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NEW GENERALIZATION OF CONTINUED FRACTION, I

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Abstract: Let three homogeneous real linear forms be given in a three-dimensional real space. Their moduli give a mapping of the space into another space. In the second space, we consider the convex hull of images of all integer points of the first space except its origin. This convex hull is called the modular polyhedron. The best integer approximations to the root subspaces of these forms are given by the integer points whose images lie on the boundary of the modular polyhedron. Here we study the properties of the modular polyhedron and use them for the construction of an algorithm generalizing continued fraction. The algorithm gives the best approximations, and it is periodic for cubic irrationalities with positive discriminant. Attempts to generalize continued fraction were made by Euler, Jacobi, Dirichlet, Hermite, Poincare, Hurwitz, Klein, Minkowski, Voronoi, and by many others.

Keywords: generalized continued fraction, lattice, modular polyhedron, face.

Introduction

Let α_0 and α_1 be natural numbers. In order to find their greatest common divisor, Euclidean division algorithm [65] (successive division with remainder) is commonly used:

$$\alpha_0 = a_0\alpha_1 + \alpha_2, \qquad \alpha_1 = a_1\alpha_2 + \alpha_3, \qquad \alpha_2 = a_2\alpha_3 + \alpha_4, \dots,$$

where natural numbers $a_0, a_1, a_2, \dots$ are incomplete quotients. This is the algorithm of expansion of the number $\alpha = \alpha_0/\alpha_1$ into the *regular continued fraction* [33]; and the algorithm is applicable to any real number α. Here $a_0 = [\alpha]$, where $[\alpha]$ is the integer part of the number α, $a_1 = [1/(\alpha - a_0)]$, …, i.e.

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}, \tag{1}$$

and

$$\begin{pmatrix} \alpha_{k+1} \\ \alpha_{k+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -a_k \end{pmatrix} \begin{pmatrix} \alpha_k \\ \alpha_{k+1} \end{pmatrix}, \qquad a_k = [\alpha_k/\alpha_{k+1}]. \tag{2}$$

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If we truncate the expansion (1) at a_k and fold the truncated continued fraction into the rational number p_k/q_k, then we obtain the *convergent fraction*, that gives the best rational approximation to the number α. Here

$$\begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix} = \begin{pmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2} \end{pmatrix} \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

and

$$\begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -a_k \end{pmatrix}, \qquad \det \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix} = \pm 1,$$

i.e. the vectors (α_k, α_{k+1}) and (p_k, q_k) belong to conjugate planes; and the pair of vectors (p_k, q_k), (p_{k-1}, q_{k-1}) may serve as a basis in one of the planes.

Continued fractions (1) and the relations (2), (3) were considered by Wallis [67] in 1655. In 1737, Euler [25] introduced the term "continued fraction" (fractio continua). Lagrange [42] proved that for quadratic irrationalities α, the expansion in continued fraction (1) is periodic (and vice versa).

Thus, the algorithm of expansion of a number into continued fraction is:

1) simple;
2) gives the best rational approximation to the number;
3) periodic for quadratic irrationalities.

Besides, it possesses a number of other remarkable properties.

In 1775, Euler [26] made the first attempt to generalize the algorithm of continued fraction for vectors. Subsequently, his approach was developed by Jacobi [32, 31], Poincaré [59], Brun [6], Perron [56], Bernstein [2], Pustilnikov [60], and others (see [61]). They, by analogy with (2), constructed *matrix algorithms* of the form

$$A_{k+1} = C_k A_k, \qquad k = 0, 1, 2, \ldots, \tag{4}$$

where A_k is an n-dimensional vector, and C_k is a square n-matrix with integer elements formed using the vector A_k, and $\det C_k = \pm 1$. These algorithms are simple, but, generally, do not give best rational approximations to vectors and not always possess an analog of the property 3) for $n = 3$:

3') periodicity for cubic irrationalities.

Already Hermite [28] criticized the Jacobi algorithm. In papers [16, 17, 49, 53, 48, 50, 51, 52], a comparison of quality of matrix algorithms was performed, and it was established that none of them possess the properties 2) and $3'$) for all vectors A_0. It appeared that Poincaré's algorithm [59] is the worst.

In 1842, Dirichlet [44] proposed for $n = 3$ instead of the three-dimensional vector $A = A_0$ to consider two linear homogeneous forms $l_1(X)$ and $l_2(X)$ such that $l_1(A) = l_2(A) = 0$. In 1850, Hermite [27], developing this approach, proposed his generalization of continued fraction. And finally, in 1895-96, Klein [34], Minkowski [45], and Voronoi [66], independently came to the idea that three homogeneous linear forms $l_1(X)$, $l_2(X)$, $l_3(X)$ should be considered in $\mathbb{R}^3$ and gave their concepts of generalization of continued fraction. However, Klein gave only some general geometric constructions, while Minkowski and Voronoi suggested concrete algorithms.

Subsequently, Klein's approach was rediscovered by Skubenko [62] and Arnold [1]; and they referred to Klein polyhedra as sails and Arnold polyhedra [40, 41]. Although in [16], there was suggested an algorithm of computation of Klein polyhedra, in [17, 49, 53, 48, 50, 51, 52] it was established that Klein-Skubenko-Arnold polyhedra do not provide a basis for a good algorithm generalizing continued fractions. Only algorithms by Minkowski and by Voronoi possess the properties 2 and $3'$, but they are very bulky. Many works were devoted to their applications and development (see [4, 20, 21]). Hurwitz proposed his approach to a generalization of continued fraction in 1894 [29], but without an algorithm.

The author's interest to the generalization of continued fraction was stimulated in connection with his study [7], 1964, that was repeated by Lang [43] in 1972 (see also Stark [63] and Appendix).

Below in Section 1, we consider various known types of continued fractions and their planar interpretations. The most appropriate for generalization is the *diagonal continued fraction*, which was first introduced (without the name) by Minkowski in 1896 [45, part I, case $\Omega = 1$]. In 1902, he introduced it once again (with the name) [46] (see also [18, 58]). In author's opinion, the name *convex* is more appropriate for these continued fractions. In Section 2, we demonstrate how to compute the diagonal continued fraction. In Section 3, we expound the three-dimensional constructions proposed by Klein and by Voronoi. In Section 4, we introduce the modular polyhedron and study its general properties. In Section 5, we study those properties of the modular polyhedron that we use for the construction of the algorithm of transition from one basis to the next. In Section 6, we describe this algorithm, which corresponds to the motion over the surface of the modular polyhedron. Points on this surface usually suffice for this algorithm; additional points lying outside this surface are necessary only for the case $\omega(\Gamma_i) = 2$. In Section 7, we discuss periodicity of the proposed algorithm for cubic irrationalities with positive discriminant.

Principle ideas and results of this study were announced in Preprints [10, 18, 11, 19] and articles [55, 13, 14]. Preliminary version of this article was published in Preprints [12, 15].

1. Continued fractions

Klein [34, 35, 36] suggested the following geometric interpretation of continued fraction of the number α (see also [37]).

Let two independent homogeneous linear forms be given in the plane with coordinates $X = (x_1, x_2)$

$$l_1(X) \stackrel{\text{def}}{=} l_{11}x_1 + l_{12}x_2, \qquad l_2(X) \stackrel{\text{def}}{=} l_{21}x_1 + l_{22}x_2; \tag{1.1}$$

$$\det \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \neq 0.$$

Straight lines $\mathbf{L}_i = \{X : l_i(X) = 0\}$, $i = 1, 2$ divide the plane $\mathbb{R}^2$ into four quadrants or angles. Consider two neighbor angles $\mathbf{O}_1 = \{X : l_1(X) \geqslant 0, l_2(X) \geqslant 0\}$ and $\mathbf{O}_2 = \{X : l_1(X) \leqslant 0, l_2(X) \geqslant 0\}$. We denote as $\mathbf{K}_i$ the convex hull of integer points X, except $X = 0$, lying in the angle $\mathbf{O}_i$ $(i = 1, 2)$. The boundaries $\partial\mathbf{K}_i$ of the sets $\mathbf{K}_i$ are the convex open polygons consisting of vertices B_j and edges R_j. If

$$l_1(X) = x_1 - \alpha x_2, \qquad l_2(X) = x_2, \tag{1.2}$$

then the vertices $B_j = (x_1, x_2)$ of these open polygons correspond to convergent fractions $x_1/x_2 = p_j/q_j$ of the continued fraction of the number α. Here the parity of the numbers j and i coincide, i.e. the vertex B_j lies on the open polygon $\partial\mathbf{K}_i$ with $i = 1$ if j is odd, and with $i = 2$ if j is even. The integer points (x_1, x_2) lying on the edges R_j of the open polygons $\partial\mathbf{K}_i$ correspond to intermediate fractions of the continued fraction of the number α (see [33, page 21]). The number of integer points on the edge minus one is equal to the incomplete quotient of the continued fraction etc. (see Fig. 1).

Voronoi [66, Part I] suggested the following interpretation of continued fraction. Let two linear forms (1.1) be given in the plane $\mathbb{R}^2$. The values of the *pair of forms* $l_1(X)$ and $l_2(X)$ at an integer point $X \neq 0$ are called the *relative minimum*, if there are no other integer point $Y \neq 0$ for which

$$|l_1(Y)| \leqslant |l_1(X)|, \quad |l_2(Y)| \leqslant |l_2(X)| \quad \text{and} \quad |l_1(Y)| + |l_2(Y)| < |l_1(X)| + |l_2(X)|.$$

Voronoi proved that all integer points X where the pair of forms (1.1) has relative minima are completely ordered, and the transitions from one pair of such neighbor points $B_{k-2} = (p_{k-2}, q_{k-2})$ and $B_{k-1} = (p_{k-1}, q_{k-1})$ to the next such pair $B_{k-1} = (p_{k-1}, q_{k-1})$ and $B_k = (p_k, q_k)$ is given by formulas of the type (2) and (3), i.e. by the algorithm of continued fraction. In particular, for the forms (1.2), the points B_k correspond to all convergents of the continued fraction of the number α, and their sequence in descending $|l_1(X)|$ and in ascending $|l_2(X)|$ corresponds to the sequence of convergents of the continued fraction. In Fig. 2, in the first quadrant of the plane $|l_1|$, $|l_2|$, for every relative minimum $(|l_1^{(k)}|, |l_2^{(k)}|) = V_k$, the quadrangle $|l_1| \leqslant |l_1^{(k)}|$, $|l_2| \leqslant |l_2^{(k)}|$ is hatched, where there are no points $(|l_1(X)|, |l_2(X)|)$ for integer $X \neq 0$.

In [10], the author suggested the following construction. Let two homogeneous linear forms (1.1) be given in $\mathbb{R}^2$, and $l_{11}\alpha + l_{12} = 0$. We consider only the points of the half-plane $l_2(X) \geqslant 0$. In the plane with coordinates

$$m_1(X) = |l_1(X)|, \qquad m_2(X) = |l_2(X)|, \qquad M(X) = (m_1(X), m_2(X)),$$

consider the set $\mathbf{Z}^2$ of the points $M(X)$ for all integer points $X \in \mathbb{Z}^2$ except zero. Let $\mathbf{M}$ be the convex hull of the set $\mathbf{Z}^2$. The boundary $\partial\mathbf{M}$ of the set $\mathbf{M}$ is the open polygon consisting of the vertices V_k and edges U_k. In Fig. 3, the set $\mathbf{M}$ is hatched, and its boundary $\partial\mathbf{M}$ is plotted by bold segments.

Let $Z_k = (z_{1k}, z_{2k}) = M(C_k)$ be the points of the set $\mathbf{Z}^2$ placed on the open polygon $\partial\mathbf{M}$ and numbered in decreasing order of the first coordinate z_{1k}, $C_k \in \mathbb{Z}^2$. The open polygon $\partial\mathbf{M}$ and the points Z_k on it possess the following properties.

1. Every point Z_k is the point of relative minimum of the pair of forms (1.1), i.e. $C_k = B_j$; but the converse, generally, does not hold.

2. Let $Z_k = M(C_k)$ and $Z_{k+1} = M(C_{k+1})$ be two neighbor points, then $\det(C_k C_{k+1}) = \pm 1$.

3. One edge U_s of the open polygon $\partial\mathbf{M}$ can not contain three points Z_k that correspond to the three vertices B_j of one Klein open polygon $\partial\mathbf{K}_i$. Indeed, none of the three vertices of one Klein open polygon $\partial\mathbf{K}_i$ lie on one straight line, but the mapping $M(X)$ for them is linear.

4. If number α is a quadratic irrationality, then the sequences Z_k and C_k are periodic beginning with some number, i.e. there exists a unimodular matrix $\mathcal{D}$ and a natural number t such that $\mathcal{D}C_l = \mu C_{l+t}$ for $l > l_0$.

A sequence of points C_k with $Z_k = M(C_k) \in \partial\mathbf{M}$ numbered in decreasing order of $|l_1|$ and increasing order of $|l_2|$ is called the *convex continued fraction*. For the forms (1.2), it coincides with the *generalized continued fraction* suggested by Minkowski in [45, Part I, Case $\Omega = 1$] (see also [37, page 39]; [58, 47]; [57, Sect. 41]) and can be obtained as convergents of the *semi-regular* continued fraction

$$\alpha = b_0 \pm \cfrac{1}{b_1 \pm \cfrac{1}{b_2 \pm \ddots}},$$

where b_i are natural numbers. The convex continued fraction for arbitrary forms (1.1) was introduced by Minkowski [46] with the name "diagonal".

Example 1.1. Let $\alpha = (1+\sqrt{3})/2 = 1.3660254\ldots$, $l_1(X) = x_1 - \alpha x_2$, $l_2(X) = x_2$. The regular continued fraction for α is periodic:

$$\alpha = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \ddots}}}. \tag{1.3}$$

Successive convergent fractions are

$$1, \quad \frac{3}{2}, \quad \frac{4}{3}, \quad \frac{11}{8}, \quad \frac{15}{11}, \quad \frac{41}{30}, \quad \frac{56}{41}, \ldots.$$

Vertices of the open polygon $\partial\mathbf{M}$ correspond only to convergent fractions with odd numbers. These convergent fractions are also convergents of the semi-regular continued fraction

$$\alpha = 1 + \cfrac{1}{3 - \cfrac{1}{4 - \cfrac{1}{4 - \cfrac{1}{4 - \ddots}}}}. \tag{1.4}$$

In Fig. 4, in the plane (m_1, m_2), the relative minima are shown that correspond to the continued fraction (1.3); and there are shown the sets $\mathbf{M}$, $\partial\mathbf{M}$ corresponding to the convex continued fraction (1.4). The edges and vertices of the open polygon $\partial\mathbf{M}$ are shown in Fig. 4 as bold segments and bold dots. The convex and diagonal continued fraction (1.4) is also the *continued fraction to the nearest integer* [37, page 39]; [30]; [57, Sect. 39]. But these continued fractions do not always coincide as the following example demonstrates.

Example 1.2. Let $\alpha = (1+\sqrt{5})/2 = 1.6180339\ldots$, $l_1(X) = x_1 - \alpha x_2$, $l_2(X) = x_2$. The regular continued fraction for α is periodic:

$$\alpha = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}, \tag{1.5}$$

the convex continued fraction has the form

$$\alpha = 2 - \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

and coincides with the regular one. But, starting with the second convergent, the continued fraction to the nearest integer is

$$\alpha = 2 - \cfrac{1}{3 - \cfrac{1}{3 - \ddots}}.$$

It corresponds only to convergents for (1.5) with even numbers.

2. Computation of the convex continued fraction

According to [57], the convex continued fraction for the forms (1.2) can be obtained from the regular continued fraction if we replace in certain places the aggregate

$$a + \cfrac{1}{1 + \cfrac{1}{b + \beta}},$$

where $\beta < 1$, by the aggregate $a + 1 - 1/(b + 1 + \beta)$, as it was done in Example 1.1 (these places include all cases with $b \geqslant 2$ and some cases with $b = 1$). However, we can compute the convex continued fraction sequentially step by step.

We describe one step of this computation. According to the property 2 of Section 1, let us take the points C_k and C_{k+1}, that correspond to the neighbor points $Z_k = M(C_k)$ and $Z_{k+1} = M(C_{k+1})$ of the open polygon $\partial\mathbf{M}$, as the basis. We will find the point C_{k+2}. We assign $E_1 = C_{k+1}$, $E_2 = C_k$ as unit vectors. Let the vector A have the form $A = (\alpha_1, \alpha_2)$ in this basis, where $\alpha_1 > |\alpha_2| > 0$. For the sake of simplicity, we assume that $|\alpha_2| = 1$. Let the linear forms $l_1(X)$ and $l_2(X)$ in this basis have the form

$$l_1(X) = \alpha_2 x_1 - \alpha_1 x_2, \qquad l_2(X) = l_{21}x_1 + l_{22}x_2, \qquad l_{21} > 0, \qquad \operatorname{sgn} l_{22} = \operatorname{sgn} \alpha_2.$$

The point C_{k+2} is to be found on the straight line

$$x_2 = \operatorname{sgn} \alpha_2 = \alpha_2. \tag{2.1}$$

Let $a = [\alpha_1/|\alpha_2|]$, then the nearest points to the point of intersection of the straight line (2.1) with the ray λA, $\lambda > 0$ are two integer points lying on the straight line (2.1): with $x_1 = a$, and with $x_1 = a + 1$ (see Fig. 5). Only at these points X, the values $|l_1(X)| < |l_1(C_{k+1})|$. Consequently, from the two points $B' = (a, \alpha_2)$ and $B'' = (a + 1, \alpha_2)$, we must choose the one that corresponds to the point Z_{k+2} on the open polygon $\partial\mathbf{M}$. Preceding two points of this open polygon are

$$Z_k = M(C_k) = (\alpha_1, |l_{22}|), \qquad Z_{k+1} = M(C_{k+1}) = (1, l_{21}).$$

Further

$$\begin{gathered} M' = M(B') = (\alpha_1 - a, l_{21}a + |l_{22}|), \\ M'' = M(B'') = (a + 1 - \alpha_1, l_{21}(a + 1) + |l_{22}|). \end{gathered} \tag{2.2}$$

Let $C = (c_1, c_2)$ and $D = (d_1, d_2)$ be two points in the plane $Y = (y_1, y_2)$. The straight line drawn through them intersects the axis y_2 at the point

$$y_2(C, D) = \frac{\begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}}{c_1 - d_1}. \tag{2.3}$$

We compute the values (2.3) for two pairs of points

$$y_2(Z_{k+1}, M') \qquad \text{and} \qquad y_2(Z_{k+1}, M''), \tag{2.4}$$

and as the point C_{k+2}, we take one of the points B', B'' for which the value of y_2 in (2.4) is less (see Fig. 6). If for both points B' and B'', the values of y_2 in (2.4) are equal, then for the point C_{k+2}, we take that where m_1 is less.

Having chosen the point C_{k+2}, we go to the basis C_{k+1}, C_{k+2}, etc.

Simplified algorithm. Computation by the formula (2.3) is rather cumbersome. It can be simplified in the following way.

Let $M = (m_1, m_2)$. We denote $|M| = m_1 m_2$. From the two points M' and M'' in (2.2), we take the one where $|M|$ is less. But if $|M'| = |M''|$, then we choose M', i.e. the point where m_1 is greater and m_2 is less. This procedure leads to the convex open polygon $\partial|\mathbf{M}|$ the vertices of which are the vertices of the open polygon $\partial\mathbf{M}$, but not all vertices of the open polygon $\partial\mathbf{M}$ are the vertices of the open polygon $\partial|\mathbf{M}|$.

3. Three-dimensional generalizations of geometric constructions

Klein [34, 35] suggested a three-dimensional analog of his two-dimensional interpretation of continued fraction. Let three independent homogeneous linear forms be given in $\mathbb{R}^3$

$$l_i(X) = \langle L_i, X\rangle, \quad i = 1,2,3, \qquad \det(L_1 L_2 L_3) \neq 0, \tag{3.1}$$

where $X \in \mathbb{R}^3$, $L_i = (l_{i1}, l_{i2}, l_{i3}) \in \mathbb{R}^3_*$, $\langle \cdot\, , \cdot\rangle$ means scalar product, and the space $\mathbb{R}^3_*$ be dual to the space $\mathbb{R}^3$. Each collection $\Sigma = (\sigma_1, \sigma_2, \sigma_3)$, where $\sigma_i = \pm 1$, corresponds to its octant

$$\mathbf{O}_\Sigma = \{X : \sigma_i \langle L_i, X\rangle \geqslant 0,\ i = 1,2,3\},$$

bounded by planes $\mathbf{L}_i = \{X : l_i(X) = 0\}$, $i = 1,2,3$. In each octant $\mathbf{O}_\Sigma$, we consider the convex hull $\mathbf{K}_\Sigma$ of all integer points $X \in \mathbf{O}_\Sigma$, $X \neq 0$. The boundary $\partial\mathbf{K}_\Sigma$ of the hull is a convex two-dimensional polyhedral surface consisting of vertices, edges, and faces. The vertices of the surface must give the best integer approximations in the octant $\mathbf{O}_\Sigma$ to the planes $\mathbf{L}_i$ and to the straight lines which are their intersections.

In studies [16, 17, 49, 53, 48, 50, 51, 52, 39, 40, 41, 38, 5], Klein polyhedra were computed for a number of homogeneous cubic forms with integer coefficients of the form

$$h(X) = \langle L_1, X\rangle\langle L_2, X\rangle\langle L_3, X\rangle; \tag{3.2}$$

each form corresponds to the roots of its cubic equation. It turned out that Klein polyhedra have a rather complex and diverse structure. A web site [5] gives many examples of plane logarithmic projections of Klein polyhedra.

Example 3.1. The equation $\lambda^3 + 9\lambda^2 + 6\lambda - 1 = 0$ has three real roots

$$\lambda_1 \approx -8.258845, \qquad \lambda_2 \approx -.878917, \qquad \lambda_3 \approx .137763.$$

The cubic form taken from [53] $g_4(X) = h(X)$ of the type (3.2) has

$$L_i = (5, -3 + 15\lambda_i + 2\lambda_i^2, 11 + 10\lambda_i + \lambda_i^2), \qquad i = 1,2,3.$$

Here $g_4(X)$ is the fourth extremal form from [64]. Fig. 7 shows the orthogonal logarithmic projection of the surface of the Klein polyhedron $\partial\mathbf{K}_{+++}$ of this form in coordinates

$$n'_1 = \frac{\log m_1 + 2\log m_2 - \log m_3}{\sqrt{6}}, \qquad n'_2 = \frac{\log m_1 - \log m_3}{\sqrt{2}}, \tag{3.3}$$

where $m_i(X) = |\langle L_i, X\rangle|$, $i = 1,2,3$. There are shown projections of vertices, edges, and integer points lying on edges and in faces of the surface $\partial\mathbf{K}_{+++}$. In Fig. 7, the absolute value of the form $h(X)$ is written near every point X. Two-periodicity of Fig. 7 is obvious. Projection of the boundary of the fundamental domain in Fig. 7 is marked by bold lines. The fundamental domain consists of 8 triangles. 5 of them are at a distance 1 from the origin, and 3 are at a distance 2. These distances are written on projections of the faces in the fundamental domain. The triangles of the distance 2 do not have interior integer points, but they have a point on the edge with $|h| = 23$. 3 faces of the distance 1 do not have interior points, have a common vertex with $|h| = 9$, and together they comprise one triangle. One triangle of the distance 1 has one interior point with $|h| = 21$. Another triangle of the distance 1 has 4 interior integer points with $|h| = 33, 49$.

The faces of Klein polyhedra may be arbitrary polygons; not only triangles as in Example 3.1. A statistics of these polygons by the number of their edges was studied in [38].

In [16], there was proposed an algorithm for computing a Klein polyhedron together with its conjugate polyhedron; but this algorithm can not be considered as a generalization of the algorithm of continued fraction. It allows to compute only a part of the surface of one Klein polyhedron. Its another part can be reconstructed only in simple cases. And the computations must be repeated for other 7 Klein polyhedra.

Let three linear homogeneous forms (3.1) be given in $\mathbb{R}^3$. They determine the vector-function

$$M(X) = (m_1(X), m_2(X), m_3(X)),$$

where

$$m_i(X) = |l_i(X)|, \qquad i = 1, 2, 3.$$

The value $M(X)$ at an integer point $X \in \mathbb{Z}^3$ is called the *relative minimum*, if there is no integer point $Y \in \mathbb{Z}^3$, $Y \neq 0$ such that

$$M(Y) \leqslant M(X) \qquad \text{and} \qquad \|M(Y)\| < \|M(X)\|,$$

where $\|M\| = m_1 + m_2 + m_3$. Voronoi [66, Part III] considered points $X \in \mathbb{Z}^3$ in $\mathbb{R}^3$ that correspond to the relative minima of the forms (3.1). He suggested a means of partial ordering of these points and constructed an algorithm of motion by these ordered points. He called this algorithm a generalization of the algorithm of continued fraction (see also [24, Chap. IV]; [23]). The studies [66, Sect. 53]; [24, Sect. 59] have each only one example of computation by the Voronoi's algorithm.

4. Modular polyhedron and its global properties

4.1. General properties

Let three real independent homogeneous linear forms be given in $\mathbb{R}^3$

$$l_i(X) = \langle L_i, X\rangle, \quad i = 1, 2, 3, \qquad \det(L_1 L_2 L_3) \neq 0, \tag{4.1}$$

where $X \in \mathbb{R}^3$, $L_i = (l_{i1}, l_{i2}, l_{i3}) \in \mathbb{R}^3_*$, $\langle\,\cdot\,,\,\cdot\,\rangle$ means the scalar product, and the space $\mathbb{R}^3_*$ be dual to the space $\mathbb{R}^3$. Each collection $\Sigma = (\sigma_1, \sigma_2, \sigma_3)$, where $\sigma_i = \pm 1$, corresponds to its octant

$$\mathbf{O}_\Sigma = \{X : \sigma_i \langle L_i, X\rangle \geqslant 0,\ i = 1, 2, 3\},$$

bounded by the planes $\mathbf{L}_i = \{X : l_i(X) = 0\}$, $i = 1, 2, 3$. Three linear homogeneous forms (4.1) determine the mapping

$$M(X) = (m_1(X), m_2(X), m_3(X)),$$

where $m_i(X) = |l_i(X)|$, $i = 1, 2, 3$.

In [10], the following construction was proposed. The vector-function $M(X)$ maps the space $\mathbb{R}^3$ with coordinates X into the space $\mathbf{R}^3$ with coordinates $M = (m_1, m_2, m_3)$, more precisely, into its non-negative octant $\mathbf{R}^3_+$. Here the set $\mathbb{Z}^3 \setminus 0$ of all integer points X, except $X = 0$, is mapped into some set $\mathbf{Z}^3$ in $\mathbf{R}^3_+$. Let $\mathbf{M}$ be the convex hull of the points of the set $\mathbf{Z}^3$ and $\partial\mathbf{M}$ be its boundary. It is clear that $\mathbf{M}$ is a convex polyhedron, which is called the *modular polyhedron*, and $\partial\mathbf{M}$ is a convex polyhedral surface consisting of vertices V_i, edges R_i, and faces Γ_i.

Let $V_1, V_2, V_3 \in \mathbf{Z}^3$ and

$$V_j = M(B_j), \qquad B_j \in \mathbb{Z}^3, \qquad j = 1, 2, 3. \tag{4.2}$$

We assign $\omega(V_1, V_2, V_3) = |\det(B_1B_2B_3)|$. Obviously, ω takes integer non-negative values. For the face Γ_i of the surface $\partial\mathbf{M}$, we define $\omega(\Gamma_i)$ as the minimum of $\omega(V_1, V_2, V_3)$ over all triples of points $V_1, V_2, V_3 \in \mathbf{Z}^3$ lying in the face Γ_i and not lying on the same straight line, i.e.

$$V_1, V_2, V_3 \in \mathbf{Z}^3 \cap \Gamma_i \qquad \text{and} \qquad \det(V_1V_2V_3) \neq 0.$$

A face Γ_i of the surface $\partial\mathbf{M}$ is called *simple*, if it is a triangle with the vertices (4.2), and it does not contain other points of the set $\mathbf{Z}^3$; and it is called *semi-simple*, if it is a triangle containing inside exactly one point of the set $\mathbf{Z}^3$, and it has $\omega(\Gamma_i) = 1$. For a simple face Γ_i with vertices (4.2), we have $\omega(\Gamma_i) = |\det(B_1B_2B_3)|$.

Theorem 4.1. *For the faces* Γ_i *of the surface* $\partial\mathbf{M}$*, it is always* $\omega(\Gamma_i) \leqslant 2$.

Proof. The proof is analogous to the proof of corollary from Theorem X of Chap. V [22], where a statement similar to Theorem 4.1 was proved for strictly convex body $\mathcal{K}$. We consider a convex body $\tilde{S}$, which is not strictly convex. Let the face Γ_i have three points (4.2). By the definition at the beginning of Subsect. 2, Sect. 2, Chap. I [22], the value $\omega(V_1, V_2, V_3) = |\det(B_1B_2B_3)|$ is the *index* I of the points B_1, B_2, B_3. Below it is more convenient to consider points B_1, B_2, B_3 and their index $I = I(B_1, B_2, B_3)$, than the points V_1, V_2, V_3 and their value $\omega(V_1, V_2, V_3)$.

Let the plane N in $\mathbf{R}^3$ pass through the face Γ_i. The plane intersects with the first octant $\mathbf{R}^3_+ = \{M \geqslant 0\}$ by some triangle T and cuts off the tetrahedron S from the first octant. Their pre-images in the space $\mathbb{R}^3$ are the octahedron $\tilde{S}$ and its surface $\partial\tilde{S}$: $M(\tilde{S}) = S$, $M(\partial\tilde{S}) = T$. Inside the octahedron $\tilde{S}$, there are no points of the lattice $\mathbb{Z}^3$, except the origin. The points $\pm B_1, \pm B_2, \pm B_3$ lie on the boundary $\partial\tilde{S}$. By Theorem X, Chap. V [22] applied to the convex body $\mathcal{K} = \tilde{S}$, the index I of the points B_1, B_2, B_3 is not greater than 6. Let us consider the cases with various values of the index I descending from 6 to 2.

Let $I = 6$. According to Corollary 2 of Theorem I(A) Sect. 2, Chap. I [22], there exist then a basis C_1, C_2, C_3 of the lattice $\mathbb{Z}^3$ such that

$$\begin{aligned} B_1 &= a_{11}C_1, \\ B_2 &= a_{21}C_1 + a_{22}C_2, \\ B_3 &= a_{31}C_1 + a_{32}C_2 + a_{33}C_3, \end{aligned} \tag{4.3}$$

where all a_{ij} are integer, $0 \leqslant a_{ij} < a_{ii}$ for $j < i$, and $a_{11}a_{22}a_{33} = 6$.

Further, $a_{11} = 1$, since otherwise $(1/2)B_1$ or $(1/3)B_1 \in \mathbb{Z}^3$, but these points lie inside the octahedron $\tilde{S}$, which is impossible. If $a_{22} = 3$, then $a_{21} = 0, 1$ or 2. If $a_{21} = 0$, then $(1/3)B_2 \in \mathbb{Z}^3$ and lies inside $\tilde{S}$. If $a_{21} = 1$, then $(1/3)(B_2 - B_1) \in \mathbb{Z}^3$ and lies inside $\tilde{S}$. If $a_{12} = 2$, then $(1/3)(B_2 + B_1) \in \mathbb{Z}^3$ and lies inside $\tilde{S}$. Consequently, these cases are impossible. If $a_{22} = 2$, then $a_{21} = 0$ or 1. If $a_{21} = 0$, then $(1/2)B_2 \in \mathbb{Z}^3$ and lies inside $\tilde{S}$. If $a_{21} = 1$, then both points $(1/2)(B_2 \pm B_1) \in \mathbb{Z}^3$ and one of them lies inside $\tilde{S}$, which is impossible. If $a_{22} = 6$, then $a_{21} \neq 0$ is not multiple of 2 or 3. Hence $a_{21} = 1$ or 5. In these cases $(B_2 \mp B_1)/6 \in \mathbb{Z}^3$ and lies inside $\tilde{S}$. There remains the case $a_{22} = 1$. Then $a_{21} = 0$, $a_{33} = 6$, and $0 \leqslant a_{31}, a_{32} < 6$. If both numbers a_{31}, a_{32} are even, then $(1/2)B_3 \in \mathbb{Z}^3$ and lies inside $\tilde{S}$, which is impossible. If only one of these numbers is even, say a_{31}, then both points $(1/2)(B_3 \pm B_2) \in \mathbb{Z}^3$, and one of them lies inside $\tilde{S}$, which is impossible. Let both numbers a_{31}, a_{32} now be odd. If both of them are equal to 3, then $(1/3)B_3 \in \mathbb{Z}^3$ and lies inside $\tilde{S}$. But if one of them is equal to 3, say $a_{31} = 3$, and $a_{32} = 1$, then $(1/3)(B_3 - B_2) \in \mathbb{Z}^3$ and lies inside $\tilde{S}$; and if $a_{32} = 5$, then $(1/3)(B_3 + B_2) \in \mathbb{Z}^3$ and lies inside $\tilde{S}$. If none of the odd numbers a_{31}, a_{32} is equal to 3, then one of the points $(1/6)(B_3 \pm B_2 \pm B_1) \in \mathbb{Z}^3$ and lies inside $\tilde{S}$, which is impossible. So, the case $I = 6$ is impossible.

Let $I = 5$. Then there exist integer numbers $\alpha_1, \alpha_2, \alpha_3$, not divisible by 5 simultaneously, such that

$$D = \frac{1}{5}(\alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3) \in \mathbb{Z}^3. \tag{4.4}$$

We can assume that α_1 is not divisible by 5, and, taking the point $2D$ instead of D if necessary, we may assume that

$$\alpha_1 \equiv \pm 1 \pmod 5.$$

Hence, without loss of generality, we can add to D integers multiple to B_1, B_2, B_3, and may assume that

$$\alpha_1 = \pm 1, \qquad |\alpha_2| \leqslant 2, \qquad |\alpha_3| \leqslant 2.$$

Then the point (4.4) lies inside the octahedron $\tilde{S}$ or in its surface $\partial\tilde{S}$. In the latter case, the points D, B_2, B_3 have index 1. So, if $I = 5$, then there are points with index 1 in the surface $\partial\tilde{S}$.

Let $I = 4$. Then there is a representation (4.3), where $a_{11}a_{22}a_{33} = 4$. Further, $a_{11} = 1$, since otherwise $(1/2)B_1 \in \mathbb{Z}^3$ and lies inside $\tilde{S}$. If $a_{22} \neq 1$, then either $(1/2)B_2$ or both points $(1/2)(B_2 \pm B_1)$ lie in $\mathbb{Z}^3$, and one of them lies inside $\tilde{S}$, which is impossible. If $a_{22} = 1$, then $a_{21} = 0$, $a_{33} = 4$, $0 \leqslant a_{31}, a_{32} < 4$. If both numbers a_{31}, a_{32} are even, then $(1/2)B_3 \in \mathbb{Z}^3$ and lies inside $\tilde{S}$. If only one of them is even, say a_{31}, then both points $(1/2)(B_3 \pm B_2) \in \mathbb{Z}^3$ and one of them lies inside $\tilde{S}$. If both numbers a_{31}, a_{32} are odd, then one of the points $(1/4)(B_3 \pm B_1 \pm B_2)$ lies in $\mathbb{Z}^3$, and all these point lie inside $\tilde{S}$. So, the case $I = 4$ is impossible.

The case $I = 3$ is considered in similar way as the case $I = 5$. In this case there exists a triple of integer points in the surface $\partial\tilde{S}$ with index 1.

Let $I = 2$. Then there is the representation (4.3) with $a_{11}a_{22}a_{33} = 2$. Similarly to the case $I = 4$, we can prove that

$$a_{11} = a_{22} = 1, \qquad a_{33} = 2, \qquad a_{21} = 0, \qquad a_{31} = a_{32} = 1. \tag{4.5}$$

Consequently, the four points

$$(1/2)(B_3 \pm B_1 \pm B_2) \in \mathbb{Z}^3, \tag{4.5$'$}$$

but they lie neither inside $\tilde{S}$ nor in $\partial\tilde{S}$. Since $\omega(\Gamma_i)$ is the smallest value of the index, then $\omega(\Gamma_i) \leqslant 2$. Theorem is proved. ■

Theorem 4.2. *The face Γ_i with $\omega(\Gamma_i) = 2$ is a simple triangle.*

Proof. The proof is by contradiction. Suppose that the face Γ_i has more than three points

$$V_i = M(B_i), \qquad B_i \in \mathbb{Z}^3 \cap \partial\tilde{S}, \qquad i = 1, 2, \ldots, m, \qquad m \geqslant 4. \tag{4.6}$$

By definition of $\omega(\Gamma_i)$, among them there are three points V_j for which the determinant from their pre-images B_j is equal to ± 2. Let them be the first three points (4.6). Then there exists a basis C_1, C_2, C_3 of the lattice $\mathbb{Z}^3$ such that the expansion (4.3) with the property (4.5) is valid. Let the expansion

$$B_4 = a_{41}C_1 + a_{42}C_2 + a_{43}C_3 \tag{4.7}$$

with integer a_{4j} exist for the point B_4. If all a_{41}, a_{42}, a_{43} are even, then the point $(1/2)B_4 \in \mathbb{Z}^3$ and lies inside the octahedron $\tilde{S}$, i.e. it is impossible. If a_{43} and one of the numbers a_{41} or a_{42} are even, say a_{41}, then the points $(1/2)(B_4 \pm B_2)$ lie in $\mathbb{Z}^3$, and at least one of them lies inside $\tilde{S}$. If a_{43} is even, and both numbers a_{41}, a_{42} are odd, then the points $(1/2)(B_4 \pm B_3)$ lie in $\mathbb{Z}^3$, and at least one of them lies inside $\tilde{S}$. Let a_{43} be odd, then $a_{4i} = k_i a_{43} + l_i$, $|l_i| < |a_{43}|/2$, $i = 1, 2$, where k_i and l_i are integers. Then the point $\tilde{B}_5 = (B_4 - l_1 B_1 - l_2 B_2)/|a_{43}|$ lies inside or in the boundary of the octahedron $\tilde{S}$, since $1 + |l_1| + |l_2| \leqslant |a_{43}|$ and have the expansion $\tilde{B}_5 = k_1C_1 + k_2C_2 + C_3$. But $\det(B_1B_2\tilde{B}_5) = 1$, which contradicts the condition $\omega(\Gamma_i) = 2$. So, the existence of four points (4.6) in the face Γ_i is impossible. Consequently, the face Γ_i contains only three points $V_i \in \mathbf{Z}^3$, and it is a simple triangle. The proof is complete. ■

Theorem 4.3. *If the points* (4.2) *lie in one face* Γ_i *and* $\omega(V_1, V_2, V_3) = 0$, *then one of the points* B_1, B_2, B_3 *is the sum of two others.*

This statement follows from Theorem XI(A), Chap. V [22]. In particular, if the face Γ_i is simple, and $\omega(\Gamma_i) = 0$, then the points (4.2) are its vertices, and one of their pre-images is the sum of two others.

Example 4.1. Fig. 8, taken from [18], in coordinates

$$n_1 = \log m_1, \qquad n_2 = \log m_2,$$

shows the logarithmic projection of the surface $\partial\mathbf{M}$ for the form $h_4 = g_4 = h$ from Example 3.1, Klein polyhedron for which is shown in Fig. 7. Fig. 8 shows projections of the vertices and edges as well as the points of relative minima. For each point $Y = M(X)$, there given the value of the modulus of the form $h(X)$, and the value of the vector X. Bold lines mark the projection of the boundary of the fundamental domain, where for each face Γ_i, there given the value of $\omega(\Gamma_i)$. The fundamental domain consists of 8 triangles; two of them have $\omega = 0$, and six have $\omega = 1$. In the fundamental domain, there is one point of relative minimum with $|h| = 21$ corresponding to the point lying at the center of the middle triangle face in Fig. 7. Consequently, not all relative minima correspond to vertices of Klein polyhedra unlike the two-dimensional case. There is one vertex in the fundamental domain with $|h| = 9$. At other vertices, $|h| = 5$. Obviously, all vertices of the polyhedron $\partial\mathbf{M}$ correspond to the vertices of Klein polyhedra, but the contrary is not true.

4.2. Points from one octant $\mathbf{O}_\Sigma$

Lemma 4.1. *If the three points* (4.2) *lie in the surface* $\partial\mathbf{M}$ *and their pre-images* B_i *lie in one octant* $\mathbf{O}_\Sigma$, *then the points* V_i *do not lie on the same straight line.*

Proof. The proof is by contradiction. Let the three points (4.2) lie on the same straight line in $\mathbf{R}^3$, then they lie in some face Γ_j or an edge R_j, i.e. also in a face. Since their pre-images B_i are from the same octant $\mathbf{O}_\Sigma$, then the pre-images also lie on the same straight line, since every octant $\mathbf{O}_\Sigma$ is mapped into $\mathbf{R}^3_+$ by its linear transformation. Hence $\omega(V_1, V_2, V_3) = 0$. By Theorem 4.3, then, one of the points B_1, B_2, B_3 is the sum of two others. Since they are from the same octant, then the same is true for points V_1, V_2, V_3. But for every two points of the convex surface $\partial\mathbf{M}$, their sum lies inside of the set $\mathbf{M}$. Consequently, all three points V_1, V_2, V_3 cannot lie in the surface $\partial\mathbf{M}$. This contradiction completes the proof of Lemma. ■

Let us consider now convex polygons in $\mathbb{R}^2$ with vertices in integer points $\mathbb{Z}^2$ modulo integer translations and unimodular transformations. Obviously, a convex polygon Δ is the convex hull of its vertices. Two sets are *equivalent* if a linear unimodular change of coordinates transforms one into another.

Lemma 4.2. *If a closed convex polygon Δ with vertices in $\mathbb{Z}^2$ does not have three integer points lying on the same straight line, then it is equivalent either to*

a) *a simple triangle with the vertices* $(0,0),(1,0),(0,1)$; *or*
b) *a semi-simple triangle with the vertices* $(0,0),(1,0),(2,3)$ *containing inside one integer point* $(1,1)$; *or*
c) *a square with the vertices* $(0,0),(1,0),(0,1),(1,1)$.

Proof. Since the polygon Δ does not contain three integer points on the same straight line, then any side of Δ does not contain other integer points except vertices. Hence we can assume that one of its sides is the segment $x_1 \in [0,1]$, $x_2 = 0$, and $x_2 \geqslant 0$ for the polygon Δ.

We begin with the triangle. Let its third vertex have integer coordinates (m,n), where $n > 0$. If $n = 1$, then this triangle is equivalent to the triangle a) (Fig. 9). If $n > 1$, then it is sufficient to consider the case $0 \leqslant m < n$. But for $m = 0$ and $m = 1$, there is a vertical side that contains more than two points of the lattice. Hence it is sufficient to assume that

$$1 < m < n, \tag{4.8}$$

i.e. $n \geqslant 3$. For $n = 3$, we have a unique case $m = 2$. This is triangle b) (Fig. 10).

For $n \geqslant 4$, every triangle with the property (4.8) contains inside the point $(1,1)$ (Fig. 11). If $m \leqslant n/2$, then the triangle Δ contains also the point $(1,2)$. In this case, it contains three points $(1,0)$, $(1,1)$, and $(1,2)$ lying on the same straight line. If $m - 1 \geqslant n/2$, then the triangle Δ contains also the point $(2,2)$, and three points $(0,0)$, $(1,1)$, and $(2,2)$ of Δ lie on the same straight line. Let us consider the remaining case

$$m > n/2 > m - 1.$$

If n is even, then $n/2$ is integer, and these inequalities have no integer solution. If n is odd, $n = 2l + 1$ with integer l, then a unique integer solution is $m = l + 1$. Hence, $m - 1 = l$, and $n - 1 = 2l$; i.e. straight line passing through points $(1,1)$ and (m,n) contains $l + 1$ integer points belonging to Δ. Since $n \geqslant 4$, then $l \geqslant 2$, and $l + 1 \geqslant 3$. Consequently, the triangle can only be of the type a) or b).

Let now the polygon Δ be not triangle, then it includes two different triangles Δ_1 and Δ_2 with common base $x_1 \in [0,1]$, $x_2 = 0$. As it was proved, the triangle Δ_1 can be one of the two triangles a) or b). Let Δ_1 be the triangle a). Let the third vertex of the triangle Δ_2 be (m,n), $n > 0$. If $n = 1$, then $m = \pm 1$; and we obtain the square c) (Fig. 12). If $n > 1$ and $n \neq 3$, then, as we proved, the triangle Δ_2 contains three points of the lattice on the same straight line. Let $n = 3$ and $m = 2 + 3k$, where $k \in \mathbb{Z}$, $k \neq -1$. Then the triangle Δ_2 suits us, but the quadrangle with the vertices $(0,0),(1,0),(0,1),(2+3k,3)$ contains three points $(0,1),(1,1),(2,1)$ on the same straight line if $k \geqslant 1$; three points $(-2,1),(-1,1),(0,1)$ if $k \leqslant 3$, and the points $(0,1),(1+3k/2,2),(2+3k,3)$ for $k=0$ and $k=-2$ (Fig. 13, $k=0$). If both triangles Δ_1 and Δ_2 are of the type b), then the segment connecting their vertices with $n=3$ contains four integer points. The proof is complete. ■

Let the face Γ_i of the surface $\partial\mathbf{M}$ contain several (more than two) points $V_i \in \mathbf{Z}^3$ the pre-images B_i of which are from the same octant $\mathbf{O}_\Sigma$. Then all points B_i lie in the same plane N. According to Lemmas 4.1 and 4.2, there may be three or four of points B_i, and their convex hull belongs to one of three cases a)–c) of Lemma 4.2. Let three vectors $C_1, C_2, C_3 \in N \cap \mathbb{Z}^3$ be such that the differences $C_1 - C_3$ and $C_2 - C_3$ form a basis in $N \cap \mathbb{Z}^3$; $\rho = |\det(C_1C_2C_3)|$ is called the *distance* of the plane N from the origin. If N does not contain the origin, then $\rho = \min|\det(C_1C_2C_3)|$ for any $C_1, C_2, C_3 \in N \cap \mathbb{Z}^3$ with $\det(C_1C_2C_3) \neq 0$.

Lemma 4.3. *If three points $B_1, B_2, B_3 \in \mathbb{Z}^3$ lie in the same octant $\mathbf{O}_\Sigma$, and in their plane N, they are vertices of a triangle, and their images $V_i = M(B_i)$, $i=1,2,3$ lie in the same face of the surface $\partial\mathbf{M}$, then $\rho(N) \leqslant 2$.*

Proof. We may assume that all points B_i, $i=1,2,3$ lie in the plane $x_3 = \rho$, and the differences $B_i - B_3$, $i=1,2$ form a basis there, i.e. they have coordinates $B_1 = (1+m,n,\rho)$, $B_2 = (m,n+1,\rho)$, $B_3 = (m,n,\rho)$. Then $\det(B_1B_2B_3) = \rho$. In this case $|\det(B_1B_2B_3)| = \omega(V_1,V_2,V_3)$, i.e. $\rho = \omega(V_1,V_2,V_3)$. But $\omega(V_1,V_2,V_3) \leqslant 2$ according to Theorem 4.1. Consequently, $\rho \leqslant 2$. The proof is complete. ■

Example 4.2. In [18, Sect. 10], there described the polyhedral surface $\partial\mathbf{M}$ for the extremal cubic form $h_7(X) = \langle L_1, X\rangle\langle L_2, X\rangle\langle L_3, X\rangle$ from [64], where

$$L_i = (1, \lambda_i^2, \lambda_i^2 - 2\lambda_i), \qquad i = 1,2,3 \tag{4.8$'$}$$

and λ_i are roots $\lambda_1 \approx -0.8019377$, $\lambda_2 \approx 0.5549581$, $\lambda_3 \approx 2.2469796$ of the cubic equation $\lambda^3 - 2\lambda^2 - \lambda + 1 = 0$, i.e.

$$\begin{aligned} L_1 &\approx (1, 0.6431041, 2.2469796), \\ L_2 &\approx (1, 0.3079785, -0.8019377), \\ L_3 &\approx (1, 5.0489173, 0.5549581). \end{aligned}$$

See also [51, 52]. Fig. 14, corresponding to Fig. 9 [18], shows the logarithmic projection of the surface $\partial\mathbf{M}$ in coordinates $n_1 = \log|l_1(X)|$, $n_2 = \log|l_2(X)|$. There we see that the triangle, where pre-images of its vertices are $B_1 = (1,0,0)$, $B_2 = (0,1,0)$, $B_3 = (1,0,1)$, is the face of the surface $\partial\mathbf{M}$, and these points lie in the same octant. It is obvious that for them $\omega = \rho = 1$. Consequently, a triangle of the type a) with $\rho = 1$ is possible.

Example 4.3. Let us demonstrate that there exist the forms (4.1) for which the triangle of the type a) has $\rho = 2$. Let our points be

$$B_1 = (1,0,0), \qquad B_2 = (0,1,0), \qquad B_3 = (-1,-1,2). \tag{4.9}$$

It is obvious that for them $\rho = 2$. Consider unperturbed forms (4.1)

$$l_1(X) = 2x_1 + x_3, \qquad l_2(X) = 2x_2 + x_3, \qquad l_3(X) = x_3. \tag{4.10}$$

Then

$$M(B_1) = (2,0,0), \qquad M(B_2) = (0,2,0), \qquad M(B_3) = (0,0,2).$$

These three points lie in the plane with the normal vector $P = (1,1,1)$, and $\langle P, M(B_i)\rangle = 2$. For remaining points $X \in \mathbb{Z}^3$, $X \neq 0$, $X \neq -B_i$, we have $\langle P, M(X)\rangle \geqslant 3$. This is easily verified for $|x_3| = 0, 1, 2,$; we need not go any further, since $m_3 = |x_3|$, i.e. $\langle P, M(X)\rangle \geqslant |x_3| \geqslant 3$.

Now we perturb the forms (4.10) in a way such that all points B_1, B_2, B_3 fall into the positive octant $\mathbf{O}_{+++}$. Namely, we assign

$$\begin{aligned} \tilde{l}_1(X) &= l_1(X) - \alpha_1 x_1 + \beta_1 x_2, \\ \tilde{l}_2(X) &= l_2(X) + \alpha_2 x_1 - \beta_2 x_2, \\ \tilde{l}_3(X) &= l_3(X) + \alpha_3 x_1 + \beta_3 x_2, \end{aligned} \tag{4.11}$$

where α_i and β_i are small positive numbers. The values $\tilde{L}(B_i) \overset{\text{def}}{=} (\tilde{l}_1(X), \tilde{l}_2(X), \tilde{l}_3(X))$ are

$$\begin{aligned} \tilde{L}(B_1) &= (2 - \alpha_1, \alpha_2, \alpha_3), \\ \tilde{L}(B_2) &= (\beta_1, 2 - \beta_2, \beta_3), \\ \tilde{L}(B_3) &= (\alpha_1 - \beta_1, \beta_2 - \alpha_2, 2 - \alpha_3 - \beta_3). \end{aligned} \tag{4.12}$$

The positiveness of the values (4.12) of the forms (4.11) mean that

$$\begin{array}{rrr} 2 - \alpha_1 > 0, & \alpha_2 > 0, & \alpha_3 > 0, \\ \beta_1 > 0, & 2 - \beta_2 > 0, & \beta_3 > 0, \\ \alpha_1 - \beta_1 > 0, & \beta_2 - \alpha_2 > 0, & 2 - \alpha_3 - \beta_3 > 0, \end{array}$$

i.e.

$$\begin{aligned} &2 > \alpha_1 > \beta_1 > 0; \\ &2 > \beta_2 > \alpha_2 > 0; \\ &2 > \alpha_3 + \beta_3 > 0; \qquad \alpha_3 > 0; \qquad \beta_3 > 0. \end{aligned} \tag{4.13}$$

The points (4.12) lie in the plane with the normal vector $\tilde{P} = P + (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, and $\langle \tilde{P}, \tilde{L}(B_i)\rangle = 2 + \varepsilon$, where ε_i and ε are small numbers. For small α_i, β_i, the scalar product $\langle \tilde{P}, \tilde{M}(X)\rangle > 2.5$ for all $X \in \mathbb{Z}^3$, $X \neq 0$, $X \neq \pm B_i$, $i = 1, 2, 3$. Here $\tilde{M} = (|\tilde{l}_1|, |\tilde{l}_2|, |\tilde{l}_3|)$.

Lemma 4.4. *If three points $B_1, B_2, B_3 \in \mathbb{Z}^3$ lie in the same octant $\mathbf{O}_\Sigma$, and in their plane N, they are vertices of the triangle of the type b), and their images $M(B_i)$ lie in the same face Γ_i of the surface $\partial\mathbf{M}$, then $\rho(N) = \omega(\Gamma_i) = 1$.*

Proof. We may assume that the plane N is given by the equation $x_3 = \rho$. Since the area of the triangle of the type b) is equal to $3/2$, then $|\det(B_1 B_2 B_3)| = 3\rho$. In the proof of Theorem 4.1, we obtained that the index of pre-images of the three points lying in the face Γ_i is no greater than 5, i.e. $3\rho \leqslant 5$. Since ρ is an integer, then $\rho \leqslant 1$. The proof is complete. ■

Example 4.4. Let us demonstrate that a triangle of the type b) with $\rho = 1$ is possible. Consider four points

$$B_1 = (1, 0, 0), \quad B_2 = (0, 1, 0), \quad B_3 = (-1, -1, 3), \quad B_4 = (0, 0, 1). \tag{4.14}$$

Obviously, $B_4 = (1/3)(B_1 + B_2 + B_3)$, and all points lie in the plane $N = \{x_1 + x_2 + x_3 = 1\}$ of the distance $\rho = 1$. The following construction is analogous to Example 4.3. First, we consider the unperturbed case

$$l_1(X) = 3x_1 + x_3, \qquad l_2(X) = 3x_2 + x_3, \qquad l_3(X) = x_3. \tag{4.15}$$

Then

$$\begin{aligned} M(B_1) &= (3, 0, 0), \qquad & M(B_2) &= (0, 3, 0), \\ M(B_3) &= (0, 0, 3), \qquad & M(B_4) &= (1, 1, 1). \end{aligned} \tag{4.16}$$

The plane passing through these points have the normal vector $P = (1, 1, 1)$, and $\langle P, M(B_i)\rangle = 3$. Now we note that for each point $X \in \mathbb{Z}^3$, $X \neq 0$, the values $l_i(X)$ are integer, and the values $\langle P, M(X)\rangle$ are natural numbers. If the point X differs from the points $\pm B_i$, $i = 1, 2, 3, 4$, then $\langle P, M(X)\rangle \geqslant 4$. This is easily verified for $|x_3| = 0, 1, 2, 3$, and we need not go further, since $m_3 = |x_3|$, i.e. $\langle P, M(X)\rangle \geqslant |x_3| \geqslant 4$.

Now we perturb the forms (4.15) in such a way that all points (4.14) fall into the positive octant. Namely, we take the forms

$$\begin{aligned} \tilde{l}_1(X) &= (3 - \alpha_1)x_1 + \beta_1 x_2 + x_3, \\ \tilde{l}_2(X) &= \alpha_2 x_1 + (3 - \beta_2)x_2 + x_3, \\ \tilde{l}_3(X) &= \alpha_3 x_1 + \beta_3 x_2 + x_3, \end{aligned} \tag{4.17}$$

where α_i, β_i are small positive numbers. The values $\tilde{L}(X) \overset{\text{def}}{=} (\tilde{l}_1(X), \tilde{l}_2(X), \tilde{l}_3(X))$ for the forms (4.17) at the points (1.14) are

$$\begin{aligned} \tilde{L}(B_1) &= (3 - \alpha_1, \alpha_2, \alpha_3), \\ \tilde{L}(B_2) &= (\beta_1, 3 - \beta_2, \beta_3), \\ \tilde{L}(B_3) &= (\alpha_1 - \beta_1, \beta_2 - \alpha_2, 3 - \alpha_3 - \beta_3), \\ \tilde{\Lambda}(B_4) &= (1, 1, 1). \end{aligned} \tag{4.18}$$

The positiveness of the values (4.18) mean the inequalities

$$\begin{array}{rrr} 3 - \alpha_1 > 0, & \alpha_2 > 0, & \alpha_3 > 0, \\ \beta_1 > 0, & 3 - \beta_2 > 0, & \beta_3 > 0, \\ \alpha_1 - \beta_1 > 0, & \beta_2 - \alpha_2 > 0, & 3 - \alpha_3 - \beta_3 > 0, \end{array}$$

i.e.

$$\begin{aligned} &3 > \alpha_1 > \beta_1 > 0;\\ &3 > \beta_2 > \alpha_2 > 0;\\ &3 > \alpha_3 + \beta_3 > 0; \qquad \alpha_3 > 0; \qquad \beta_3 > 0. \end{aligned} \tag{4.19}$$

The points (4.18) lie in the same plane with the normal vector $\tilde{P} = P + (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, and $\langle \tilde{P}, \tilde{\Lambda}(B_i)\rangle = 3 + \varepsilon$, where ε_i and ε are small numbers. For small α_i, β_i, the scalar product $\langle \tilde{P}, \tilde{M}(X)\rangle > 3.5$ for all X, $X \neq 0$, $X \neq \pm B_i$, $i = 1, 2, 3, 4$.

Remark 4.1. The situations of Examples 4.3 and 4.4 did not occur for the forms (4.1), that we computed in [18] with $(L_1 L_2 L_3) = SW$, where S is a nonsingular rational matrix, and W is the Vandermonde matrix of the cubic polynomial $\lambda^3 + a\lambda^2 + b\lambda + c$ with integer coefficients. Although these situations are generic.

Lemma 4.5. *The situation when four points $B_1, B_2, B_3, B_4 \in \mathbb{Z}^3$ lie in the same octant $\mathbf{O}_\Sigma$, and when in the same plane N, they are vertices of a quadrangle of the type c), and their images $M(B_i)$, $i = 1, 2, 3, 4$ lie in the same face Γ_i of the surface $\partial\mathbf{M}$, is impossible.*

Proof. Using a unimodular change of basis, we introduce a coordinate system such that the plane N is given by the equation $x_3 = \rho$, and the points B_i are these

$$B_1 = (k, l, \rho), \quad B_2 = (k+1, l, \rho), \quad B_3 = (k, l+1, \rho), \quad B_4 = (k+1, l+1, \rho), \tag{4.20}$$

where k and l are integers. Obviously, in (4.20), using a unimodular change of coordinates, we can always achieve inequalities

$$0 \leqslant k \leqslant l < \rho.$$

Now we note that the convex cone

$$X = \sum_{i=1}^{4} \mu_i B_i, \qquad \mu_i > 0$$

contains the integer point

$$(k, l, \rho - 1).$$

in the intersection with the plane $x_3 = \rho - 1$. Indeed, this intersection is the square

$$\frac{k(\rho-1)}{\rho} \leqslant x_1 \leqslant \frac{(k+1)(\rho-1)}{\rho}, \qquad \frac{l(\rho-1)}{\rho} \leqslant x_2 \leqslant \frac{(l+1)(\rho-1)}{\rho},$$

and

$$\frac{k(\rho-1)}{\rho} < k \leqslant \frac{(k+1)(\rho-1)}{\rho},$$

since $0 \leqslant \rho - k - 1$. Similarly for l. Consequently, the situation described in Lemma 4.5 is impossible for $\rho \geqslant 2$.

Consider this situation for $\rho = 1$. The proof is by contradiction. Let the points B_i be

$$B_1 = (1,0,0), \qquad B_2 = (0,1,0), \qquad B_3 = (0,0,1), \qquad B_4 = (1,1,-1), \quad (4.21)$$

and the forms (4.1) be

$$l_i(X) = x_1 + \alpha_i x_2 + \beta_i x_3, \qquad i = 1,2,3. \quad (4.22)$$

Then the points B_i lie in the plane $N = \{x_1 + x_2 + x_3 = 1\}$ of the distance 1, and they form in it a square of the type c). Since at the point B_1, all forms $l_i(X)$ are positive, then they must be positive at the remaining three points (4.21) as well. Consequently,

$$\alpha_i > 0, \qquad \beta_i > 0, \qquad 1 + \alpha_i > \beta_i, \qquad i = 1,2,3. \quad (4.23)$$

Here

$$B_1 + B_2 = B_3 + B_4. \quad (4.24)$$

Let $M_i = M(B_i)$, $i = 1,2,3,4$. According to (4.21), (4.22), we have

$$\begin{aligned} &M_1 = (1,1,1), \qquad M_2 = (\alpha_1, \alpha_2, \alpha_3), \qquad M_3 = (\beta_1, \beta_2, \beta_3), \\ &M_4 = (1 + \alpha_1 - \beta_1, 1 + \alpha_2 - \beta_2, 1 + \alpha_3 - \beta_3) \stackrel{\text{def}}{=} (\gamma_1, \gamma_2, \gamma_3). \end{aligned} \quad (4.25)$$

Let the vector $P = (p_1, p_2, p_3) > 0$ with $p_1 + p_2 + p_3 = 1$ be normal to the plane W where the points (4.25) lie. Then

$$\begin{aligned} \langle P, M_1 \rangle &= p_1 + p_2 + p_3 = 1, \\ \langle P, M_2 \rangle &= p_1\alpha_1 + p_2\alpha_2 + p_3\alpha_3 = 1, \\ \langle P, M_3 \rangle &= p_1\beta_1 + p_2\beta_2 + p_3\beta_3 = 1, \\ \langle P, M_4 \rangle &= p_1\gamma_1 + p_2\gamma_2 + p_3\gamma_3 = 1, \\ 1 + \alpha_i &= \beta_i + \gamma_i; \qquad p_i, \alpha_i, \beta_i, \gamma_i > 0, \qquad i = 1,2,3. \end{aligned} \quad ■$$

Remark 4.2. If $p_1 + p_2 + p_3 = 1$, and $p_1, p_2, p_3 > 0$, then

$$\min_i p_i < 1/3 < \max_i p_i$$

and among p_i, no more than one $p_i > 1/2$.

In each of the four sums $\langle P, M_i \rangle$ in (4.26), we mark the addend that is greater than one half. Let us consider different cases.

Case 1. There is the marked addend in each sum (4.26). Then there are at least two sums where these addends have the same index i. Let it be for definiteness $\langle P, M_1 \rangle$, and $\langle P, M_2 \rangle$, and the marked addends have index $i = 1$. Let $\alpha_1 > 1$. Let us consider subcases.

Subcase $\alpha_2 > 1$, $\alpha_3 < 1$. According to (4.26), we have

$$\begin{aligned} M(B_2 - B_1) &= (\alpha_1 - 1)p_1 + (\alpha_2 - 1)p_2 + (1 - \alpha_3)p_3 \\ &= \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 - (p_1 + p_2 + p_3) - 2\alpha_3 p_3 + 2p_3 \\ &= 2p_3(1 - \alpha_3). \end{aligned} \tag{4.27}$$

Since $1 > \alpha_3 > 0$, then $0 < 1 - \alpha_3 < 1$ and $p_3 < 1/2$ by the Remark 1.2. Consequently,

$$2p_3(1 - \alpha_3) < 1,$$

i.e. $M(B_2 - B_1) < 1$, and the plane $\langle P, M\rangle = 1$ is not supporting for the set $\mathbf{M}$. We came to contradiction.

Subcase $\alpha_2, \alpha_3 < 1$. According to (4.26), we have

$$\begin{aligned} M(B_2 - B_1) &= (\alpha_1 - 1)p_1 + (1 - \alpha_2)p_2 + (1 - \alpha_3)p_3 \\ &= M(B_1) - M(B_2) + 2(\alpha_1 p_1 - p_1) = 2p_1(\alpha_1 - 1). \end{aligned} \tag{4.28}$$

Since p_1 and $\alpha_1 p_1$ are marked addends, then $p_1 > 1/2$ and $\alpha_1 p_1 > 1/2$. Consequently, $\alpha_1 p_1 - p_1 < 1/2$ and

$$M(B_2 - B_1) < 1.$$

Again we obtain that the plane $\langle P, M\rangle = 1$ is not supporting for the set $\mathbf{Z}^3$, i.e. we have contradiction with the supposition.

Remaining subcases of this case are obtained by reversal of the inequalities between 1 and α_i and interchanging indices.

Case 2. Among the four sums (4.26), there are at least two without the marked addends, i.e. all addends in these sums are less than one half. Let them be the first two sums, and $\alpha_1, \alpha_2 > 1$, $\alpha_3 < 1$. We apply the formula (4.27): $M(B_2 - B_1) = 2(p_3 - \alpha_3 p_3)$. Since $p_3 < 1/2$, and $\alpha_3 p_3 < 1/2$, then $p_3 - \alpha_3 p_3 < 1/2$, and $M(B_2 - B_1) < 1$. We came to contradiction.

Remaining subcases of this case are obtained by reversal of the inequalities between 1 and α_i and interchanging indices.

Case 3. Only one of the sums (4.26) does not have the marked addend, but every other of the three sums (4.26) do have the marked addend, and all these marked addends have different indices. We assume that the marked addend is absent in the first sum, and in the sum $\langle P, M_i\rangle$ with $i > 1$, the addend with index $i - 1$ is marked.

Subcase $p_1 = \max\limits_{1 \leqslant i \leqslant 3} p_i$. Then $1/3 < p_1 < 1/2$ according to Remark 4.2. Since $p_1 + \alpha_1 p_1 = \beta_1 p_1 + \gamma_1 p_1$, and $\beta_1 p_1, \gamma_1 p_1 < 1/2$, then

$$\alpha_1 p_1 < 1 - p_1 < 2/3. \tag{4.29}$$

According to (4.28), $M(B_2 - B_1) = 2(\alpha_1 p_1 - p_1)$, but $p_1 > 1/3$, and according to (4.29), $\alpha_1 p_1 < 2/3$. Consequently, $\alpha_1 p_1 - p_1 < 1/3$, and $M(B_2 - B_1) < 2/3$. We came to contradiction.

Subcase $p_2 = \max_{1\leqslant i\leqslant 3} p_i$, i.e. $1/3 < p_2 < 1/2$. Since $p_2 + \alpha_2 p_2 = \beta_2 p_2 + \gamma_2 p_2$, and $\beta_2 p_2 > 1/2$, but $\alpha_2 p_2, \gamma_2 p_2 < 1/2$, then

$$\beta_2 p_2 - p_2 = \alpha_2 p_2 - \gamma_2 p_2 < 1/2.$$

Consequently, we have

$$M(B_2 - B_1) = 2(\beta_2 p_2 - p_2) < 1.$$

We again came to contradiction.

Remaining subcases of this case are obtained by interchanging coordinates p_i and vectors B_i. The proof of Lemma is complete.

Lemma 4.6. *If the points* (4.2) *lie in the same face* Γ_i, *and* $\omega(V_1, V_2, V_3) = 0$, *then the points* B_1, B_2, B_3 *do not lie in the same octant* $\mathbf{O}_\Sigma$.

Proof. This is an obvious corollary of Theorem 4.3. ∎

4.3. Generic case

The triple of forms (4.1) is *generic*, if its coefficients l_{ij} do not satisfy an algebraic equation with integer coefficients, which is homogeneous in components of each vector L_i. For instance, none of the relations l_{ij}/l_{ik} is a rational number.

We note without proof the following properties of the surface $\partial\mathbf{M}$ in generic case.

1. For every face Γ of the surface $\partial\mathbf{M}$, its exterior normal vector is strictly negative, i.e. all its components are negative.
2. For every edge R of the surface $\partial\mathbf{M}$, its directing vector has components of various signs.

Theorem 4.4. *If the forms* (4.1) *are generic, then no three points from* $\mathbf{Z}^3 \cap \partial\mathbf{M}$ *lie on the same straight line.*

Proof. The proof is by contradiction. Suppose that there are three points (4.2) from the set $\mathbf{Z}^3 \cap \partial\mathbf{M}$ that lie on the same straight line R. Three points

$$U = (u_0, u_1, u_2), \qquad V = (v_0, v_1, v_2), \qquad W = (w_0, w_1, w_2) \tag{4.30}$$

on the same straight line satisfy the conditions

$$\begin{vmatrix} u_i & v_i \\ u_{i+1} & v_{i+1} \end{vmatrix} + \begin{vmatrix} v_i & w_i \\ v_{i+1} & w_{i+1} \end{vmatrix} + \begin{vmatrix} w_i & u_i \\ w_{i+1} & u_{i+1} \end{vmatrix} = 0, \qquad i = 1, 2, 3, \tag{4.31}$$

where indices are taken modulo 3. If not all the points lie in the same octant $\mathbf{O}_\Sigma$, then conditions (4.31) give at least one nontrivial quadratic equation for coefficients of the forms (4.1). But that is not possible in generic case.

If all points B_i lie in the same octant, then conditions (4.31) mean that they lie on the same straight line. But according to Lemma 4.1, that is not possible. Theorem is proved. ∎

Theorem 4.5. *Generically, if one face* Γ_i *of the surface* $\partial\mathbf{M}$ *has more than three points of the set* $\mathbf{Z}^3$, *then this face* Γ_i *is a semi-simple triangle.*

Proof. The proof is by contradiction. Let four points from $\mathbf{Z}^3$

$$V_i = M(B_i), \qquad B_i \in \mathbb{Z}^3 \cap \pi\mathbb{R}^3, \qquad i = 1, 2, 3, 4 \tag{4.32}$$

lie in the same plane. Here $\pi\mathbb{R}^3$ means the semi-space $l_3(X) \geqslant 0$ of the space $\mathbb{R}^3$. Then

$$\det(V_1V_2V_3) + \det(V_2V_3V_4) + \det(V_3V_4V_1) + \det(V_4V_1V_2) = 0. \tag{4.33}$$

If not all points B_1, B_2, B_3, B_4 lie in the same octant $\mathbf{O}_\Sigma$, then the equation (4.33) has coefficients l_{ij} of the forms (4.1) in unavoidable way. And this equation for l_{ij} has integer coefficients, i.e. the forms (4.1) are not generic.

But if all 4 points B_1, B_2, B_3, B_4 are in the same octant $\mathbf{O}_\Sigma$, then according to Lemmas 4.2, 4.4, and 4.5, this is possible only if these points belong to a triangle of the type b) of Lemma 4.2 with $\omega = 1$. The proof is complete. ■

Corollary 4.1. *Generically, all faces* Γ_i *of the surface* $\partial\mathbf{M}$ *are simple with* $\omega(\Gamma_i) \leqslant 2$ *or semi-simple with* $\omega(\Gamma_i) = 1$.

A semi-simple face Γ_i is naturally split into three triangles. Each of these triangles has two vertices which are the vertices of the face Γ_i, and the third vertex is an interior point of the face Γ_i, and the point belongs to the set $\partial\mathbf{M} \cap \mathbf{Z}^3$. Consequently, a generic surface $\partial\mathbf{M}$ has a natural triangulation.

According to the definition of generic case given at the beginning of this Subsection, the cubic forms h_4, h_7 of Examples 4.1, 4.2, and the remaining cubic forms from [18] and [64] are not generic, since their coefficients are integers, and every coefficient of the cubic form

$$h(X) = \langle L_1, X\rangle\langle L_2, X\rangle\langle L_3, X\rangle \tag{4.34}$$

is a symmetric polynomial of third degree in components l_{ij} of vectors L_i. Consequently, equations of third degree with integer coefficients are satisfied for components l_{ij}. Therefore we can narrow the definition of generic case if we require only the absence of equations of the type (4.31) and (4.33). Thus defined, the cubic forms from [18] will become generic, but there are possible cubic forms of the type (4.34) with integer coefficients which are not generic even under this narrowed definition. The surfaces $\partial\mathbf{M}$ corresponding to them may have faces that are not triangles or may have only triangle faces that are not simple or semi-simple. Examples of surfaces $\partial\mathbf{M}$ of this type are considered in next Subsection.

4.4. Special cases

Here we consider some examples of singularities of construction of the polyhedron $\mathbf{M}$.

Example 4.5. Let us demonstrate that a polyhedral surface $\partial\mathbf{M}$ can have a face Γ with five points $V_i \in \mathbf{Z}^3$. Consider the forms (4.1) of the type

$$\begin{aligned} l_1(X) &= x_1 - 2x_2 + (1+\delta)x_3, \\ l_2(X) &= -2x_1 + x_2 + (1+\delta)x_3, \\ l_3(X) &= x_1 + x_2 + 2(1-\delta)x_3, \end{aligned} \tag{4.35}$$

where $0 < \delta < 1$. Let $L(X) = (l_1(X), l_2(X), l_3(X))$ and $M(X) = (|l_1(X)|, |l_2(X)|, |l_3(X)|)$. In Table 1, in the first column, we cite the number k of vectors B_k; in the second column, we cite the values of vectors B_k; the third column cites the corresponding values $L(B_k)$, the fourth column cites the values $M(B_k) = V_k$, and the fifth column cites scalar products $\langle P, M(B_k)\rangle \overset{\text{def}}{=} \Delta_k$ for $P = (1,1,1)$. Table 1 demonstrates that the points $V_1, \ldots, V_5$ are placed in the plane $\langle P, V\rangle = 4$. We also see that the points $V_6, \ldots, V_9$ are placed above that plane. It is easy to demonstrate that the remaining points from $\mathbf{Z}^3$ are placed above that plane. Consequently, this plane is supporting to the set $\mathbf{M}$, i.e. all points $V_1, \ldots, V_5$ lie in the same face Γ, which is their convex hull. Fig. 15 shows projections of points $V_1, \ldots, V_5$ and the face Γ on the plane (m_1, m_2) for $\delta = 1/4$ (a), $\delta = 1/2$ (b), and $\delta = 3/4$ (c). From the Fig. 15, we see that in cases (a) and (b), the face Γ is the triangle with the vertices V_1, V_2, V_5, but it contains also two points V_3 and V_4 either both inside (a) or one point V_3 in the boundary. In the case (c), the face Γ is the quadrangle with the vertices V_1, V_2, V_3, V_5 and an interior point V_4. We remark also that in all three cases, three points V_3, V_4, V_5 lie on the same straight line, and in the case (b), other three points V_1, V_2, V_3 also lie on the same straight line. Consequently, the forms (4.35) are strongly degenerate.

The following problems concerning the structure of $\partial\mathbf{M}$ remain yet open: what is the greatest number of points from $\mathbf{Z}^3$ that may be in the same face? What is the greatest number of vertices that the face Γ_i of the surface $\partial\mathbf{M}$ may have? The author's guess is that both these numbers are no greater than seven. But it is not essential for the algorithm proposed below in Section 6.

Table 1. Values $M(X)$ for the forms (4.35).

k	B_k	$L(B_k)$	$M(B_k) = V_k$	Δ_k
1	1, 0, 0	$1, -2, 1$	1, 2, 1	4
2	0, 1, 0	$-2, 1, 1$	2, 1, 1	4
3	0, 0, 1	$1+\delta, 1+\delta, 2(1-\delta)$	$1+\delta, 1+\delta, 2(1-\delta)$	4
4	1, 1, 0	$-1, -1, 2$	1, 1, 2	4
5	1, 1, 1	$\delta,\ \delta,\ 2(2-\delta)$	$\delta,\ \delta,\ 2(2-\delta)$	4
6	$1, -1, 0$	$3, -3, 0$	3, 3, 0	6
7	1, 0, 1	$2+\delta, -1+\delta, 3-2\delta$	$2+\delta, 1-\delta, 3-2\delta$	$6-2\delta$
8	2, 1, 1	$1+\delta, -2+\delta, 5-2\delta$	$1+\delta, 2-\delta, 5-2\delta$	$8-2\delta$
9	2, 2, 1	$-1+\delta, -1+\delta, 6-2\delta$	$1-\delta, 1-\delta, 6-2\delta$	$8-4\delta$

We note yet another singularity of the face Γ of this example. Since $\det(B_1 B_2 B_4) = 0$, so $\omega(\Gamma) = 0$. However, $\det(B_1 B_2 B_3) = 1$. But in the case (b) $V_3 = (1/2)(V_1 + V_2)$ and $\det(V_1 V_2 V_3) = 0$. Consequently, for the points $V_i, V_j, V_k \in \Gamma \cap \mathbf{Z}^3$ the function $\omega(V_i, V_j, V_k)$ takes two different values: zero and one.

Example 4.6 (continuation of Example 4.3)**.** Let us study the situation near a simple face Γ with $\omega(\Gamma) = 2$. As it was mentioned in Example 4.3, unperturbed forms (4.10) have a simple face Γ the vertices of which are the points $V_i = M(B_i)$, $i = 1, 2, 3$. $\langle P, V_i \rangle = 2$, $i = 1, 2, 3$ for $P = (1, 1, 1)$, and $\omega(\Gamma) = 2$. For other points $V \in \mathbf{Z}^3$, the scalar product $\langle P, V_i \rangle > 2$. However, there are only four points B_k with $l_3(B_k) \geqslant 0$ for which $\langle P, M(B_k) \rangle = 3$. These are points

$$B_4 = (0, 0, 1), \quad B_5 = (-1, 0, 1), \quad B_6 = (0, -1, 1), \quad B_7 = (-1, -1, 1). \tag{4.36}$$

For the points (4.9), the points (4.36) coincide with the points (4.5′). For all of them $M(B_k) = (1, 1, 1)$. However, for perturbed forms (4.11), all four points $\tilde{M}(B_k) = V_k$, $k = 4, 5, 6, 7$, in general, are different. Here for small arbitrary perturbations α_i, β_i (that may even not satisfy inequalities (4.13)), the values of the scalar product

$$\langle \tilde{P}, \tilde{M}(B_k) \rangle > \begin{cases} \approx 2 & \text{for } \ k = 1, 2, 3, \\ \approx 3 & \text{for } \ k = 4, 5, 6, 7, \\ > 3.5 & \text{for remaining points } B \in \mathbb{Z}^3 \text{ with } l_3(B) \geqslant 0. \end{cases}$$

The face Γ has three edges

$$R_1 = [V_1, V_2], \qquad R_2 = [V_2, V_3], \qquad R_3 = [V_3, V_1]. \tag{4.37}$$

Each point V_4, V_5, V_6, V_7 corresponds to the tetrahedron T_k with the base Γ and the vertex V_k, $k = 4, 5, 6, 7$. Each tetrahedron T_k has three faces Δ_{ik} spanned on edges R_i and the point V_k. From the type of points (4.9), (4.36) and edges (4.37), we obtain that

$$\omega(\Delta_{ik}) \stackrel{\text{def}}{=} |\det(B_i B_{i+1} B_k)| = 1,$$

where $i = 1, 2, 3$ and $B_{3+1} = B_1$, and $k = 4, 5, 6, 7$. Consequently, the point V_k, $k = 4, 5, 6, 7$ can be assigned to the corresponding polyhedron $\mathbf{M} \setminus T_k$, which is convex-concave. Here the face Γ with $\omega(\Gamma) = 2$ is replaced by three faces Δ_{ik} with $\omega(\Delta_{ik}) = 1$.

A similar situation takes place for an arbitrary surface $\partial \mathbf{M}$. Namely, each face Γ_j with $\omega(\Gamma_j) = 2$ can be replaced by three faces Δ_{ij} with $\omega(\Delta_{ij}) = 1$. As a result, we obtain a convex-concave polyhedral surface all the faces of which have $\omega \leqslant 1$. Generally speaking, for every face Γ_i with $\omega(\Gamma_i) = 2$, this change can be done in four ways. However, some of the points V_4, V_5, V_6, V_7 may coincide. Then this change can be done in less than four ways.

5. Local properties of modular polyhedron

5.1. Plane geometry

Lemma 5.1. *Let three points $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$ be given in the plane $\mathbb{R}^2$. We assign*

$$æ = |AB| + |BC| + |CA|, \tag{5.1}$$

where

$$|AB| = \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}.$$

If $æ = 0$, then the points A, B, C lie on the same straight line.

If $æ \neq 0$, then the points A, B, C are vertices of a triangle. When we circuit this triangle in positive direction, i.e. anticlockwise, these vertices are placed in direct succession A, B, C, if $æ < 0$, and in reverse succession A, C, B, if $æ > 0$.

Proof. The straight line $\mathcal{L}$ passing through the points A and B has the directing vector $L = (a_1 - b_1, a_2 - b_2)$, and the normal vector $D = (b_2 - a_2, a_1 - b_1)$. On the straight line $\mathcal{L}$

$$\langle D, A\rangle = \langle D, B\rangle = a_1(b_2 - a_2) + a_2(a_1 - b_1) = |AB|. \tag{5.2}$$

At the same time,

$$\langle D, C\rangle = c_1(b_2 - a_2) + c_2(a_1 - b_1) = |CB| - |CA|. \tag{5.3}$$

According to (5.1), (5.2), and (5.3),

$$æ = \langle D, A\rangle - \langle D, C\rangle.$$

Hence when $æ = 0$, the point C lies on the straight line $\mathcal{L}$, and when $æ \neq 0$, it lies outside this straight line.

For the vector D, we denote the angle between the positive semi-axis of abscissa and this vector as $\arg D$. Then

$$\arg D = \arg L + \pi/2.$$

If $æ < 0$, then the point C lies on the same side of the straight line $\mathcal{L}$ where the vector D is directed, i.e. the points A, B, C are placed in succession of positive circuit.

If $æ > 0$, then the point C lies on the side of the straight line $\mathcal{L}$ which is contrary to the vector D (Fig. 16). Consequently, the point A, B, C are placed in succession of their negative circuit, i.e. as the angle decreases with respect to the interior point of the triangle ABC. The proof is complete. ■

Let two real homogenous forms

$$l_i(X) = l_{i1}x_1 + l_{i2}x_2, \qquad l_{ij} \neq 0, \qquad i, j = 1, 2; \qquad \det(l_{ij}) \neq 0 \tag{5.4}$$

be given in the plane $\mathbb{R}^2$ with coordinates $X \stackrel{\text{def}}{=} (x_1, x_2)$. In the unit vectors $E_1 = (1,0)$ and $E_2 = (0,1)$, we have

$$l_i(E_j) = l_{ij}, \qquad i,j = 1,2.$$

We assign

$$m_i(X) = |l_i(X)|, \qquad i,j = 1,2; \qquad M(X) = (m_1(X), m_2(X))$$

and denote

$$M_j = M(E_j) = (|l_{1j}|, |l_{2j}|) \stackrel{\text{def}}{=} (m_{1j}, m_{2j}), \qquad j = 1,2. \tag{5.5}$$

In the plane $\mathbf{R}^2$ with coordinates $M = (m_1, m_2)$, the straight line $\mathcal{L}$ passing through the points (5.5) is given by the equation

$$\lambda M_1 + (1-\lambda) M_2 = M, \qquad \lambda \in \mathbb{R}. \tag{5.6}$$

Lemma 5.2. *The straight line $\mathcal{L}$ crosses coordinate axes m_1 and m_2 at the points $(m_1^*, 0)$ and $(0, m_2^*)$ respectively, where*

$$m_1^* = \frac{|M_1 M_2|}{m_{22} - m_{21}}, \qquad m_2^* = \frac{|M_2 M_1|}{m_{11} - m_{12}}. \tag{5.7}$$

Proof. (see Fig. 17). According to (5.5) and (5.6), the point $M = (m_1^*, 0)$ satisfies the system of two equations

$$\begin{aligned} \lambda\, m_{11} + (1-\lambda)\, m_{12} &= m_1^*, \\ \lambda\, m_{21} + (1-\lambda)\, m_{22} &= 0. \end{aligned} \tag{5.8}$$

From the second equation, we find

$$\lambda = \frac{m_{22}}{m_{22} - m_{21}}.$$

Consequently,

$$1 - \lambda = -\frac{m_{21}}{m_{22} - m_{21}}.$$

Substituting these values in the first equation (5.8), we obtain

$$\frac{m_{22} m_{11} - m_{21} m_{12}}{m_{22} - m_{21}} = m_1^*,$$

i.e. the value m_1^* in (5.7). Similarly, we compute m_2^*. The proof is complete. ■

Further we assume that the normal vector $D = (d_1, d_2)$ of the straight line $\mathcal{L}$ is positive: $d_1, d_2 > 0$. Then the numbers m_1^* and m_2^* are also positive (Fig. 17).

Lemma 5.3. *The points* $(m_1^*, 0)$ *and* $(0, m_2^*)$ *in the plane* M, *i.e.* $\mathbf{R}^2$, *correspond to* 4 *points* $\pm Y$ *and* $\pm Z$ *in the plane* X, *i.e.* $\mathbb{R}^2$, *where*

$$Y = \frac{\delta}{|l_{22}| - |l_{21}|}(l_{22}, -l_{21}), \qquad Z = \frac{\delta}{|l_{11}| - |l_{12}|}(-l_{12}, l_{11}), \tag{5.9}$$

and

$$\delta = \det \begin{pmatrix} |l_{11}| & |l_{21}| \\ |l_{12}| & |l_{22}| \end{pmatrix} \Big/ \det \begin{pmatrix} l_{11} & l_{21} \\ l_{12} & l_{22} \end{pmatrix}. \tag{5.10}$$

Proof. The point $Y = (y_1, y_2)$ satisfies the system of equations

$$l_1(Y) = m_1^*, \qquad l_2(Y) = 0.$$

According to (5.4), if we solve this system by Cramer's rule, we obtain

$$y_1 = l_{22} m_1^* / \Delta, \qquad y_2 = -l_{21} m_1^* / \Delta,$$

where $\Delta = l_{11} l_{22} - l_{12} l_{21}$. If we express m_1^* by the formula (5.7), we obtain an expression for Y by the formula (5.9). Similarly we find an expression for Z in (5.9). The proof is complete. ∎

Lemma 5.4. *The triangle* Δ *with the vertices* $(0,0)$, $(m_1^*, 0)$, $(0, m_2^*)$ *corresponds to the parallelogram* Π *with the vertices* $\pm Y$, $\pm Z$ *in the plane* $\mathbf{R}^2$. *The side of the triangle* Δ *spanned on the points* $(m_1^*, 0)$ *and* $(0, m_2^*)$ *corresponds to the boundary of the parallelogram* Π.

Proof. The proof is obvious. ∎

Lemma 5.5. *All integer points* X *lying inside the parallelogram* Π *lie on one of the two straight lines*

$$x_1 = \pm x_2. \tag{5.11}$$

Proof. We will distinguish two cases: $l_{11} l_{12} l_{21} l_{22} > 0$, and $l_{11} l_{12} l_{21} l_{22} < 0$. In the first case, the vectors (l_{11}, l_{12}) and (l_{21}, l_{22}) lie either in the same quadrant or in opposite quadrants; and in the plane $\mathbb{R}^2$, the straight lines $l_1(X) = 0$ and $l_2(X) = 0$ lie in the same pair of opposite quadrants (Fig. 18). In the second case, these vectors lie in neighboring quadrants; and in the plane $\mathbb{R}^2$, the straight lines $l_1(X) = 0$ and $l_2(X) = 0$ lie in different quadrants (Fig. 19). We assign

$$\sigma_{ij} = \operatorname{sgn} l_{ij}, \qquad i, j = 1, 2. \tag{5.12}$$

Now we note that

$$\begin{aligned} \begin{vmatrix} l_{11} & l_{21} \\ l_{12} & l_{22} \end{vmatrix} &= \sigma_{11}\sigma_{21} \begin{vmatrix} |l_{11}| & |l_{21}| \\ \sigma_{11}\sigma_{21}|l_{12}| & \sigma_{21}\sigma_{22}|l_{22}| \end{vmatrix} \\ &= \sigma_{11}\sigma_{22} \begin{vmatrix} |l_{11}| & |l_{21}| \\ \sigma|l_{12}| & |l_{22}| \end{vmatrix}, \end{aligned} \tag{5.13}$$

where $\sigma = \sigma_{11}\sigma_{12}\sigma_{21}\sigma_{22}$; and we will consider both cases separately.

First case ($\sigma = 1$). From (5.13) and (5.10), we see that $|\delta| = 1$. Besides,

$$\begin{aligned} (l_{22}, -l_{21}) &= \sigma_{22}(|l_{22}|, -\sigma_{21}\sigma_{22}|l_{21}|), \\ (-l_{12}, l_{11}) &= \sigma_{12}(-|l_{12}|, \sigma_{12}\sigma_{11}|l_{11}|). \end{aligned} \tag{5.14}$$

In this case $\sigma_{21}\sigma_{22} = \sigma_{12}\sigma_{11}$. We take the vector $P = (\sigma_{21}, \sigma_{22})$ and consider its scalar products with the vectors $\pm Y$ and $\pm Z$. According to (5.9), we obtain

$$\langle P, \pm Y\rangle = \pm\delta\,\sigma_{22}, \qquad \langle P, \pm Z\rangle = \pm\delta\,\sigma_{12}, \qquad \text{i.e.} \qquad |\langle P, \pm Y\rangle| = |\langle P, \pm Z\rangle| = 1.$$

Since the parallelogram Π is the convex hull of the points $\pm Y$ and $\pm Z$, then for its points X, we have

$$|\langle P, X\rangle| \leqslant 1.$$

In the plane $\mathbb{R}^2$, this inequality isolates the band of width 2 along the straight line

$$\langle P, X\rangle = 0. \tag{5.15}$$

Consequently, the integer points lying inside this band lie on the straight line (5.15), which is one of the straight lines (5.11).

Second case ($\sigma = -1$). From (5.13) and (5.10), we see that $|\delta| < 1$, since the determinant in (5.13) is equal to $|l_{11}||l_{22}| + |l_{12}||l_{21}|$. In addition, formulas (5.14) are valid, but now $\sigma_{21}\sigma_{22} = -\sigma_{12}\sigma_{11}$. Hence the points $\pm Y$ and $\pm Z$ lie in two different bands

$$|x_1 + x_2| < 1 \qquad \text{and} \qquad |x_1 - x_2| < 1.$$

Since four sides of the parallelogram Π pass through the four points $\pm E_1$ and $\pm E_2$, then there are points $X \in \Pi$ with $|x_1| + |x_2| \geqslant 2$ only in one of these bands. But all integer points X lying in this band lie on its axis, i.e. on the corresponding straight line (5.11). Lemma is proved. ∎

Corollary 5.1. *Out of the two points $M(E_1 + E_2)$ and $M(E_1 - E_2)$, no more than one lies on the same side of the straight line $\mathcal{L}$ as the origin.*

5.2. Space geometry

For a three-dimensional point $M = (m_1, m_2, m_3)$, the upper bar will denote its projection on the plane $(m_1, m_2) : \bar{M} = (m_1, m_2)$.

Lemma 5.6. *In $\mathbb{R}^3$, let three points $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, and $C = (c_1, c_2, c_3)$ be given such that their projections $\bar{A}, \bar{B}, \bar{C}$ do not lie on the same straight line. The plane passing through the points A, B, C is crossing the third coordinate axis m_3 at the point*

$$m_3^*(A, B, C) = \frac{|ABC|}{|\bar{A}\bar{B}| + |\bar{B}\bar{C}| + |\bar{C}\bar{A}|}, \tag{5.16}$$

where $|ABC| \stackrel{\text{def}}{=} \det(ABC)$.

Proof. The points M of a plane passing through the points A, B, C have the form

$$M = A + \lambda(B - A) + \mu(C - A), \qquad \lambda, \mu \in \mathbb{R}. \tag{5.17}$$

The point $(0, 0, m_3^*)$ of this plane, lying on the axis m_3, satisfies the system of two equations

$$\begin{aligned} m_1 &= 0 = a_1 + \lambda(b_1 - a_1) + \mu(c_1 - a_1), \\ m_2 &= 0 = b_2 + \lambda(b_2 - a_2) + \mu(c_2 - a_2). \end{aligned}$$

From this system, we find that

$$\lambda = \frac{a_2c_1 - a_1c_2}{æ} = \frac{|\bar{C}\bar{A}|}{æ}, \qquad \mu = \frac{a_1b_2 - a_2b_1}{æ} = \frac{|\bar{A}\bar{B}|}{æ},$$

where $æ = (b_1 - a_1)(c_2 - a_2) - (b_2 - a_2)(c_1 - a_1) = |\bar{A}\bar{B}| + |\bar{B}\bar{C}| + |\bar{C}\bar{A}|$. For these values λ and μ, the third component m_3 in (5.17) is

$$\begin{aligned} m_3^* &= a_3 + \lambda(b_3 - a_3) + \mu(c_3 - a_3) \\ &= [(|\bar{A}\bar{B}| + |\bar{B}\bar{C}| + |\bar{C}\bar{A}|)a_3 + |\bar{C}\bar{A}|(b_3 - a_3) + |\bar{A}\bar{B}|(c_3 - a_3)]/æ \\ &= [|\bar{B}\bar{C}|a_3 + |\bar{C}\bar{A}|b_3 + |\bar{A}\bar{B}|c_3]/æ. \end{aligned}$$

The last formula coincides with (5.16), since in square brackets stands the expansion of the determinant $|ABC|$ over the last line. Lemma is proved. ■

In $\mathbb{R}^3$, let three linear homogeneous forms (4.1) and the mapping $M(X) = (m_1(X), m_2(X), m_3(X))$ be given, where $m_i(X) = |l_i(X)|$, $i = 1, 2, 3$. Let $A = M(E_1)$, $B = M(E_2)$, and $\mathcal{L}$ be the straight line in the plane $\mathbf{R}^2$ passing through the points $\bar{A}$ and $\bar{B}$. We assume that the normal to $\mathcal{L}$ planar vector $D > 0$ is positive: $D > 0$. The set of points

$$F = kE_1 + lE_2, \qquad k, l \in \mathbb{Z}, \qquad |k| + |l| > 1$$

for which the points $\bar{M}(F)$ lie on the left of the straight line $\mathcal{L}$ is denoted as $\mathcal{F}$.

Theorem 5.1. *The minimum of positive values of*

$$m_3^*(A, B, M(F)) \qquad \textit{over} \qquad F \in \mathcal{F}$$

is attained at that point $F = E_1 + E_2$ or $F = E_1 - E_2$ which belongs to the set $\mathcal{F}$.

Proof. According to Lemma 5.5, if the set $\mathcal{F}$ is not empty, then all its points lie on one of the straight lines (5.11), i.e. have the form

$$F = k\,G, \qquad k \in \mathbb{Z}, \qquad |k| > 0,$$

where $G = E_1 + E_2$ or $G = E_1 - E_2$. Since $M(kG) = |k|M(G)$, then for these F, by Lemma 5.6,

$$m_3^*(A, B, M(F)) = \frac{|k|\,|ABC|}{|\bar{A}\bar{B}| + |k|\,(|\bar{B}\bar{C}| + |\bar{C}\bar{A}|)}, \tag{5.18}$$

where $C = M(G)$. Now we note that the determinants $|ABC|$ and $|\bar{A}\bar{B}|$ have the same sign, and the sum $|\bar{B}\bar{C}| + |\bar{C}\bar{A}|$ has the opposite sign.

Indeed, for the disposition of Fig. 16, where $A = \bar{A}$, $B = \bar{B}$, $C = \bar{C}$, we have $|\bar{A}\bar{B}| > 0$, and by Lemma 5.1, $æ = |\bar{A}\bar{B}| + |\bar{B}\bar{C}| + |\bar{C}\bar{A}| < 0$. Consequently, $|\bar{B}\bar{C}| + |\bar{C}\bar{A}| < 0$. Since for Fig. 16, the vectors A, B, C preserve orientation of the basis vectors, then $|ABC| > 0$.

If we interchange the vectors A and B, then all determinants will change sign. So, the formula (5.18) takes the form

$$m_3^*(A, B, M(F)) = \frac{|k|\,\alpha}{\beta - |k|\,\gamma}, \tag{5.19}$$

where $\alpha, \beta, \gamma > 0$. In addition, by condition of the theorem, this value is positive, i.e. $\beta - |k|\gamma > 0$. Obviously, the minimum of the value (5.19) is attained for $|k| = 1$. The proof is complete. ■

5.3. Plane non-homogeneous geometry

Now, along with the homogeneous forms (5.4), we consider the corresponding nonhomogenious forms

$$\tilde{l}_i(X) = l_i + \gamma_i, \qquad \gamma_i = \text{const} \in \mathbb{R}, \qquad i = 1, 2.$$

We put

$$\tilde{m}_i(X) = |\tilde{l}_i(X)|, \quad i = 1, 2; \qquad \tilde{M}(X) = (\tilde{m}_1(X), \tilde{m}_2(X)).$$

Let the vector $\Lambda = (\lambda_1, \lambda_2)$ be such that $\tilde{l}_1(\Lambda) = \tilde{l}_2(\Lambda) = 0$. We put $a_i = [\lambda_i]$, $i = 1, 2$, and consider 4 points

$$W_1 = (a_1, a_2), \quad W_2 = (a_1 + 1, a_2), \quad W_3 = (a_1, a_2 + 1), \quad W_4 = (a_1 + 1, a_2 + 1).$$

Theorem 5.2. *From the four points*

$$V_j = \tilde{M}(W_j), \qquad j = 1, 2, 3, 4,$$

at least one is lying on the same side of the straight line $\mathcal{L}$ as the origin.

Proof. According to Subsect. 5.1, it is sufficient to prove that one of the points W_j is inside the set $\Lambda + \Pi$. From Figures 18 and 19, we see that the square without vertices

$$\tilde{\Pi} = \{X \colon |x_i| \leqslant 1/2,\ i = 1, 2;\ |x_1| + |x_2| < 1\}$$

lies inside the parallelogram Π. In case of Figure 18, one of the two pairs of vertices $\pm(1/2, 1/2)$ and $\pm(1/2, -1/2)$ of the quadrangle $\tilde{\Pi}$ lies on the border of the parallelogram Π, and another pair lies inside Π. In case of Figure 19, all vertices of square $\tilde{\Pi}$ lie inside Π.

We consider 5 cases.

1. Let $\lambda_i = a_i + 1/2$, $i = 1, 2$. Then in case of Figure 18, one of the pairs of points W_1, W_4 and W_2, W_3 lies inside the set $\Lambda + \Pi$; and in case of Figure 19, all points W_j lie in this set.

Further, we consider the cases that differ from the case 1.

2. If $\lambda_i \geqslant a_i + 1/2$, $i = 1, 2$, then the point $W_4 \in \Lambda + \tilde{\Pi}$.
3. If $\lambda_i \leqslant a_i + 1/2$, $i = 1, 2$, then the point $W_1 \in \Lambda + \tilde{\Pi}$.
4. If $\lambda_1 \geqslant a_1 + 1/2$ and $\lambda_2 \leqslant a_2 + 1/2$, then the point $W_2 \in \Lambda + \tilde{\Pi}$.
5. If $\lambda_1 \leqslant a_1 + 1/2$ and $\lambda_2 \geqslant a_2 + 1/2$, then the point $W_3 \in \Lambda + \tilde{\Pi}$.

Since $\tilde{\Pi} \subset \Pi$, then in cases 2-5, that differ from the case 1, the mentioned point $W_j \in \Lambda + \Pi$. Since the cases 1-5 exhaust all possibilities, then the theorem is proved. ■

We suggest below a generalization of continued fraction that is a directed motion over the surface $\partial\mathbf{M}$; one step of the motion gives a transition from one triangle with $\omega = 1$ to the nearest one with the same property.

6. Algorithm of motion over the surface of modular polyhedron

6.1. Statement of the problem

In $\mathbb{R}^3$, let three linear forms (4.1) be given and the vector $A = (\alpha_1, \alpha_2, \alpha_3)$ such that $l_1(A) = l_2(A) = 0$. We denote the semi-space $l_3(X) \geqslant 0$ as $\pi\mathbb{R}^3$. Let $A \in \pi\mathbb{R}^3$ (this can always be achieved by changing sign of the vector A). Our goal is to construct integer approximations to the ray μA, $\mu > 0$.

First, we assume that in the surface $\partial\mathbf{M}$:

$$\text{all faces are simple or semi-simple} \tag{6.1}$$

and they have

$$\omega \leqslant 1. \tag{6.2}$$

According to Corollary 4.1, the property (6.1) holds generically.

Let integer vectors $B_1, B_2, B_3 \in \pi\mathbb{Z}^3$ form a basis, i.e. $\det(B_1B_2B_3) = \pm 1$. Then we have $l_{ij} \overset{\text{def}}{=} l_i(B_j)$, $i, j = 1, 2, 3$ and the vector $\Lambda = (\lambda_1, \lambda_2, \lambda_3)$ such that $\lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 B_3 = \mu A$, $\mu \neq 0$. Here $\sum\limits_{j=1}^{3} l_{ij}\lambda_j = 0$, $i = 1, 2$. This initial data is conveniently written in the following table

$$\begin{matrix} B_1 & l_{11} & l_{21} & l_{31} & \lambda_1 \\ B_2 & l_{12} & l_{22} & l_{32} & \lambda_2 \\ B_3 & l_{13} & l_{23} & l_{33} & \lambda_3. \end{matrix} \tag{6.3}$$

Here $M_i = M(B_i)$, i.e. $m_{ij} = |l_{ij}|$, $i, j = 1, 2, 3$. For the point M or the set of points, for example Γ_i, the upper bar will denote their orthogonal projections on the plane (m_1, m_2) in the space $\mathbf{R}^3$, i.e. $\bar{M}$ and $\bar{\Gamma}_i$ respectively. Let points M_i be the vertices of the triangle Γ.

6.2. Transition to another basis

Transition to another basis B_1', B_2', B_3' consists of successive execution of the following 5 steps.

Step 1. First, we need to clarify mutual disposition of three points $\bar{M}_j = (|l_{1j}|, |l_{2j}|) \stackrel{\text{def}}{=} (m_{1j}, m_{2j})$, $j = 1, 2, 3$ in the plane (m_1, m_2). Namely, these points are the vertices of the triangle. Each side of the triangle has the outer normal vector. We take the side of the triangle where both components of the normal vector are negative. The *pair of points* from $\bar{M}_1$, $\bar{M}_2$, $\bar{M}_3$ lying on this side is *marked*. Technically, marking such pair of points can be done either in the figure or using computations described in Subsection 6.3. If there are no such points, then this is the *special case*, which we treat in Subsection 6.4. If there are two such sides, i.e. there are two pairs of marked points, then the algorithm bifurcates, and it must be executed for each pair separately. Let for definiteness $\bar{M}_1$ and $\bar{M}_2$ be marked. The straight line passing through them is denoted as $\mathcal{L}$ (see Fig. 20).

Step 2. We compute $a_i = [\lambda_i / |\lambda_3|]$, $i = 1, 2, 3$. Here $a_3 = \pm 1$.

Step 3. For each of six points

$$
\begin{aligned}
M(B_1 + B_2) &\stackrel{\text{def}}{=} U_1, \\
M(B_1 - B_2) &\stackrel{\text{def}}{=} U_2; \\
M(a_1 B_1 + a_2 B_2 + a_3 B_3) &\stackrel{\text{def}}{=} V_1, \\
M((a_1 + 1) B_1 + a_2 B_2 + a_3 B_3) &\stackrel{\text{def}}{=} V_2, \\
M(a_1 B_1 + (a_2 + 1) B_2 + a_3 B_3) &\stackrel{\text{def}}{=} V_3, \\
M((a_1 + 1) B_1 + (a_2 + 1) B_2 + a_3 B_3) &\stackrel{\text{def}}{=} V_4,
\end{aligned}
\tag{6.4}
$$

we clarify the disposition of its projection $\bar{U}_i$, $\bar{V}_j$ with respect to the straight line $\mathcal{L}$. We discard the points U_i, V_j projections of which $\bar{U}_i$, $\bar{V}_j$ are separated from the origin by the straight line $\mathcal{L}$. We keep only the points projections of which lie on the same side of the straight line $\mathcal{L}$ as the origin.

In (6.4), as arguments, we took all integer non-basis vectors closest to the straight line μA and lying in the planes $x_3 = 0$ (Fig. 21) and $x_3 = a_3$ (Fig. 22). Here we do not distinguish between the vectors that differ only in sign.

According to Corollary 5.1, from the two points $\bar{U}_1$, $\bar{U}_2$, no more than one lies on the same side of the straight line $\mathcal{L}$ as the origin.

According to Theorem 5.2, from the four points $\bar{V}_1$–$\bar{V}_4$, at least one lies on the same side of the straight line $\mathcal{L}$ as the origin.

Step 4. For each of the points (6.4) that we kept, by Lemma 5.6, we compute the point of intersection of the plane drawn through the points M_1, M_2 and this point with the third coordinate axis. From these values, we take the smallest, and take the point U_i or V_j corresponding to that value. If there are several such points, then the procedure bifurcates, since we consider further each of them.

According to Theorem 5.1, among all integer points in the planes $x_3 = 0$ and $x_3 = a_3$, the minimum we seek is attained at some points among (6.4).

Step 5. If the point chosen at step 4 corresponds to one of the points U_i, then we execute step 5a. If it corresponds to one of the points V_i, then we execute step 5b.

Step 5a. Let the point U_1 be chosen. We consider the triangle with the vertices $\bar{M}_1$, $\bar{M}_2$, $\bar{U}_1$. From its two edges incident with the vertex $\bar{U}_1$, we choose the edge that has negative outer normal vector (i.e. lying in the III-rd quadrant). If there are no such edges, then it is the *special case*, which we treat in Subsection 6.4. If there are two such edges, then the algorithm bifurcate. Let the point $\bar{M}_2$ belong to the chosen edge. Then from the two vectors B_1 and B_2, we keep B_2. From the basis B_1, B_2, B_3, we go to the basis $B_4 = B_1 + B_2$, B_2, B_3. We write down the matrix N for which

$$\begin{pmatrix} B_1' \\ B_2' \\ B_3' \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} B_4 \\ B_2 \\ B_3 \end{pmatrix} = N \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \qquad N = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix N^{*-1} gives the transformation

$$\Lambda' = N^{*-1}\Lambda, \qquad N^{*-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where asterisk means transposition. If we have chosen the point U_2, then $B_4 = B_1 - B_2$ or $B_2 - B_1$ in order that $B_4 \in \pi\mathbb{R}^3$; the matrices N and N^{*-1} are changed respectively.

Step 5b. Let the point $V_j = M(B_3')$ be chosen, where $B_3' = \tilde{a}_1 B_1 + \tilde{a}_2 B_2 + a_3 B_3$ according to (6.4), i.e. $\tilde{a}_i$ $(i = 1, 2)$ are either a_i or $a_i + 1$ according to formula for V_j in (6.4). Then from the basis B_1, B_2, B_3, we go to the basis $B_1' = B_1$, $B_2' = B_2$, $B_3' = \tilde{a}_1 B_1 + \tilde{a}_2 B_2 + a_3 B_3$ by the formulas

$$\begin{pmatrix} B_1' \\ B_2' \\ B_3' \end{pmatrix} = N \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \qquad N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \tilde{a}_1 & \tilde{a}_2 & a_3 \end{pmatrix}.$$

Here

$$\Lambda' = N^{*-1}\Lambda, \qquad N^{*-1} = \begin{pmatrix} 1 & 0 & -a_3\tilde{a}_1 \\ 0 & 1 & -a_3\tilde{a}_2 \\ 0 & 0 & a_3 \end{pmatrix}.$$

The vectors B_1', B_2', B_3' give the new basis; the vector Λ' gives the value of the vector A in this basis. Thus, transition from the old basis B_1, B_2, B_3 to the new basis B_1', B_2', B_3' is complete.

Under assumptions (6.1) and (6.2), if the points M_1 and M_2 were corresponding to the edge of natural triangulation of the surface $\partial\mathbf{M}$, then the described transition to another basis gives the triangle of the surface $\partial\mathbf{M}$ with the vertices M_1, M_2, and U_i or V_j.

Moreover, under assumptions (6.1) and (6.2), if the basis B_1, B_2, B_3 is such that the pair of marked points from $M(B_i)$, $i = 1, 2, 3$ corresponds to the edge of natural triangulation of the surface $\partial\mathbf{M}$, and we obtained the basis $\tilde{B}_1, \tilde{B}_2, \tilde{B}_3$ after several such transitions to new bases, and in the last transition, we have chosen the point of the type V_j in (6.4), then the new basis $\tilde{B}_1, \tilde{B}_2, \tilde{B}_3$ also corresponds to the triangle of the surface $\partial\mathbf{M}$ with the vertices $M(\tilde{B}_i), i = 1, 2, 3$.

Remark 6.1. We gave above the simplest and most general formulation of the transition to another basis. In addition to the exposed theory, we can specify several cases and subcases when this transition simplifies. For example, if $l_{11}l_{12}$ and $l_{21}l_{22} < 0$, then we can omit the point U_2; if $l_{11}l_{12}$ and $l_{21}l_{22} > 0$, then we can omit the point U_1. However, this general form of transition is most convenient for programming.

Remark 6.2. If initial basis does not correspond to a triangle of the surface $\partial\mathbf{M}$, then the proposed algorithm leads to a face of the surface $\partial\mathbf{M}$ after several transitions to another basis.

If the assumption (6.1) is not satisfied, i.e. there is a face that is not simple or semi-simple, then the proposed algorithm allows to come to such a face, to move over it, and to go to another face from that one. Here the condition of correspondence of a pair of marked points to an edge of natural triangulation of the surface $\partial\mathbf{M}$ can be substituted by the condition that the segment connecting these points was lying in the surface $\partial\mathbf{M}$; the statement itself is that the obtained three points lie in the same face of the surface $\partial\mathbf{M}$.

If the assumption (6.2) is not satisfied, then for a triangle Γ_i of the surface $\partial\mathbf{M}$ with $\omega(\Gamma_i) = 2$, the algorithm gives one of the points of the type (4.36), (4.5′), and the sequence of triangles Δ_{ik} described in Example 4.6, since, as it is easy to see, pre-images of the four points V_j in (6.4) are the four points of the type (4.36), i.e. (4.5′). And in addition, each of these triangles Δ_{ik} has a common edge with the face Γ_i. After two steps, the algorithm comes to another edge of the triangle Γ_i and continues to move over triangles of the surface $\partial\mathbf{M}$.

6.3. Ordering of three points and examples

From Lemma 5.1, obviously follows

Corollary 6.1. *The vertices A, B of the triangle with the vertices A, B, C are marked, if the values $|AB|$ and æ from* (5.1) *have different signs, and the components of the difference $A - B$ have different signs.*

Therefore in order to find the separating edge, we need to compute the determinants $|AB|$, $|BC|$, $|CA|$, and their sum æ. If the sign of a determinant is opposite to the sign of æ, then the vertices corresponding to this determinant are marked if their difference has components of different signs.

Example 6.1. For the form (3.2) of Examples 3.1 and 4.1, the logarithmic projection of the surface of the modular polyhedron $\partial\mathbf{M}$ is shown in Fig. 8. The vertices of this polyhedron correspond to bold points. At each of these points we give: the vector $X = (x_1, x_2, x_3)$ that is the pre-image of the vertex, and the value $|h(X)|$ at this vector. The edges of the surface $\partial\mathbf{M}$ correspond to curves connecting the projections of the vertices. For the sake of simplicity, we begin with the basis

$$B_1 = (1, 0, 0), \qquad B_2 = (-1, 0, 1), \qquad B_3 = (3, 1, -1).$$

This basis corresponds to the face Γ_1 of the surface $\partial\mathbf{M}$ with $\omega(\Gamma_1) = 1$. Here the points $\bar{M}(B_1)$ and $\bar{M}(B_2)$ are marked. In the first transition, we replace the vector B_1 by the vector $B_4 = B_1 + B_2 = (0, 0, 1)$ and obtain the basis B_1, B_2, B_4. The point $M(B_4)$ has the form U_1 in (6.4). For this basis, the points $\bar{M}(B_2)$ and $\bar{M}(B_4)$ are marked. In the second transition, we replace the vector B_3 by the vector $B_5 = 2B_2 + 2B_4 + B_3 = (1, 1, 3)$. Here the point $M(B_5)$ is the point V_1 in (6.4). For the basis B_2, B_4, B_5, the points $\bar{M}(B_4)$ and $\bar{M}(B_5)$ are marked. In the third transition, we replace B_4 by the vector $B_6 = B_4 + B_5 = (1, 1, 4)$. Here the point $M(B_6)$ is the point U_1 in (6.4). For the basis B_2, B_5, B_6, the points $\bar{M}(B_5)$ and $\bar{M}(B_6)$ are marked. In the fourth transition, we replace B_2 by the vector $B_7 = B_5 + B_6 + B_2 = (1, 2, 8)$. Here the point $M(B_7)$ is the point V_1 in (6.4). For the basis B_5, B_6, B_7, the points $\bar{M}(B_6)$ and $\bar{M}(B_7)$ are marked. In the fifth transition, we replace B_5 by $B_8 = B_7 + B_5 = (2, 3, 11)$. Here $M(B_8)$ is V_1 in (6.4). For the basis B_6, B_7, B_8, there are two pairs of marked points. The first pair $\bar{M}(B_7)$ and $\bar{M}(B_8)$ and the second pair $\bar{M}(B_6)$ and $\bar{M}(B_8)$. The first pair gives the sixth transition with the replacement of B_6 by $B_9 = B_7 + B_8 + B_6 = (4, 6, 23)$, which corresponds to V_1 in (6.4). The second pair gives the transition with the replacement of B_7 by B_9, but the triangle $\{\bar{M}(B_6), \bar{M}(B_8), \bar{M}(B_9)\}$ is of the special case, so we will not use this transition. For the basis B_7, B_8, B_9, the points $\bar{M}(B_7)$ and $\bar{M}(B_9)$ are marked. They give the seventh transition with the replacement of B_8 by $B_{10} = B_9 + B_8 = (6, 9, 34)$, which corresponds to V_1 in (6.4). From Fig. 8, we see that in logarithmic coordinates

$$n_i = \log m_i, \qquad i = 1, 2, \tag{6.5}$$

the triangle $\{\bar{M}(B_9), \bar{M}(B_{10}), \bar{M}(B_7)\}$ can be obtained by parallel translation of the triangle $\{\bar{M}(B_1), \bar{M}(B_2), \bar{M}(B_3)\}$. Consequently, the linear transformation $X = YT$ with

$$\begin{pmatrix} B_9 \\ B_{10} \\ B_7 \end{pmatrix} = T \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$$

gives linear automorphism of the surface $\partial\mathbf{M}$, where

$$T = \begin{pmatrix} B_9 \\ B_{10} \\ B_7 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}^{-1} = \begin{pmatrix} 15 & 29 & 6 \\ 22 & 43 & 9 \\ 5 & 10 & 2 \end{pmatrix}.$$

Example 6.2 (continuation of Example 4.2)**.** Let for forms (4.1), (4.8′) (see Fig. 14), we start from the basis $B_1 = (1, 0, 0)$, $B_2 = (0, 1, 0)$, $B_3 = (1, 0, 1)$, then the marked points are $\bar{M}(B_2)$ and $\bar{M}(B_3)$. In the first transition, we replace B_3 by the vector $B_4 = B_2 - B_3 = (-1, 1, -1)$. Now the marked points are $\bar{M}(B_2)$ and $\bar{M}(B_4)$. In the second transition, we replace B_1 by the vector $B_5 = -B_1 + 3B_2 = (-1, 3, 0)$. Vectors B_2, B_4, B_5 form a basis, and points $M(B_2), M(B_4), M(B_5)$ are vertices of the surface $\partial\mathbf{M}$. Now the points $\bar{M}(B_2)$, $\bar{M}(B_5)$ are marked, and the new vector $B_6 = B_5 - B_2 = (-1, 2, 0)$ should be taken instead of B_2. Now the points $\bar{M}(B_5), \bar{M}(B_6)$ are marked and the new vector $B_7 = B_4 + 3B_5 = (-4, 10, -1)$. The final basis is B_6, B_5, B_7. According to Fig. 14, in logarithmic coordinates (6.5), the triangle $\{\bar{M}(B_6), \bar{M}(B_7), \bar{M}(B_5)\}$ can be obtained from the initial triangle $\{\bar{M}(B_1), \bar{M}(B_2), \bar{M}(B_3)\}$ by a parallel translation. Thus, the linear transformation $X = YT$ with

$$\begin{pmatrix} B_6 \\ B_7 \\ B_5 \end{pmatrix} = T \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$$

induces the linear automorphism of the surface $\partial\mathbf{M}$, i.e.

$$T = \begin{pmatrix} B_6 \\ B_7 \\ B_5 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 2 & 0 \\ -3 & 10 & -1 \\ -1 & 3 & 0 \end{pmatrix}.$$

6.4. Special case

If none of the edges of the triangle with the vertices $\bar{M}_1$, $\bar{M}_2$, $\bar{M}_3$ has negative outer normal vector, then this *case* is called *special.*

We will use the following obvious statements.

Lemma 6.1. *If three points M_1, M_2, M_3 lie in the surface $\partial\mathbf{M}$ and do not lie on the same straight line, then all components p_i of the vector $P = (p_1, p_2, p_3)$ that is normal to the plane passing through these points are of the same sign.*

Lemma 6.2. *If the normal vector P of the triangle Γ with the vertices M_1, M_2, M_3 is positive: $P > 0$ and the projection $\bar{\Gamma}$ is classified as special case, then both orthogonal projections of the triangle Γ on the planes $m_1 = 0$ and $m_2 = 0$ are not of the special case.*

If we are in the situation of special case, and the triangle Γ has the normal vector $P > 0$, then, according to Lemma 6.2, we perform a transition to other basis using described algorithm for the corresponding approximation of the ray $\mu\tilde{A}$, where $l_2(\tilde{A}) = l_3(\tilde{A}) = 0$, or the ray $\mu\tilde{\tilde{A}}$, where $l_3(\tilde{\tilde{A}}) = l_1(\tilde{\tilde{A}}) = 0$. Here we take the expansion $\tilde{\Lambda}$ or $\tilde{\tilde{\Lambda}}$ of the vector $\tilde{A}$ or $\tilde{\tilde{A}}$ by the basis B_1, B_2, B_3 as the vector Λ. For example, $\tilde{A} = \tilde{\lambda}_1 B_1 + \tilde{\lambda}_2 B_2 + \tilde{\lambda}_3 B_3$, and $\tilde{\Lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3)$.

If the triangle Γ has the normal vector with components of different signs, then we need to go to other basis in an arbitrary way taking this basis in such a way as to obtain a positive normal vector. The simplest way for that is to replace one of basis vectors B_i by its sum or difference with the other basis vectors.

If initial basis B_1, B_2, B_3 is of the special case, then we need not make projections on other coordinate planes, but we better go immediately to another basis such that is not of the special case.

6.5. Simplified algorithm

Computation by the formula (5.16) is rather cumbersome especially by hand. But the algorithm described above can be simplified in the following way. Let $M = (m_1, m_2, m_3)$.We denote $|M| = m_1 m_2 m_3$. Steps 1 and 2 are executed as earlier. Step 3 is substituted by

Step 3s. We compute the points (6.4) and the values $|U_i|$ and $|V_j|$ for them. From these values, we choose the smallest and keep only those points U_i and V_j for which $|U_i|$ or $|V_j|$ is equal to this smallest value.

Step 3's. If among the points that we kept, there are points U_i, then we discard those among them which have projections $\bar{U}_i$ separated from the origin by the straight line $\mathcal{L}$. If after that, there still remain the point U_i (no more than one according to Corollary 5.1), then we execute step 5a.

Step 3"s. If among the points that we kept in step 3s, there are no points U_i, then for the remaining several points V_j, we execute step 4 and then step 5b. But if there is only one point V_j left, then step 4 is not necessary, we execute step 5b immediately.

All vertices of the surface $\partial|\mathbf{M}|$ obtained in this way are the vertices of the surface $\partial\mathbf{M}$, but some of the vertices of the surface $\partial\mathbf{M}$ can be overlooked.

Example 6.3. According to Example 6.1, if we execute the general algorithm, then for the forms of Examples 3.1 and 4.1, we obtain all vertices of the surface $\partial\mathbf{M}$ with $|h| = 5$ and $|h| = 9$ (see Fig. 8). But if we use the simplified algorithm, then we obtain only all the vertices with $|h| = 5$. Here three triangular faces of the surface $\partial\mathbf{M}$ with the common vertex with $|h| = 9$ are replaced by one triangle that have $|h| = 5$ at the vertices and for which $\omega = 1$.

7. Periodicity

Definition 7.1. *A unimodular 3×3 matrix T is called a period of the surface $\partial\mathbf{M}$ if the surface $\partial\tilde{\mathbf{M}}$ corresponding to forms $\tilde{l}_i(Y) = \langle L_i, TY^* \rangle$, $i = 1, 2, 3$, is transformed to $\partial\mathbf{M}$ by the linear transformation*

$$m_i = \mu_i \tilde{m}_i, \qquad i = 1, 2, 3, \tag{7.1}$$

where $\mu_i \in \mathbb{R}$, and $\mu_1 \mu_2 \mu_3 = 1$.

Here the asterisk means the transposition. So the periodicity of $\partial\mathbf{M}$ means that for $\partial\mathbf{M}$ two linear transformations $X^* = TY^*$ in $\mathbb{R}^3$ and (7.1) in $\mathbb{R}^3_+$ give the same result.

Let the cubic polynomial irreducible in $\mathbb{Q}$

$$P(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c \tag{7.2}$$

have positive discriminant. Then it has three real roots $\lambda_1, \lambda_2, \lambda_3$. Let the vectors L_i in the forms (4.1) be such that

$$(L_1^* L_2^* L_3^*) = SW, \tag{7.3}$$

where S is non-singular matrix, and W is the Vandermonde matrix for the polynomial (7.2).

Theorem 7.1. *If all coefficients of the polynomial* (7.2) *are integers, and all elements of the matrix S are rational numbers, then the surface $\partial\mathbf{M}$ corresponding to the forms* (4.1) *is doubly periodic, i.e. its logarithmic projection* (6.5) *is invariant under two independent parallel translations.*

Proof. The proof is based on Ch. II and Section 2 of the Algebraic Supplement of the book [3]. Let λ be a root of the polynomial (7.2), $\Lambda = \left(1, \lambda, \lambda^2\right)$, $L^* = S\Lambda^*$ and $L = (r_1, r_2, r_3)$. Let us consider the module Ω consisting of numbers $\langle L, X\rangle$, where $X \in \mathbb{Z}^3$, i.e. numbers r_1, r_2, r_3 are generators of the module Ω. Each number æ $\in \Omega$ corresponds to an integer matrix $\mathcal{D}_{æ} = (d_{ij})$ such that

$$æ r_i = \sum_{j=1}^{3} d_{ij} r_j, \qquad i = 1, 2, 3.$$

According to this formula, the vector L is the eigenvector of the matrix $\mathcal{D}_{æ}$ with the eigenvalue æ, because

$$\mathcal{D}_{æ} L^* = æ L^*.$$

The vector L does not depend on the number æ $\in \Omega$, so L is the eigenvector of all matrices $\mathcal{D}_{æ}$. It is true for any root $\lambda_i = \lambda$, i.e.

$$\mathcal{D}_{æ} L_i^* = æ_i L_i^*, \qquad i = 1, 2, 3,$$

where $æ_i = \langle L_i, X\rangle$ if $æ = \langle L, X\rangle$. Here

$$\det \mathcal{D}_{æ} = æ_1 æ_2 æ_3.$$

In the module Ω, the multiplication by the number æ $\in \Omega$ corresponds to the linear transformation

$$X^* = \mathcal{D}_{æ}^* Y^*,$$

i.e.

$$\langle L_i, X\rangle = \langle L_i, \mathcal{D}_{æ}^* Y^*\rangle = \langle \mathcal{D}_{æ} L_i^*, Y\rangle = æ_i \langle L_i, X\rangle, \qquad i = 1, 2, 3.$$

If the number æ is a unit of the field $\mathbb{Q}(\lambda)$, then $\det \mathcal{D}_{æ} = \pm 1$, i.e. the matrix $\mathcal{D}_{æ}$ is unimodular. Here

$$|\langle L_i, X\rangle| = |æ_i| |\langle L_i, Y\rangle|, \qquad i = 1, 2, 3,$$

i.e.

$$m_i(X) = |æ_i| m_i(Y), \qquad i = 1, 2, 3.$$

As $|æ_1 æ_2 æ_3| = |\det \mathcal{D}_{æ}| = 1$, then the matrix $\mathcal{D}_{æ}^*$ is a period of the surface $\partial\mathbf{M}$ according to Definition. Here

$$\log m_i(X) = \log |æ_i| + \log m_i(Y), \qquad i = 1, 2.$$

According to the Dirichlet theorem, in this case exactly two independent fundamental units u and v of the field $\mathbb{Q}(\lambda)$ are in the module Ω, i.e. each unit of the field has the form $\zeta u^m v^n$, where $\zeta^k = 1$ and k, l, $m \in \mathbb{Z}$. Hence the surface $\partial\mathbf{M}$ has two independent periods $\mathcal{D}_u^*$ and $\mathcal{D}_v^*$. In the logarithmic projection (6.5), each of the periods corresponds to its own parallel translation. Proof is complete. ■

The corresponding unit of the field is easily found by the automorphism (or the period) [66, 4, 3]. In order to compute two independent units, we must apply the described algorithm to the forms (4.1), (7.3) twice: for the approximation to the straight line μA (where the vector $A \neq 0$ is a common root of two forms $l_1(A) = l_2(A) = 0$, $\mu \in \mathbb{R}$), and to the straight line $\mu\tilde{A}$ (where the vector $\tilde{A} \neq 0$ is a common root of two forms $l_2(\tilde{A}) = l_3(\tilde{A}) = 0$, $\mu \in \mathbb{R}$).

Similar theorem for relative minima is proved in the work by Voronoi [66]; but all vertices of the surface $\partial\mathbf{M}$ are relative minima of the triple of forms (4.1). Although there may be relative minima that are not vertices of the surface $\partial\mathbf{M}$ but lie inside the polyhedral set $\mathbf{M}$. For example, in Fig. 8, these are points with $|h| = 21$, and in Fig. 14, these are points with $|h| = 8$. These figures make obvious the double periodic structure.

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Appendix

In May–June 1964, the author had a correspondence with C.L. Siegel concerning the results of the article [7].

One year later the author [8] improved the Siegel condition on small divisors

$$|\langle Q, \Lambda\rangle| \geqslant \mathrm{const}||Q||^{-\nu} \tag{$*$}$$

for all $Q \in \mathbb{Z}^n \setminus 0$ by the following condition ω. Let

$$\omega_k = \min|\langle Q, \Lambda\rangle| \text{ for } Q: \ Q \in \mathbb{Z}^n, \qquad \langle Q, \Lambda\rangle \neq 0, \qquad \|Q\| < 2^k.$$

Condition ω. The series $\sum\limits_{k=1}^{\infty} 2^{-k} \log\omega_k > -\infty$, i.e. it converges.

If $n = 2$ and q_k are denominators of the k's convergents of the continued fraction of the number λ_2/λ_1, then the condition ω is equivalent to convergency of the series

$$\sum_{k=1}^{\infty} \frac{\log q_{k+1}}{q_k}.$$

See details in [9]. The condition ω was called “the Bruno condition” and was very popular in 1980–1995 among specialists on differential equations.

The condition (*) is based on the formula for the measure (or exponent) of irrationality.

Figures

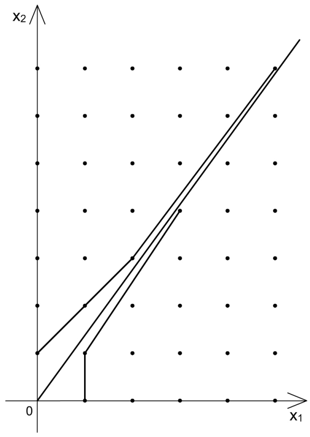


Figure 1

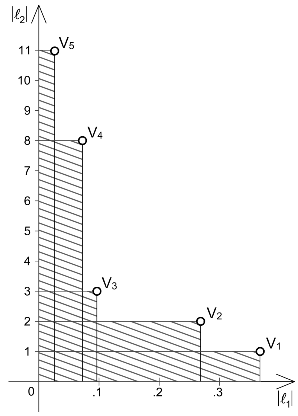


Figure 2

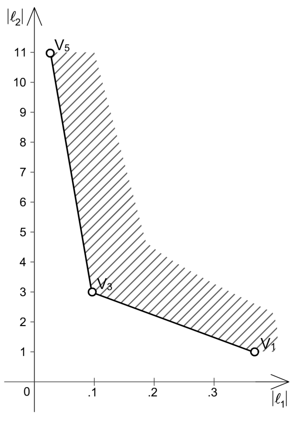


Figure 3

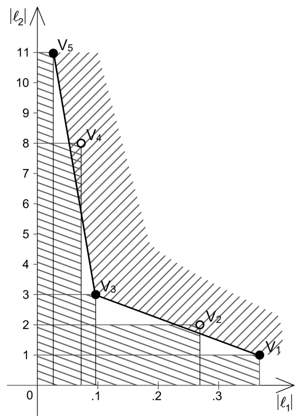
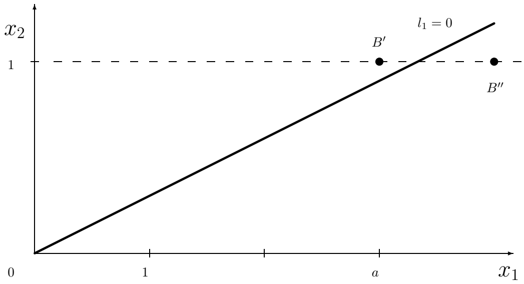


Figure 4



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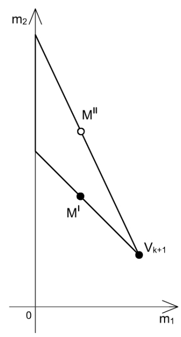


Figure 6

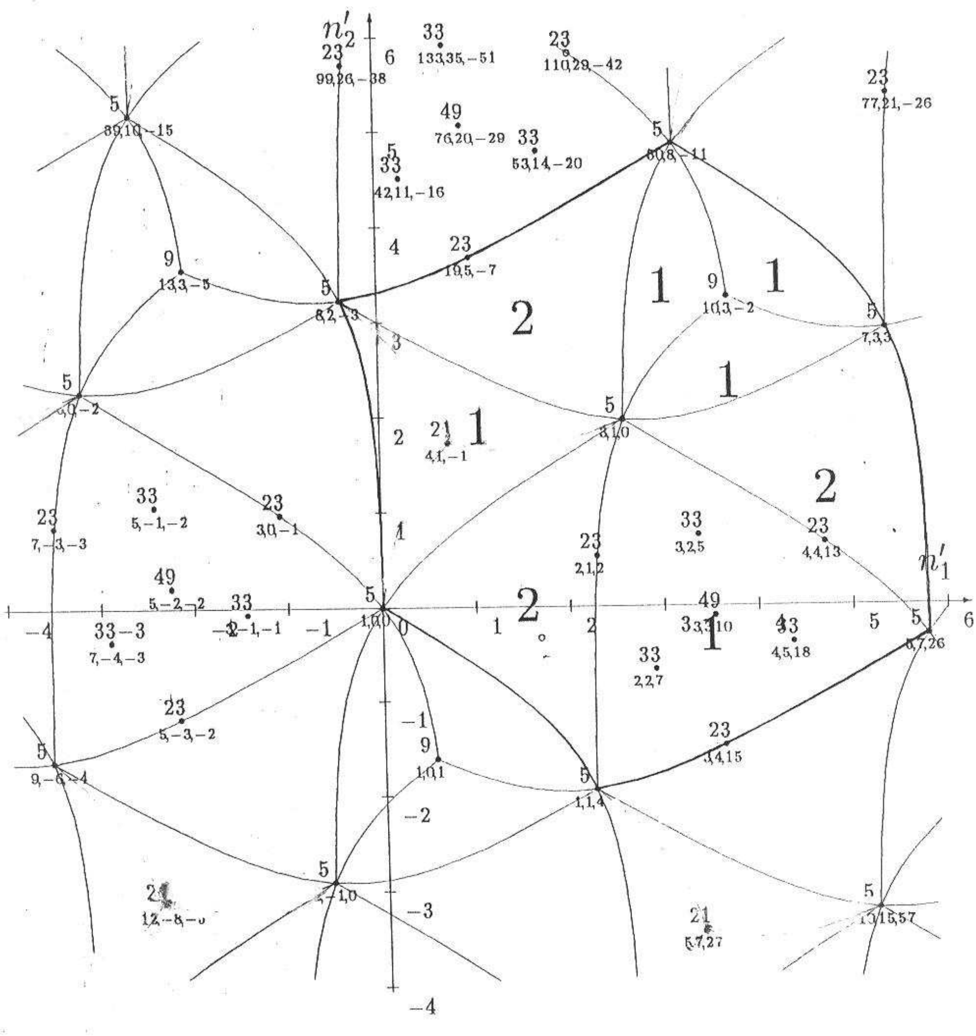


Figure 7

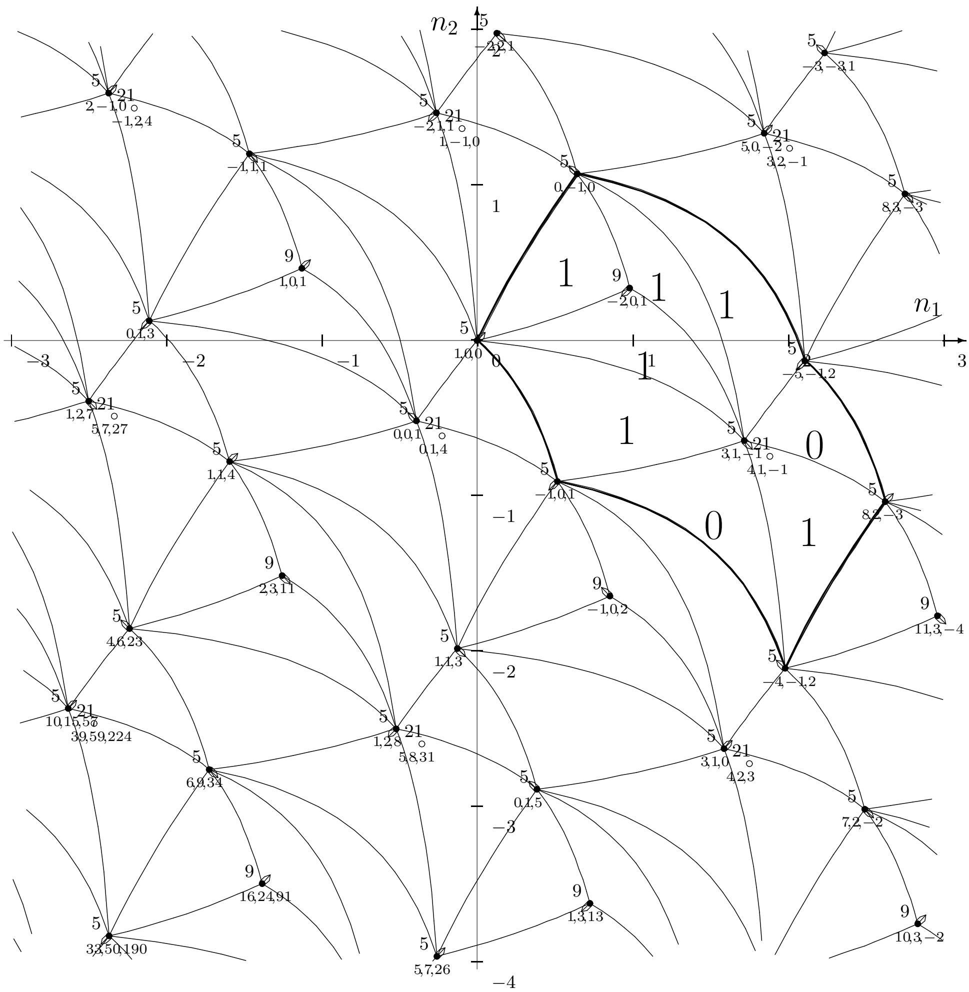


Figure 8

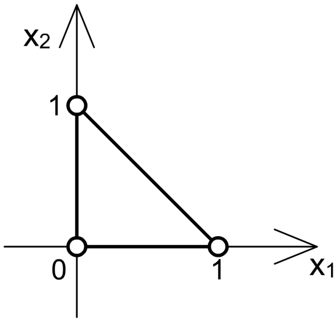


Figure 9

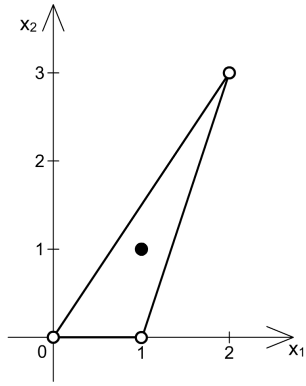
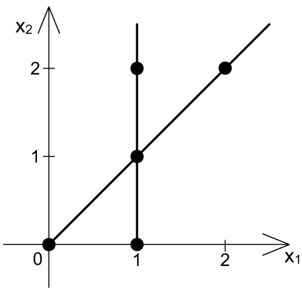


Figure 10



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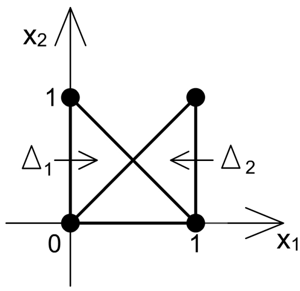

Figure 12

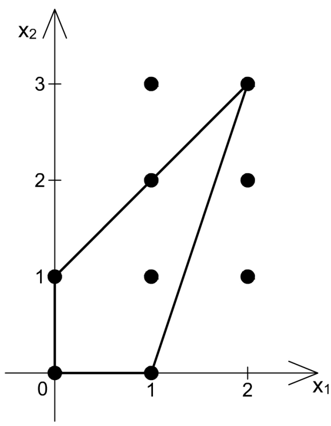

Figure 13

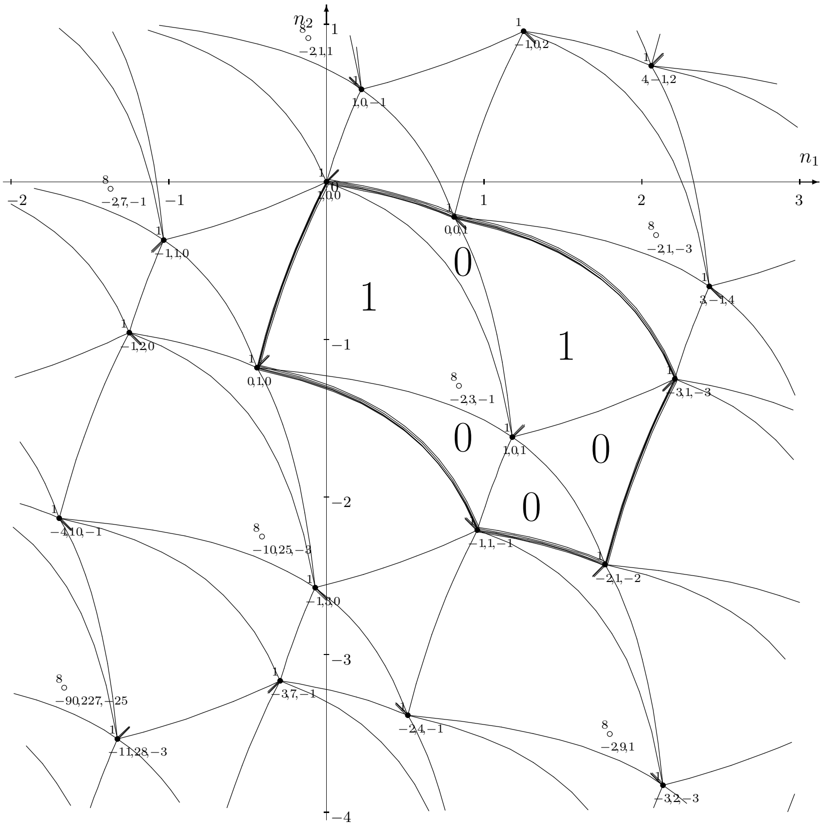

Figure 14

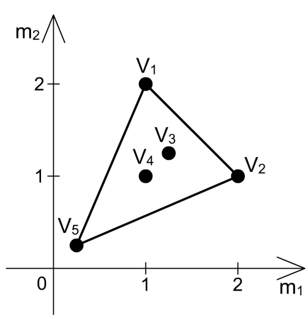

Figure 15a

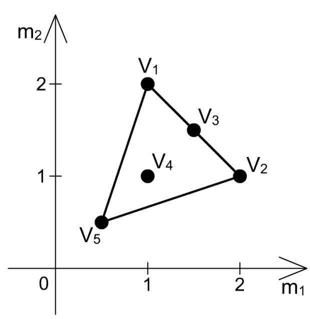

Figure 15b

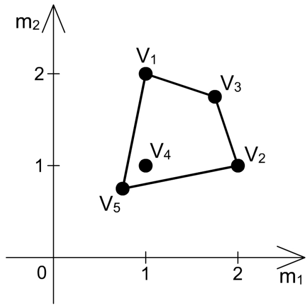

Figure 15c

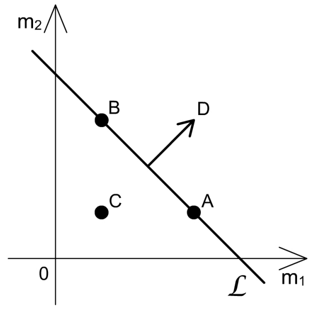

Figure 16

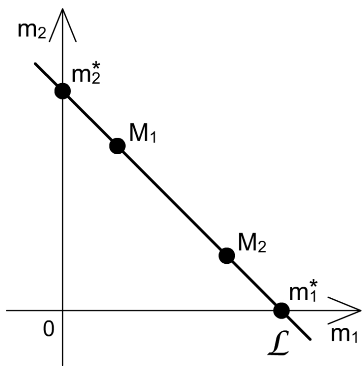

Figure 17

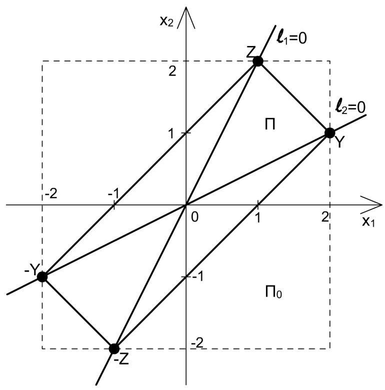

Figure 18

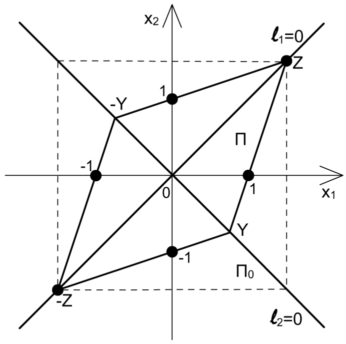

Figure 19

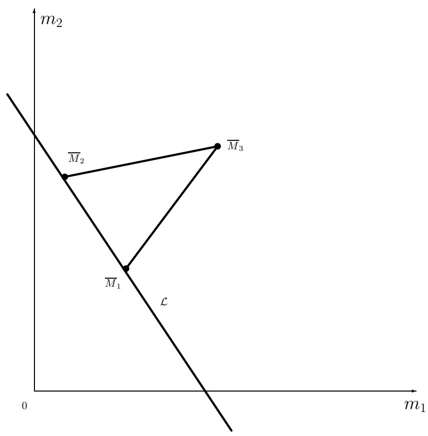

Figure 20

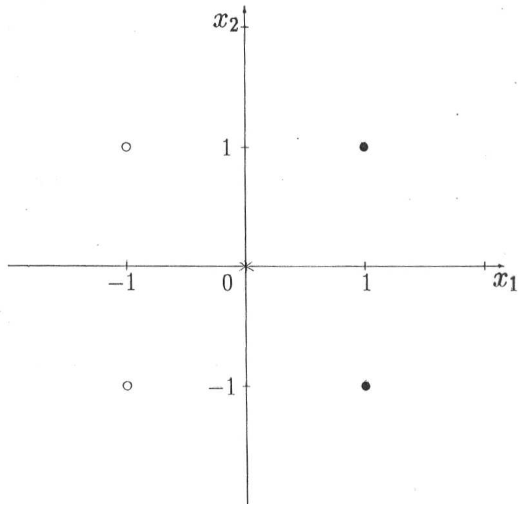

Figure 21

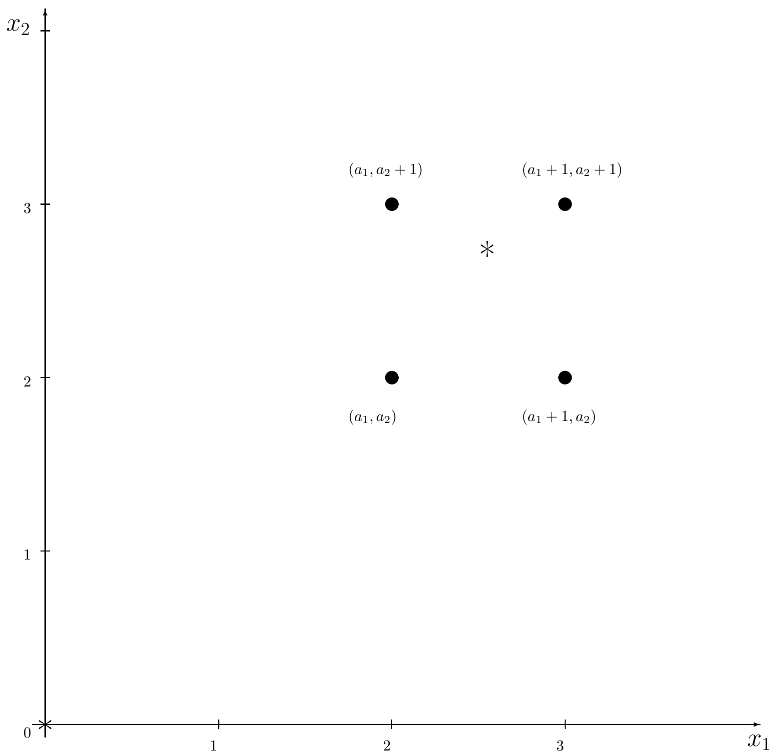

Figure 22

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