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HAUSDORFF DIMENSION OF THE RECURRENCE SETS OF GAUSS TRANSFORMATION ON THE FIELD OF LAURENT SERIES

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Abstract: Define the recurrence set of Gauss transformation T on the field of Laurent series as following

$$E(x_0) = \{x \in I : T^n(x) \in I_{t_n}(x_0) \text{ for infinitely many } n\},$$

where $I_{t_n}(x_0)$ denotes t_n-th order cylinder of x_0. In this paper, the Hausdorff dimension of the set $E(x_0)$ is determined.

Keywords: continued fraction, recurrence set, formal Laurent series, Hausdorff dimension.

1. Introduction

It is known that the continued fraction of a real number can be generated by the Gauss transformation $T : [0, 1) \to [0, 1)$ defined by

$$T(x) := \frac{1}{x} - [\frac{1}{x}], \qquad T0 := 0.$$

where $[x]$ denotes the integer part of x. Fernández and Melián [4] have considered with quantitative recurrence properties in continued fraction dynamical system.

In this paper, we consider the analogous problem for the continued fraction expansion on the field of formal Laurent series. We study the Hausdorff dimensions of the recurrence sets of Gauss transformation on the field of Laurent series. The Hausdorff dimensions of some other sets occurring in the continued fraction expansion of Laurent series have been discussed in [6], [8], [9] and [12].

2. Preliminaries

Let p be a prime, q be a power of p, and $\mathbb{F}_q$ be a finite field of q elements. Let $\mathbb{F}_q((z^{-1}))$ denote the field of all formal Laurent series $B = \sum_{n=v}^{\infty} c_n z^{-n}$ in an

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indeterminate z, with coefficients c_n all lying in the field $\mathbb{F}_q$. Recall that $\mathbb{F}_q[z]$ denotes the ring of polynomials in z with coefficients in $\mathbb{F}_q$. For the above formal Laurent series B, we may assume that $c_v \neq 0$. Then the integer $v = v(B)$ is called the order of B. The norm (or valuation) of B is defined to be $\|B\| = q^{-v(B)}$. It is well known that $\|\cdot\|$ is a non-Archimedean valuation on the field $\mathbb{F}_q((z^{-1}))$ and $\mathbb{F}_q((z^{-1}))$ is a complete metric space under the metric ρ defined by $\rho(B_1, B_2) = \|B_1 - B_2\|$.

For $B = \sum\limits_{n=v}^{\infty} c_n z^{-n} \in \mathbb{F}_q((z^{-1}))$, let $[B] = \sum\limits_{v \leqslant n \leqslant 0} c_n z^{-n} \in \mathbb{F}_q[z]$. We call $[B]$ the integral part of B. It is evident that the integer $-v(B) := -v$ is equal to the degree $\deg[B]$ of the polynomial $[B]$ provided $v \leqslant 0$, *i.e.*, $[B] \neq 0$. Let I denote the valuation ideal of $\mathbb{F}_q((z^{-1}))$. It consists of all formal series $\sum\limits_{n=1}^{\infty} c_n z^{-n}$. The ideal I is compact because it is isomorphic to $\prod\limits_{n=1}^{\infty} \mathbb{F}_q$. A natural measure on I is the normalized Haar measure on $\prod\limits_{n=1}^{\infty} \mathbb{F}_q$, which we denote by $\mathbf{P}$.

Consider the following transformation from I to I defined by

$$T(x) := \frac{1}{x} - [\frac{1}{x}], \qquad T0 := 0.$$

this map describes the regular continued fraction over the field of Laurent series and has been introduced by Artin [1]. As in the classical theory, every $x \in I$ has the following continued fraction expansion

$$x = \cfrac{1}{A_1(x) + \cfrac{1}{A_2(x) + \cfrac{1}{A_3(x) + \ddots}}} := [0; A_1(x), A_2(x), \cdots], \tag{1}$$

where the digits $A_n(x)$ are polynomials of a strictly positive degree and are defined by

$$A_n(x) = [\frac{1}{T^{n-1}(x)}], \qquad n \geqslant 1.$$

The metric and ergodic properties of continued fraction of Laurent series have been studied by Niederreiter [8], Niederreiter and Vielhaber [9] and Berthé and Nakada [2].

The following results will be used frequently. For more details, we refer to the results in [2], [5], [7] and [11].

As in the real case, $P_n(x)$ and $Q_n(x)$ are obtained by the following recurrence formulae

$$P_{-1} = 1, \qquad P_0 = 0, \qquad P_n = A_n P_{n-1} + P_{n-2}, \quad n \geqslant 2.$$

$$Q_{-1} = 0, \qquad Q_0 = 1, \qquad Q_n = A_n Q_{n-1} + Q_{n-2}, \quad n \geqslant 2.$$

We call $\frac{P_n(x)}{Q_n(x)}$ the n-th convergents of x, since

$$\frac{P_n(x)}{Q_n(x)} = [0; A_1(x), A_2(x), \cdots, A_n(x)], \qquad (n \geqslant 1).$$

Lemma 2.1 ([2]). *Let $A_1, A_2, \cdots, A_n$ be given polynomials with a strictly positive degree and put*

$$I_n(A_1, A_2, \cdots, A_n) = \{x \in I : A_1(x) = A_1, A_2(x) = A_2, \cdots, A_n(x) = A_n\}.$$

Then $I_n(A_1, A_2, \cdots, A_n)$ is a closed disc with diameter equal to

$$|I_n(A_1, A_2, \cdots, A_n)| = q^{-2\sum\limits_{k=1}^{n} \deg A_k - 1}$$

and

$$\boldsymbol{P}(I_n(A_1, A_2, \cdots, A_n)) = q^{-2\sum\limits_{k=1}^{n} \deg A_k}.$$

Remark 2.2. *We call $I_n(A_1, A_2, \cdots, A_n)$ in Lemma 2.1 an n-th digital cylinder. Since the valuation $\|\cdot\|$ is non-Archimedean, if two cylinders intersect, one contains the other.*

Lemma 2.3 ([6]). *Let s_α be the unique solution of*

$$f_\alpha(s) := \sum_{k=1}^{\infty} (q-1) q^k \left(\frac{1}{q^{2k+\alpha}}\right)^s = 1. \tag{2}$$

Then s_α is continuous with respect to α. Furthermore,

$$\lim_{\alpha \to 1} s_\alpha = 1, \qquad \lim_{\alpha \to \infty} s_\alpha = \frac{1}{2}.$$

3. Hausdorff dimension of $E(x_0)$

Now, we are ready to study the Hausdorff dimension of the set $E(x_0)$, which is the main result of this paper.

Theorem 3.1. *Let $x_0 \in I$ have continued fraction expansion $x_0 = [A_1(x_0), A_2(x_0), \cdots]$ and t_n be a non decreasing sequence of natural numbers. Write*

$$\liminf_{n \to \infty} \frac{2\sum\limits_{k=1}^{t_n} \deg A_k(x_0)}{n} = \alpha.$$

If $1 < \alpha < +\infty$, then we have

$$\dim_H E(x_0) = s_\alpha.$$

Before proving Theorem 3.1, we state the mass distribution principle (see[3]), which shall be applied to obtain a lower bound for $\dim_H E(x_0)$.

Lemma 3.2 ([3]). *Suppose $E \subset I$ and μ is a measure with $\mu(E) > 0$. If there exist constants $c > 0$ and $\delta > 0$ such that*

$$\mu(D) \geqslant c|D|^s,$$

for all discs D with diameter $|D| \leqslant \delta$, then

$$\dim_H E \geqslant s.$$

Proof of Theorem 3.1. Firstly we give an upper bound for $\dim_H E(x_0)$. Notice that

$$\begin{aligned} E(x_0) &= \{x \in I : T^n(x) \in I_{t_n}(x_0) \text{ for infinitely many } n\} \\ &= \limsup_{n\to\infty}\{x \in I : T^n(x) \in I_{t_n}(x_0)\} = \bigcap_{N=1}^{\infty} \bigcup_{n \geqslant N} \{x \in I : T^n(x) \in I_{t_n}(x_0)\} \\ &= \bigcap_{N=1}^{\infty} \bigcup_{n\geqslant N} \bigcup_{A_1,\cdots,A_n} \{x \in I : A_k(x) = A_k \in \mathbb{N},\ 1 \leqslant k \leqslant n; \\ &\qquad\qquad A_{n+j}(x) = A_{n+j}(x_0),\ 1 \leqslant j \leqslant t_n\}, \end{aligned}$$

where the fourth union takes over all $(A_1, \cdots, A_n) \in (\mathbb{F}_q[z])^n$ with strictly positive degree.

For any $\epsilon > 0$ and $\tau > 0$, when n is large enough, we have $s_\alpha < s_{\alpha-\epsilon} + \tau$ and $\frac{2\sum_{k=1}^{t_n} \deg A_k(x_0)}{n} > \alpha - \epsilon$. Then

$$\begin{aligned} \mathbf{H}^{s_{\alpha-\epsilon}+\tau}(E(x_0)) &\leqslant \liminf_{N\to\infty} \sum_{n\geqslant N} \sum_{A_1,\cdots,A_n} \left(\frac{1}{q^{2\sum_{k=1}^{t_n} \deg A_k(x_0)}\, q^{2\sum_{k=1}^{n} \deg A_k}} \right)^{s_{\alpha-\epsilon}+\tau} \\ &\leqslant \liminf_{N\to\infty} \sum_{n\geqslant N} \sum_{A_1,\cdots,A_n} \left(\frac{1}{q^{n(\alpha-\epsilon)} q^{2\sum_{k=1}^{n} \deg A_k}} \right)^{s_{\alpha-\epsilon}+\tau} \\ &= \liminf_{N\to\infty} \sum_{n\geqslant N} \frac{1}{q^{n(\alpha-\epsilon)(s_{\alpha-\epsilon}+\tau)}} \prod_{j=1}^{n} \sum_{A_j} \left(\frac{1}{q^{2\deg A_j}} \right)^{s_{\alpha-\epsilon}+\tau} \\ &= \liminf_{N\to\infty} \sum_{n\geqslant N} \frac{1}{q^{n(\alpha-\epsilon)(s_{\alpha-\epsilon}+\tau)}} \left(\sum_{k=1}^{\infty} (q-1)q^k \frac{1}{q^{2k(s_{\alpha-\epsilon}+\tau)}} \right)^n \\ &\leqslant \liminf_{N\to\infty} \sum_{n\geqslant N} \left(\frac{q^{(\alpha-\epsilon)s_{\alpha-\epsilon}}}{q^{(\alpha-\epsilon)(s_{\alpha-\epsilon}+\tau)}} \right)^n = 0. \end{aligned}$$

Therefore

$$\dim_H E(x_0) \leqslant s_{\alpha-\epsilon} + \tau.$$

Since τ and ϵ are arbitrary positive constants, from Lemma 2.3, we have

$$\dim_H E(x_0) \leqslant s_\alpha.$$

Now we give a lower bound of $\dim_H E(x_0)$. Let Γ denote the set $\{n_k, n_k+1, n_k+2, \cdots, n_k+t_{n_k}-1, k \geqslant 1\}$.

Step I. In this part, we will construct a subset $E_\beta(x_0) \subset E(x_0)$.

Fix $\beta \in \mathbb{N}$ and a sequence $\{n_k\} \subset \mathbb{N}$, which satisfy $\lim\limits_{k\to\infty} \frac{2\sum_{k=1}^{t_{n_k}} \deg A_k(x_0)}{n_k} = \alpha$ and

$$n_1 + \cdots + n_k + t_{n_k} < \frac{1}{k+1} n_{k+1}, \qquad \forall_{k\geqslant 1}. \tag{3}$$

Assume that $n_k(\alpha - \epsilon) \leqslant 2\sum_{j=1}^{t_{n_k}} \deg A_j(x_0) \leqslant n_k(\alpha+\epsilon)$, $\forall_{k\geqslant 1}$.

Let

$$E_\beta(x_0) = \{x \in I : x = [a_1, a_2, \cdots, a_n, \cdots],\ \deg a_{n_k+j} = \deg A_j(x_0),\ k \geqslant 1, \\ 1 \leqslant j \leqslant t_{n_k};\ 1 \leqslant \deg a_j \leqslant \beta,\ j \notin \{n_k+1, n_k+2, \cdots, n_k+t_{n_k},\ k \geqslant 1\}\}.$$

Obviously

$$E_\beta(x_0) \subseteq E(x_0).$$

Let $s_{\alpha+\epsilon}(\beta)$ be the unique solution of

$$\sum_{k=1}^{\beta} (q-1) q^k q^{-2ks} = q^{(\alpha+\epsilon)s},$$

we will show that $\dim_H E_\beta(x_0) \geqslant s_{\alpha+\epsilon}(\beta)$.

For any $n \geqslant 1$, define

$$D_n = \{(\sigma_1, \cdots, \sigma_n) \in (\mathbb{F}_q[z])^n : E_\beta(x_0) \cap I_n(\sigma_1, \cdots, \sigma_n) \neq \emptyset\}.$$

$$D = \bigcup_{n=0}^{\infty} D_n, \quad (D_0 := \emptyset).$$

For any $n \geqslant 1$ and $(\sigma_1, \cdots, \sigma_n) \in D_n$, we call

$$J(\sigma_1, \cdots, \sigma_n) := \bigcup_{\sigma_{n+1}} \mathrm{cl}\, I_{n+1}(\sigma_1, \cdots, \sigma_{n+1}) \tag{4}$$

a basic interval of order n with respect to $E_\beta(x_0)$, where the union in (4) is taken over all σ_{n+1} such that $(\sigma_1, \cdots, \sigma_n, \sigma_{n+1}) \in D_{n+1}$ and cl stands for the closure.

For any $x, y \in J(\sigma_1, \cdots, \sigma_n)$, let $x \in I(\sigma_1, \cdots, \sigma_n, \sigma_{n+1})$ and $y \in I(\sigma_1, \cdots, \sigma_n, \sigma'_{n+1})$ with $\deg \sigma_{n+1} \leqslant \deg \sigma'_{n+1}$. Then

$$|x-y| = |\frac{\sigma_{n+1} - \sigma'_{n+1}}{\sigma_{n+1}\sigma'_{n+1} Q_n(x)^2}| \leqslant \frac{1}{|\sigma_{n+1} Q_n(x)^2|}. \tag{5}$$

Equality holds if $\deg \sigma_{n+1} \neq \deg \sigma'_{n+1}$.

Combining with Lemma 2.1, we have

Case I. If $n \in \Gamma$, i.e., $n_k \leqslant n < n_k + t_{n_k}$, for some $k \geqslant 1$.

$$|J(\sigma_1, \cdots, \sigma_n)| = q^{-2\sum\limits_{k=1}^{n+1} \deg \sigma_k - 1}. \tag{6}$$

Case II. If $n \notin \Gamma$, i.e., $n_k + t_{n_k} \leqslant n < n_{k+1}$, for some $k \geqslant 1$.

$$|J(\sigma_1, \cdots, \sigma_n)| = q^{-2\sum\limits_{k=1}^{n} \deg \sigma_k - 1}. \tag{7}$$

It is clear that

$$E_\beta(x_0) = \bigcap_{n \geqslant 1} \bigcup_{(\sigma_1, \cdots, \sigma_n) \in D_n} J(\sigma_1, \cdots, \sigma_n). \tag{8}$$

Step II. For the lower bound, we define a probability measure supported on $E_\beta(x_0)$.

Let $m_k = n_k - n_{k-1} - t_{n_{k-1}}$ with $k \geqslant 1$ and $n_0 = t_{n_0} := 0$. Now we define a set function $\mu : \{J(\sigma), \sigma \in D \backslash D_0\} \to \mathbb{R}^+$ as follows .

Denote

$$q_{m_i} := q_{m_i}(\sigma_{n_{i-1}+t_{n_{i-1}}+1}, \cdots, \sigma_{n_i}) = q^{\left(2 \sum\limits_{j=\sigma_{n_{i-1}+t_{n_{i-1}}+1}}^{\sigma_{n_i}} \deg \sigma_j\right)}.$$

For any $n \geqslant 1$ and $(\sigma_1, \cdots, \sigma_n) \in D_n$, let

$$\mu(J(\sigma_1, \cdots, \sigma_n)) := \begin{cases} \prod_{j=1}^k \left(\frac{1}{q^{m_j(\alpha+\epsilon)} q_{m_j}}\right)^{s_{\alpha+\epsilon}(\beta)}, & \text{if } n \in \{n_k, \cdots, n_k + t_{n_k}, k \geqslant 1\} \\ \sum\limits_{\deg \sigma_{n+1}, \cdots, \deg \sigma_{n_k} \leqslant \beta} \mu(J(\sigma_1, \cdots, \sigma_{n_k})), & \text{if } n_{k-1} + t_{n_{k-1}} < n < n_k \end{cases} \tag{9}$$

Until now, the set function $\mu : \{J(\sigma), \sigma \in D \backslash D_0\} \to \mathbb{R}^+$ is well defined. By Lemma 2.3, for any $n \geqslant 1$ and $(\sigma_1, \cdots, \sigma_n) \in D_n$, we have

$$\mu(J(\sigma_1, \cdots, \sigma_n)) = \sum_{\sigma_{n+1}} \mu(J(\sigma_1, \cdots, \sigma_{n+1})),$$

where the summation is taken over all σ_{n+1} such that $(\sigma_1, \cdots, \sigma_n, \sigma_{n+1}) \in D_{n+1}$. Notice that

$$\sum_{(\sigma_1, \cdots, \sigma_{n_1}) \in D_{n_1}} \mu(J(\sigma_1, \cdots, \sigma_{n_1})) = 1.$$

By Kolmogorov extension theorem, the set function μ can be extended into a probability measure supported on $E_\beta(x_0)$, which is still denoted by μ.

Step III. We now give the estimation of $\mu(J(\sigma_1,\cdots,\sigma_n))$, for each $(\sigma_1,\cdots,\sigma_n)\in D_n$.

Fix $0<t<s_{\alpha+\epsilon}(\beta)$, take $\tau=\frac{s_{\alpha+\epsilon}(\beta)-t}{2}$. We claim that there is an integer N such that $n\geqslant N$ and $(\sigma_1,\cdots,\sigma_n)\in D_n$ implies

$$\mu(J(\sigma_1,\cdots,\sigma_n))\leqslant c\cdot|J(\sigma_1,\cdots,\sigma_n)|^{t-2\tau}, \tag{10}$$

where $c>0$ is an absolute constant.

Choose k_0 sufficiently large such that

$$\frac{t}{t+\tau}\leqslant\frac{m_k}{n_k},\quad \forall\ k>k_0. \tag{11}$$

Take $c_0=q^{2n_{k_0}\beta}\prod_{j=1}^{k_0}q^{n_j(\alpha+\epsilon)}$. Then

$$\prod_{j=1}^{k_0}\Big(\frac{1}{q^{m_j(\alpha+\epsilon)}q_{m_j}}\Big)^{s_{\alpha+\epsilon}(\beta)}\leqslant 1\leqslant c_0\prod_{j=1}^{k_0}\left(\frac{1}{q^{n_j(\alpha+\epsilon)}q_{m_j}}\right)^{t}. \tag{12}$$

For any $n>n_{k_0}$ and $(\sigma_1,\cdots,\sigma_n)\in D_n$, we will distinguish two cases to establish $\mu(J(\sigma_1,\cdots,\sigma_n))$.

Case I. $n_k\leqslant n\leqslant n_k+t_{n_k}$, for some $k\geqslant k_0$.

$$\begin{aligned}
\mu(J(\sigma_1,\cdots,\sigma_n)) &= \prod_{j=1}^{k}\Big(\frac{1}{q^{m_j(\alpha+\epsilon)}q_{m_j}}\Big)^{s_{\alpha+\epsilon}(\beta)}\\
&=\prod_{j=1}^{k_0}\left(\frac{1}{q^{m_j(\alpha+\epsilon)}q_{m_j}}\right)^{s_{\alpha+\epsilon}(\beta)}\prod_{j=k_0+1}^{k}\left(\frac{1}{q^{m_j(\alpha+\epsilon)}q_{m_j}}\right)^{s_{\alpha+\epsilon}(\beta)}\\
&\leqslant c_0\prod_{j=1}^{k_0}\left(\frac{1}{q^{n_j(\alpha+\epsilon)}q_{m_j}}\right)^{t}\prod_{j=k_0+1}^{k}\left(\frac{1}{q^{m_j(\alpha+\epsilon)}q_{m_j}}\right)^{t+\tau}\quad\text{(by (12))}\\
&\leqslant c_0\prod_{j=1}^{k_0}\left(\frac{1}{q^{n_j(\alpha+\epsilon)}q_{m_j}}\right)^{t}\prod_{j=k_0+1}^{k}\left(\frac{1}{q_{m_j}}\right)^{t}\prod_{j=k_0+1}^{k}\left(\frac{1}{q^{n_j(\alpha+\epsilon)}}\right)^{t}\quad\text{(by (11))}\\
&=c_0\prod_{j=1}^{k}\left(\frac{1}{q^{n_j(\alpha+\epsilon)}q_{m_j}}\right)^{t}\leqslant c_0\cdot\prod_{j=1}^{k}\frac{1}{q^{2t\sum_{l=1}^{t_{n_k}}\deg A_l(x_0)}(q_{m_j})^t}\\
&\leqslant c_0|J(\sigma_1,\cdots,\sigma_n)|^{t-\tau}\quad\text{(by (6) and (7))}.
\end{aligned}\tag{13}$$

Case II. $n_{k-1}+t_{n_{k-1}}<n<n_k$, for some $k\geqslant k_0$.

Let $\ell'=n_k-n$. By the definition of μ, proceeding in the similar way as in Case I, we have

$$\begin{aligned}
\mu(J(\sigma_1,\cdots,\sigma_n)) &= \sum_{\sigma_{n+1},\cdots,\sigma_{n_k}} \mu(J(\sigma_1,\cdots,\sigma_n,\sigma_{n+1},\cdots,\sigma_{n_k}))\\
&= \prod_{j=1}^{k-1}\left(\frac{1}{q^{m_j(\alpha+\epsilon)}q_{m_j}}\right)^{s_{\alpha+\epsilon}(\beta)} \sum_{\sigma_{n+1},\cdots,\sigma_{n_k}} \left(\frac{1}{q^{m_k(\alpha+\epsilon)}q_{m_k}}\right)^{s_{\alpha+\epsilon}(\beta)}\\
&\leqslant c_0\frac{1}{q^{2(t-\tau)\sum_{i=1}^{n_{k-1}+t_{n_{k-1}}}\sigma_i}} \sum_{\sigma_{n+1},\cdots,\sigma_{n_k}} \left(\frac{1}{q^{m_k(\alpha+\epsilon)}q^2_{m_k}}\right)^{s_{\alpha+\epsilon}(\beta)}\\
&\leqslant c_0\frac{1}{q^{2(t-\tau)\sum_{i=1}^{n}\deg\sigma_i}} \sum_{\sigma_1,\cdots,\sigma_{\ell'}} \left(\frac{1}{q^{\ell'(\alpha+\epsilon)}q^2_{\ell'}}\right)^{s_{\alpha+\epsilon}(\beta)}\\
&\leqslant c_0 q|J(\sigma_1,\cdots,\sigma_n)|^{t-2\tau} \quad \text{(by (6) and (7))}.
\end{aligned}$$

Step IV. In this part, we will estimate the measure of $B(x,r)$.

For any $x\in E_\beta(x_0)$, there exists an infinite sequence $\{\sigma_1,\sigma_2,\cdots\}$ with $\sigma_{n_k+j}=A_j(x_0)$, $k\geqslant 1$, $1\leqslant j\leqslant t_{n_k}$; $1\leqslant\sigma_j\leqslant\beta$, $j\notin\{n_k+1,n_k+2,\cdots,n_k+t_{n_k},k\geqslant 1\}$, such that $x\in J(\sigma_1,\cdots,\sigma_n)$, for all $n\geqslant 1$. Let $r_0=\min_{\sigma\in D_{n_{k_0}}}|J(\sigma)|$, for any $0<r<r_0$, there exists an integer $n\geqslant n_{k_0}$ such that

$$|J(\sigma_1,\cdots,\sigma_n,\sigma_{n+1})|\leqslant r<|J(\sigma_1,\cdots,\sigma_n)|. \tag{14}$$

Now we distinguish two cases to estimate the measure of $B(x,r)$.

Case I. $n\in\Gamma$, i.e., $n_k\leqslant n<n_k+t_{n_k}$, for some $k\geqslant 1$.

In this case, the ball $B(x,r)$ can only intersect with one basic interval of order n, which is just $J(\sigma_1,\cdots,\sigma_n)$ and can intersect at most one basic interval of order $n+1$. From the dimension of the measure μ and (10), we have

$$\begin{aligned}
\mu(B(x,r)) &\leqslant \mu(J(\sigma_1,\cdots,\sigma_n))\\
&= \mu(J(\sigma_1,\cdots,\sigma_{n+1}))\\
&\leqslant c_0|J(\sigma_1,\cdots,\sigma_{n+1})|^{t-2\tau}\\
&\leqslant c_0|r|^{t-2\tau}
\end{aligned} \tag{15}$$

Case II. $n\notin\Gamma$.

By the dimension of the measure μ, we have

$$\mu(J(\sigma_1,\cdots,\sigma_n,\sigma_{n+1}))\geqslant\frac{1}{q^{(\alpha+\epsilon)+2\beta}}\mu(J(\sigma_1,\cdots,\sigma_n)).$$

From (10) and (14), we obtain

$$\begin{aligned}\mu(B(x,r)) &\leqslant q^{(\alpha+\epsilon)+2\beta}\mu(J(\sigma_1,\cdots,\sigma_{n+1})) \\ &\leqslant c_0\cdot q^{(\alpha+\epsilon)+2\beta}|J(\sigma_1,\cdots,\sigma_{n+1})|^{t-2\tau} \\ &\leqslant 4c_0 q^{(\alpha+\epsilon)+2\beta}\cdot r^{t-2\tau}.\end{aligned} \tag{16}$$

Combining these two cases with Lemma 3.2, we can get

$$\dim_H E_\beta(x_0) \geqslant t-2\tau = 2t - s_{(\alpha+\epsilon)}(\beta).$$

Since $t < s_{(\alpha+\epsilon)}(\beta)$ is arbitrary, we have

$$\dim_H E(x_0) \geqslant \dim_H E_\beta(x_0) \geqslant s_{(\alpha+\epsilon)}(\beta).$$

Therefore Theorem 3.1 is proved . ■

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