi|_{m_j} \qquad \forall h \in E, \varphi \in \mathcal{D}_{(\lambda)}(K_{j+1}),$$

where

$$|\varphi|_m := \sup_{x\in\mathbb{R}^d} \sup_{\alpha\in\mathbb{N}_0^d} \left|\varphi^{(\alpha)}(x)\right| \exp\big(-m\varphi^*_\lambda(\frac{|\alpha|}{m})\big).$$

As in the proof of Theorem 9, there are an entire function $g \in H(\mathbb{C})$ without zeros on the real line and a conic neighborhood Γ of $\mathbb{R}\setminus\{0\}$, defined by $|\mathrm{Im}\ z| < \epsilon|\mathrm{Re}\ z|$, such that

$$|g(z)| \leqslant Ae^{B\omega(|z|^{\frac{1}{m}})} \quad \forall z \in \mathbb{C} \qquad \text{and} \qquad |g(z)| \geqslant ae^{b\omega(|z|^{\frac{1}{m}})} \quad \forall z \in \Gamma.$$

We now define $G \in H(\mathbb{C}^d)$ by $G(z) := g\big(P(z)\big)$ and we check that $G(D)$ is $(\omega)-$hypoelliptic. In fact, for some constants $C, D > 0$,

$$|G(z)| \leqslant Ae^{B\omega(|P(z)|^{\frac{1}{m}})} \leqslant Ce^{D\omega(|z|)}$$

for every $z \in \mathbb{C}^d$ and

$$|G(\xi)| \geqslant ae^{b\omega(|\xi|^r)} = ae^{b\lambda(|\xi|)}$$

for some $a, b > 0$ and every $\xi \in \mathbb{R}^d$. We now check that

$$\lim_{\substack{|z|\to\infty \\ G(z)=0}} \frac{|\mathrm{Im}\ z|}{\omega(|z|)} = \infty.$$

To this end we observe that $G(z) = 0$ implies $|\text{Im } P(z)| \geqslant \epsilon |\text{Re } P(z)|$. Hence, it suffices to prove that, for any $A > 0$, the inequality $|\text{Im } z| < A\omega(|z|)$ implies $|\text{Im } P(z)| < \epsilon |\text{Re } P(z)|$ whenever $|z|$ is large enough. Since $\omega(t) = o(t)$ we may assume $|\text{Im} z| \leqslant A\omega(|\text{Re} z|)$ for some different constant A. Using Taylor's formula we have, for $z = x + iy$, $x, y \in \mathbb{R}^d$,

$$P(z) = P(x) + \sum_{\alpha \neq 0} \frac{P^{(\alpha)}(x)}{\alpha!} (iy)^{(\alpha)}.$$

Now, since P is hypoelliptic get, for some positive constant C (which depends on A),

$$\Big|\sum_{\alpha \neq 0} \frac{P^{(\alpha)}(x)}{\alpha!} (iy)^{(\alpha)}\Big| \leqslant C\big|P(x)\big| \sum_{\alpha \neq 0} \frac{1}{\alpha!} \Big(\frac{\omega(|x|)}{|x|^r}\Big)^{|\alpha|}.$$

Therefore, using $\lim\limits_{t \to \infty} \dfrac{\omega(t)}{t^r} = 0$ and $\lim_{x \in \mathbb{R}^d, |x| \to \infty} \frac{\text{Im} P(x)}{\text{Re} P(x)} = 0$ we deduce that

$$\Big|\sum_{\alpha \neq 0} \frac{P^{(\alpha)}(x)}{\alpha!} (iy)^{(\alpha)}\Big| < \min\Big(\frac{\epsilon}{2}, \frac{1}{2}\Big) |\text{Re} P(x)|$$

for $|x|$ large enough (equivalently, for $|z|$ large enough, since $|y| \leqslant A\omega(|x|)$ and $\omega(|x|) = o(|x|)$ as $|x|$ tends to ∞). Therefore, for $|z|$ large enough,

$$|\text{Im } P(z)| < \frac{\epsilon}{2} |\text{Re } P(x)|$$

while

$$|\text{Re } P(z)| \geqslant |\text{Re } P(x)| - \Big|\sum_{\alpha \neq 0} \frac{P^{(\alpha)}(x)}{\alpha!} (iy)^{(\alpha)}\Big| \geqslant \frac{1}{2} |\text{Re } P(x)|.$$

Hence,

$$|\text{Im } P(z)| < \epsilon |\text{Re } P(z)|,$$

as we wanted to check. Consequently, $G(D)$ is $(\omega)-$hypoelliptic [2, Theorem 2.1]. Proceeding as in Theorem 9 we conclude that $G(D)E \subset E$ and $G(D) : E \to E$ is continuous and linear map. To finish the proof, let $\mu \in X_j$ be given and take $l_j \in \mathbb{N}$ with

$$bl_j - \overline{m}_j > j + 1$$

(where $\overline{m}_j$ is as large as needed later) and define $G_j(D) = G^{l_j}(D)$ and

$$f_j(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\widehat{\mu}(t)}{G_j(t)} e^{i<x,t>} \, dt.$$

Since, for every $\alpha \in \mathbb{N}_0^d$,

$$\Big|\frac{t^\alpha \widehat{\mu}(-t)}{G_j(t)}\Big| \leqslant \frac{||\mu||_j^*}{a} e^{(-bl_j + j)\lambda(t) + |\alpha| \log(t)} \leqslant \frac{||\mu||_j^*}{a} e^{-\lambda(t)} e^{\overline{m}_j \varphi_\lambda^*(\frac{|\alpha|}{\overline{m}_j})}$$

then

$$f_j \in \mathcal{C}^\infty(\mathbb{R}^d) \subset \mathcal{D}'_{(\omega)}(\mathbb{R}^d)$$

and

$$\sup_{x\in\mathbb{R}^d}\sup_{\alpha\in\mathbb{N}_0^d}\left|f_j^{(\alpha)}(x)\right| e^{-\overline{m}_j\varphi^*_\lambda(\frac{|\alpha|}{\overline{m}_j})} < +\infty.$$

Moreover

$$G_j(-D)f_j = \mu$$

in $\mathcal{D}'_{(\omega)}(\mathbb{R}^d)$. Let $\psi_j \in \mathcal{D}_{(\omega)}(\overset{\circ}{K}_{j+1})$ be such that $\psi_j = 1$ on a neighborhood of K_j. As in Theorem 9 we will prove that each ultradistribution $h \in E$ can be extended to a continuous and linear map

$$T_h : \mathcal{E}'_{(\lambda)}(\mathbb{R}^d) \to \mathbb{C}.$$

Since $G_j(-D)$ is $(\omega)-$hypoelliptic then

$$\mathrm{sing}_{(\omega)}\mathrm{supp} f_j \subset \mathrm{conv}\ (\mathrm{sing}_{(\omega)}\mathrm{supp} f_j) \subset K_j$$

and the mapping

$$X_j \to \mathcal{E}_{(\omega)}(\mathbb{R}^d), \mu \mapsto (1-\psi_j(x))\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\widehat{\mu}(t)}{G_j(t)}e^{i<x,t>}\ dt$$

is well-defined. Moreover, as in the proof of Theorem 5, the mapping $X_j \to \mathcal{E}_{(\lambda)}(\mathbb{R}^d)$ is continuous. Hence, by the closed graph theorem, also

$$X_j \to \mathcal{E}_{(\omega)}(\mathbb{R}^d)$$

is continuous. We now fix $h \in E$ and consider a regularizing family (η_ϵ) of test functions in $\mathcal{D}_{(\omega)}(\mathbb{R}^d)$. If $\overline{m}_j$ is big enough, we may guarantee that

$$\lim_{\epsilon\to 0}|(\psi_j f_j)*\eta_\epsilon - (\psi_j f_j)|_{m_j} = 0,$$

from where it follows that

$$\big(\langle G_j(D)h, (\psi_j f_j)*\eta_\epsilon\rangle\big)_{\epsilon\downarrow 0}$$

is a Cauchy net. Define

$$T_j : X_j \to \mathbb{C}$$

by

$$T_j(\mu) = \lim_{\epsilon\to 0}\langle G_j(D)h, (\psi_j f_j)*\eta_\epsilon\rangle + \langle h, G_j(-D)((1-\psi_j)f_j)\rangle\,.$$

The same argument as in Theorem 5 gives that T_j is a continuous linear form, the restriction of T_{j+1} to X_j coincides with T_j and $T_j(\mu) = \langle h,\mu\rangle$ for every $\mu \in X_j \cap \mathcal{D}_{(\omega)}(\mathbb{R}^d)$. That is, there is $f \in \mathcal{E}_{(\lambda)}(\mathbb{R}^d)$ such that

$$\langle f,\mu\rangle = \langle h,\mu\rangle$$

for every $\mu \in \mathcal{D}_{(\omega)}(\mathbb{R}^d)$. Since $\mathcal{D}_{(\omega)}(\mathbb{R}^d)$ is dense in $\mathcal{D}_{(\lambda)}(\mathbb{R}^d)$ then $h = f \in \mathcal{E}_{(\lambda)}(\mathbb{R}^d)$. ■

Corollary 12. *We assume that the differential operator $P(D)$ of degree m satisfies*

(1) *P is hypoelliptic and has real coefficients*

or

(2) *P is semi-elliptic with real generalized principal part.*

If the Fréchet space $E \subset \mathcal{D}'_{(\lambda)}(\mathbb{R}^d)$ is $(\omega, P(D))-$stable then $E \subset \mathcal{E}_{(\lambda)}(\mathbb{R}^d)$ with continuous inclusion.

Proof. It is straightforward to check that in both cases

$$\lim_{x\in\mathbb{R}^d, |x|\to\infty} \frac{\mathrm{Im}P(x)}{\mathrm{Re}P(x)} = 0$$

(see the proof of [9, Theorem 11.1.11] for the semi-elliptic case). Then Proposition 11 applies. ■

A weight ω is called a *strong weight* if it satisfies the additional condition:

(ϵ) there exists $C \geqslant 1$ such that for all $y > 0$,

$$\int_1^\infty \frac{\omega(ty)}{t^2}\, dt \leqslant C\omega(y) + C.$$

Examples of weight functions with and without property (ϵ) can be found in [15].

Corollary 13. *Let ω be a strong weight and $P(D)$ a $(\omega)-$hypoelliptic differential operator of degree m such that*

$$\lim_{x\in\mathbb{R}^d, |x|\to\infty} \frac{\mathrm{Im}P(x)}{\mathrm{Re}P(x)} = 0.$$

Then $\mathcal{E}_{P,(\omega_m)}(\mathbb{R}^d)$ is continuously contained in $\mathcal{E}_{(\lambda)}(\mathbb{R}^d)$ for some non-quasianalytic weight λ.

Proof. According to [7, 3.7], there is $0 < r < 1$ such that $P(D)$ is $\{t^r\}-$hypoelliptic and $\lim_{t\to\infty} \frac{\omega(t)}{t^r} = 0$. Then Proposition 11 implies that $E = \mathcal{E}_{P,(\omega_m)}(\mathbb{R}^d)$ is continuously contained in $\mathcal{E}_{(\lambda)}(\mathbb{R}^d)$ for $\lambda(t) := \omega(t^r)$. ■

Under the additional assumption that there exists a constant $H \geqslant 1$ such that for all $t \geqslant 0$

$$2\omega(t) \leqslant \omega(Ht) + H, \tag{3.6}$$

the third author proved in [12, Theorem 4.12] that the ellipticity of the polynomial is necessary in order to achieve the equality $\mathcal{E}_{P,(\omega_m)}(\mathbb{R}^d) = \mathcal{E}_{(\omega)}(\mathbb{R}^d)$. However, it follows from Corollary 15 below that this is not the case for arbitrary weights. The growth condition (3.6) was considered by J. Bonet, R. Meise and S.N. Melikhov in [3]. Observe that the weights $\omega(t) = \log^\beta(1+t)$, $\beta > 1$, do not satisfy (3.6).

We recall that two hypoelliptic polynomials P and Q are *equally strong* if $\frac{Q(\xi)}{P(\xi)}$ and $\frac{P(\xi)}{Q(\xi)}$ are bounded at infinity in $\mathbb{R}^d$. The next Lemma can be proved proceeding as in [12, Theorems 4.4 and 4.5].

Lemma 14. *Let P and Q be hypoelliptic polynomials and let Ω be an open subset of $\mathbb{R}^d$.*

(1) *If P and Q are equally strong then there is m_0 such that $m \geqslant m_0$ implies $\mathcal{E}_{P,(\omega_m)}(\Omega) = \mathcal{E}_{Q,(\omega_m)}(\Omega)$.*
(2) *The converse is true under the additional assumption that ω satisfies (3.6).*

Corollary 15. *Let $\omega(t) = \log^\beta(1+t)$, $\beta > 1$, be given and $P(D)$ a differential operator of degree m. Then $\mathcal{E}_{P,(\omega_m)}(\mathbb{R}^d) = \mathcal{E}_{(\omega)}(\mathbb{R}^d)$ if, and only if, P is hypoelliptic.*

Proof. It follows from [12, Lemma 4.11] that the hypoellipticity of P is a necessary condition in order to get the identity $\mathcal{E}_{P,(\omega_m)}(\mathbb{R}^d) = \mathcal{E}_{(\omega)}(\mathbb{R}^d)$. We now assume that P is hypoelliptic and denote $\overline{P}(\xi) := \overline{P(\xi)}$ the conjugate of P. The polynomials P and $\overline{P}$ are equally strong and ω_m is an equivalent weight to ω, hence we deduce from the previous Lemma that $\mathcal{E}_{P,(\omega)}(\mathbb{R}^d) = \mathcal{E}_{\overline{P},(\omega)}(\mathbb{R}^d)$. In particular, $\mathcal{E}_{P,(\omega)}(\mathbb{R}^d)$ is an $(\omega, P(D) \circ \overline{P}(D))-$stable Fréchet space. Since the hypoelliptic polynomial (of degree $2m$) $P\overline{P}$ has real coefficients we can apply Proposition 11 to finally conclude that $\mathcal{E}_{P,(\omega)}(\mathbb{R}^d)$ is contained in $\mathcal{E}_{(\omega)}(\mathbb{R}^d)$. The converse inclusion always holds. ■

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