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ON THE DIOPHANTINE EQUATION $X^2 - (p^{2m} + 1)Y^6 = -p^{2m}$

Bo He, Alain Togbé, Pingzhi Yuan

Abstract: Let p be a prime and m a positive integer. In this paper, it is shown that the equation in the title has at most four solutions in positive integers (X, Y).

Keywords: algebraic approximations, Thue's equations, elliptic curves.

1. Introduction

In [9] - [14], Ljunggren obtained absolute upper bounds for the number of positive integer solutions to equations of the form

$$aX^4 - bY^2 = c, \tag{1}$$

where $c \in \{\pm 1, -2, \pm 4\}$. One can rewrite equation (1) into the form

$$X^2 - dY^4 = k. \tag{2}$$

Particularly, in [11], Ljunggren proved that the quartic equation

$$X^2 - 2Y^4 = -1 \tag{3}$$

has only the positive integer solutions $(x, y) = (1, 1), (239, 13)$. Also, in [14], he proved that the only positive integer solution to

$$X^2 - 5Y^4 = -4 \tag{4}$$

is $(X, Y) = (1, 1)$.

In 2008, the first, second authors, and Walsh [6] used a result of Akhatari to generalize the equations (3), (4) and to prove that the Diophantine equation

$$X^2 - (2^{2m} + 1)Y^4 = -2^{2m} \tag{5}$$

has at most 12 solutions in odd positive integers X, Y. In 2009, their result was improved by Stoll, Walsh, and the third author who showed that equation (5) has

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at most 3 solutions in odd positive integers X, Y. In the same spirit, Yuan and Zhang [19, 20, 21] considered other generalizations of equation (5). In fact, under certain conditions, they found sharp bounds for the number of solutions of the following Diophantine equations

$$X^2-(a^2+p^{2n})Y^4=-p^{2n}, \tag{6}$$

$$X^2-(a^2+4p^{2n})Y^4=-4p^{2n}, \tag{7}$$

$$X^2-(1+a^2)Y^4=-2a. \tag{8}$$

In fact, they proved that if $a, n \geqslant 1$ are integers and $p \geqslant 3$ is a prime such that $\gcd(a,p)=1$ and if the diophantine equation $x^2-(a^2+p^{2n})y^2=-1$ has a solution, then equation (6) has at most two coprime positive integer solutions (X, Y). For equation (7), the conditions are: $a, n \geqslant 1$ are integers and $p \geqslant 3$ is a prime such that $\gcd(a, 2p)=1$, the diophantine equation $x^2-(a^2+4p^{2n})y^2=-1$ has a solution, and the equation $u^2-(a^2+4p^{2n})u^2=4$ has no coprime solution. In the same way, they showed that if $a \geqslant 1$ is an integer, then equation (8) has at most 3 solutions in positive integers X, Y. The method is the hypergeometric method that is based on Padé approximations using hypergeometric functions. It is a successful method having many flavors. For examples, one can see [1], [3]-[5], [8], [15]-[21].

The aim of this present paper is to consider the family of equations

$$X^2-(p^{2m}+1)Y^6=-p^{2m} \tag{9}$$

and to prove the following result.

Theorem 1.1. *Let p be a prime and m a positive integer. Then the equation $X^2-(p^{2m}+1)Y^6=-p^{2m}$ has at most four solutions in positive integers (X, Y).*

The paper is organized as follows. In Section 2, we will recall some results related to the hypergeometric method. Moreover, we will improve Yuan's result in [17, 18] and adapt it to degree 6. In Section 3, we prove some preliminary results related to the solutions. The last section is devoted to the proof of Theorem 1.1.

2. Effective algebraic approximations of algebraic numbers

In this section, like the third author [17, 18] and Voutier [16], we give some effective irrationality measures for numbers over imaginary quadratic fields. We will apply these results to prove the main result of the present paper in Section 4.

Let $D>0$ be a positive integer, x_0, y_0 rational numbers such that $|x_0|>\sqrt{3}|y_0|\sqrt{D}$ and $z=x_0-y_0\sqrt{-D}$ and $u=x_0+y_0\sqrt{-D}$ are algebraic integers of the field $\mathbb{Q}(\sqrt{-D})$. Put $\omega=z/u$, then it is easy to see that $\omega=e^{i\varphi}$ with $0<|\varphi|<\pi/3$ and $|\omega-1|<1$.

Suppose that m, n are positive integers with $0<m<n$, $(m,n)=1$, $v=m/n$. Put $\omega^v=e^{iv\varphi}$, $\sqrt{\omega}=e^{i\varphi/2}$ and

$$\mu_n=\prod_{p \mid n} p^{1/(p-1)},$$

$$\varepsilon_1 = \sqrt{x_0^2 + y_0^2 D} + |x_0|, \qquad \varepsilon_2 = \sqrt{x_0^2 + y_0^2 D} - |x_0|.$$

In this section we recall some basic definitions and results on hypergeometric functions. Suppose that α, β, γ are given complex numbers. The hypergeometric function $F(\alpha, \beta, \gamma, z)$ is defined to be the power series of the complex variable z as

$$F(\alpha, \beta, \gamma, z) = 1 + \sum_{i=1}^{\infty} \left(\prod_{j=0}^{i-1} \frac{(\alpha+j)(\beta+j)}{(\gamma+j)} \right) \frac{z^i}{i!}.$$

It is easy to see that the radius of convergence of $F(\alpha, \beta, \gamma; z)$ is 1. Let r be a positive integer, v a real number with $0 < v < 1$. Put

$$Y_r(z) = F(-r-v, -r, , 1-v, z), \qquad X_r(z) = z^r Y_r(z^{-1})$$

and

$$R_r(z) = \frac{\Gamma(r+1+v)}{r!\Gamma(v)} \int_1^z (1-t)^r (t-z)^r t^{-r-1+v} dt,$$

where the path of integration does not pass through 0, and $(1-u)^{-r-1+v} = 1$ for $u = 0$.

We observe that $w = \frac{x_0 - y_0\sqrt{-D}}{x_0 + y_0\sqrt{-D}}$. So the following lemma is a slight extension of Lemma 3.1 in [18]. For the proof, we refer to Lemmas 2.3, 2.5, and 2.6 of [4], Lemmas 1 and 2 of [17].

Lemma 2.1.

(i) *If* $|\omega - 1| < 1$, *then we have*

$$\omega^v X_r(\omega) - Y_r(\omega) = R_r(\omega)$$

and

$$X_r(\omega) Y_{r+1}(\omega) \neq X_{r+1}(\omega) Y_r(\omega).$$

(ii) *We have*

$$|R_r(\omega)| \leqslant \frac{\Gamma(r+1+v)}{r!\Gamma(v)} |\varphi| |1 - \sqrt{\omega}|^{2r}$$

and

$$|X_r(\omega)| = |Y_r(\omega)| \leqslant \frac{4r!\Gamma(1-v)}{\Gamma(r+1-v)} |1 + \sqrt{\omega}|^{2r-2}.$$

Let $r \in \mathbb{N}$, $\Delta_{n,r}$ be the least common denominator of the coefficients of $X_r(z)$ and $Y_r(z)$, $N_{n,r}$ the greatest common divisor of the numerators of the coefficients of $X_r(1 - n\mu_n z)$ and $Y_r(1 - n\mu_n z)$, and S_r the n-part of $r!$ (as defined in [5]). Then we have.

Lemma 2.2. (Proposition 5.1, [5]; Proposition 2, [8])

(i) $N_{n,r}$ *is divisible by* $n^r S_r$.

(ii) *For $n = 6$, we have*

$$\frac{16^r \Delta_{6,r}\Gamma(5/6)r!}{N_{6,r}\Gamma(r+5/6)} < 1.2e^{2.56r}, \quad \frac{27^r \Delta_{6,r}\Gamma(r+7/6)}{N_{6,r}\Gamma(1/6)r!} < 0.16e^{3.09r}.$$

Now, we can prove the following result.

Theorem 2.1. *Let A, B be nonzero integers with*

$$|A| > \sqrt{3}|B|, \qquad 2e^{3.09}(\sqrt{A^2+B^2} - |A|)/27 < 1,$$

and let

$$\omega = \frac{A - Bi}{A + Bi}, \qquad \varepsilon_1 = \sqrt{A^2+B^2} + |A|, \qquad \varepsilon_2 = \sqrt{A^2+B^2} - |A|,$$

$$w_1 = e^{2.56}\varepsilon_1/8, \qquad w_2 = 2e^{3.09}\varepsilon_2/27.$$

Then for any nonzero algebraic integers p, q of $\mathbb{Q}(i)$ with

$$\frac{|qB|}{2|A|} > C_1, \qquad 0 < C_1 \leqslant \frac{25}{16},$$

we have

$$|q\omega^{1/6} - p| > \frac{1 - w_2}{C|q|^\lambda},$$

where

$$\lambda = |\log w_1 / \log w_2|, \qquad C = 1.3 w_1 (w_1 - w_2)|f|^\lambda \tag{10}$$

and

$$f \geqslant \frac{|B|}{2C_1|A|}\frac{(w_1 - w_2)}{(w_1 - 1)}. \tag{11}$$

Proof. Let $u = A + Bi$, $v = 1/6$, $\omega = e^{i\phi}$, then $|\phi| < \pi/3$ since $|A| > \sqrt{3}|B|$, thus $|\omega - 1| < 1$ and $|\phi| = 2|\arctan\frac{B}{A}| < \frac{2|B|}{|A|}$.

Put

$$A_{6,r} = \frac{\Delta_{6,r}}{N_{6,r}}X_r(\omega)u^r, \qquad B_{6,r} = \frac{\Delta_{6,r}}{N_{6,r}}Y_r(\omega)u^r, \tag{12}$$

$$R_{6,r} = \frac{\Delta_{6,r}}{N_{6,r}}R_r(\omega)u^r.$$

Then by Lemmas 2.1, 2.2 and the definition of $\Delta_{6,r}$ we know that $A_{6,r}$ and $B_{6,r}$ are algebraic integers of $\mathbb{Q}(i)$, and

$$|A_{6,r}| = |B_{6,r}| \leqslant \frac{4\Delta_{6,r}\Gamma(5/6)r!}{N_{6,r}\Gamma(r+5/6)}|u|^r|1+\sqrt{\omega}|^{2r-2} < 1.3w_1^r, \tag{13}$$

$$|R_{6,r}| \leqslant \frac{\Delta_{6,r}\Gamma(r+7/6)}{N_{6,r}\Gamma(1/6)r!}|u|^r|1-\sqrt{\omega}|^{2r}|\phi| < 0.16w_2^r|\phi|. \tag{14}$$

In fact, we have used $|u|^r|1+\sqrt{\omega}|^{2r}=(2\varepsilon_1)^r$, $|u|^r|1-\sqrt{\omega}|^{2r}=(2\varepsilon_2)^r$, and $|1+\sqrt{\omega}|^2=4\cos^2\frac{\phi}{4}>4\cos^2\frac{\pi}{12}=2+\sqrt{3}>3.73$.

Let $R=q\omega^v-p$, we have

$$X_r(\omega)R=q\omega^vX_r(\omega)-pX_r(\omega).$$

Since $\omega^vX_r(\omega)=Y_r(\omega)+R_r(\omega)$, we have

$$X_r(\omega)R=qY_r(\omega)-pX_r(\omega)+qR_r(\omega). \tag{15}$$

We multiply both sides of (15) by $\frac{\Delta_{6,r}}{N_{6,r}}u^r$ and we put

$$\Delta_r=qB_{6,r}-pA_{6,r},$$

where $A_{6,r}$ and $B_{6,r}$ are defined by (12). Then we obtain

$$A_{6,r}R=\Delta_r+qR_{6,r}. \tag{16}$$

Notice that Δ_r is an algebraic integer of $\mathbb{Q}(i)$. So if $\Delta_r\neq 0$, then $|\Delta_r|\geqslant 1$. By Lemma 2.1 we have that $X_rY_{r+1}\neq X_{r+1}Y_r$, and by the definitions of $A_{6,r}$ and $B_{6,r}$ we have $A_{6,r}B_{6,r+1}\neq A_{6,r+1}B_{6,r}$. Further it is easy to see that if $pq\neq 0$, then at least one of Δ_r and Δ_{r+1} is not zero. Since $|fq|>1, 0<w_2<1$, we can define r_0 to be the positive integer with

$$w_2^{1-r_0}\leqslant |fq|<w_2^{-r_0}.$$

Let $r=r_0$ or r_0+1 with $\Delta_r\neq 0$, then using (13) and (14) we have

$$1.3w_1^r|R|>1-0.16|q\phi|w_2^r.$$

It follows from the choice of r that if $r=r_0$, then

$$\begin{aligned}1.3w_1^{r_0}|R|&\geqslant 1-0.16\times\frac{2|Bq|}{|A|}w_2^{r_0}>1-0.16\times 4C_1|qf|\frac{w_1-1}{w_1-w_2}w_2^{r_0}\\&>1-0.64C_1\frac{w_1-1}{w_1-w_2}>\frac{1-w_2}{w_1-w_2};\end{aligned} \tag{17}$$

and if $r=r_0+1$, then

$$1.3w_1^{r_0+1}|R|\geqslant 1-0.64C_1\frac{(w_1-1)w_2}{w_1-w_2}>\frac{w_1(1-w_2)}{w_1-w_2}. \tag{18}$$

From (17) and (18) we obtain

$$|R|>\frac{1-w_2}{1.3w_1^{r_0}(w_1-w_2)}.$$

Now $(w_1)^{r_0-1}=(w_2)^{\lambda(1-r_0)}\leqslant |fq|^\lambda$, where $\lambda=|\log w_1/\log w_2|$. Therefore we have concluded

$$|q\omega^{1/6}-p|>\frac{1-w_2}{C|q|^\lambda},$$

where $C=1.3w_1(w_1-w_2)|f|^\lambda$. This is the desired result of Theorem 2.1. ■

3. Preliminary Results

We begin our analysis with the following useful observation.

Lemma 3.1. *Let p be a prime and m a positive integer. If $(X,Y)\neq(1,1)$ is a solution in positive integers to*

$$X^2-(p^{2m}+1)Y^6=-p^{2m},$$

then we have

$$\pm X\pm p^m i=(1+p^m i)(s\pm ri)^6, \qquad Y=s^2+r^2,$$

for some coprime non-negative integers r and s.

Proof. All coprime integer solutions (x,y) to the quadratic equation

$$x^2-(p^{2m}+1)y^2=-p^{2m}$$

are given by

$$x+y\sqrt{1+p^{2m}}=\pm(\pm1+\sqrt{1+p^{2m}})(p^m+\sqrt{1+p^{2m}})^{2j} \tag{19}$$

for some integer j, see Theorems 11.4.1 and 11.4.2 in [7].

For brevity, let $b=p^m$ and $\alpha=T+U\sqrt{1+b^2}=b+\sqrt{1+b^2}$. For $j\geqslant 0$, we define sequences $\{T_j\}$ and $\{U_j\}$ by

$$\alpha^j=T_j+U_j\sqrt{1+b^2}.$$

Therefore, a solution in positive integers $(X,Y)\neq(1,1)$ to $X^2-(p^{2m}+1)Y^6=-p^{2m}$ is equivalent to a solution to

$$Y^3=T_{2k}\pm U_{2k}, \qquad X=(1+b^2)U_{2k}\pm T_{2k} \tag{20}$$

for some $k\geqslant 1$, since $(1+b^2)U_{2k}>T_{2k}>U_{2k}$.

By the well known identities $T_{2k}=T_k^2+(1+b^2)U_k^2$ and $U_{2k}=2T_kU_k$, equation (20) shows that

$$Y^3=(T_k\pm U_k)^2+(bU_k)^2,$$

and the terms involved in this equality are pairwise coprime since $b=p^m$ and $\gcd(T_k,U_k)=1$. There exist coprime integers s,u such that $Y=s^2+u^2$ and

$$(T_k\pm U_k)+bU_k\sqrt{-1}=(s+u\sqrt{-1})^3.$$

It follows that

$$T_k\pm U_k=s(s^2-3u^2) \qquad \text{and} \qquad bU_k=u(3s^2-u^2).$$

Now, from

$$\begin{aligned} X &= (1+b^2)U_{2k} \pm T_{2k} = \pm\left(T_k^2+(1+b^2)U_k^2\right) + 2(1+b^2)T_kU_k \\ &= \pm(T_k \pm U_k)^2 + b^2U_k(\pm U_k + 2T_k) \\ &= \pm(T_k \pm U_k)^2 + bU_k(2b(T_k \pm U_k) \mp bU_k) \\ &= \pm s^2(s^2-3u^2)^2 + u(3s^2-u^2)\left(2bs(s^2-3u^2) \mp u(3s^2-u^2)\right) \\ &= \pm s^6 + 6bs^5u \mp 15s^4u^2 - 20bs^3u^3 \pm 15s^2u^4 + 6bsu^5 \mp u^6, \end{aligned}$$

and by taking $r=\mp u$, we get

$$\pm X = s^6 - 6bs^5r - 15s^4r^2 + 20bs^3r^3 + 15s^2r^4 - 6bsr^5 - r^6.$$

It follows that

$$2|X| = |(1+bi)(s+ri)^6 + (1-bi)(s-ri)^6|.$$

Since $X^2+b^2=(1+b^2)y^6=(1+b^2)(s^2+r^2)^6$, we obtain

$$(1+bi)(s+ri)^6 - (1-bi)(s-ri)^6 = \pm 2bi.$$

This completes the proof of Lemma 3.1. ∎

Lemma 3.2. *Let the assumptions be as in Lemma 3.1 and $T+U\sqrt{1+b^2}=b+\sqrt{1+b^2}$ the fundamental solution of the diophantine equation $x^2-(1+b^2)y^2=-1$. Suppose that (X,Y) is a coprime positive integer solution to $X^2-(1+b^2)Y^6=-b^2$, $Y^3=T_{2k}\pm U_{2k}$, $k\geqslant 1$. Then $Y^3>4b^4$.*

Proof. If $k=1$, then we have $Y^3=T_2\pm U_2=2b^2+1\pm 2b=b^2+(b\pm 1)^2$. There exist integers u,v such that

$$b = v(v^2-3u^2), \qquad b\pm 1 = u(3v^2-u^2),$$

and so $v^3-3v^2u-3u^2v+u^3=(u+v)(u^2-4uv+v^2)=(u+v)((u+v)^2-6uv)=\mp 1$. Then we have $u+v=\pm 1$, and $-6uv=0$ or -2. This implies $u=0$ (notice v is a divisor of b). It follows that $b=\pm 1$, then we have a contradiction to our assumption.

Otherwise, $k>1$ and then we get

$$Y^3 = T_{2k} \pm bU_{2k} = (T_k \pm bU_k)^2 + (bU_k)^2 > (bU_2)^2 = 4b^2T^2U^2 > 4b^4.$$

This proves Lemma 3.2. ∎

Now, suppose that $(X,Y)\neq(1,1)$ is a solution in coprime positive integers to $X^2-(1+b^2)Y^6=-b^2$. By Lemma 3.1, there exist integers r,s such that

$$\pm X \pm bi = (1+bi)(s\pm ri)^6, \qquad Y = r^2+s^2.$$

We will assume that
$$X \pm bi = (1+bi)(s+ri)^6 \tag{21}$$
as the argument for the other cases are identical. It follows that
$$(1+bi)(s+ri)^6 - (1-bi)(s-ri)^6 = \pm 2bi. \tag{22}$$
Let $\bar{\omega} = \frac{1-bi}{1+bi} = e^{i\theta}, \bar{\omega}^{1/6} = e^{i\theta/6}$. Using Lemma 3.2 and equality (22) we have
$$\left|\bar{\omega} - \left(\frac{s+ri}{s-ri}\right)^6\right| = \frac{2b}{\sqrt{1+b^2}Y^3} < \frac{2}{Y^3} < \frac{2}{4b^4} \leqslant \frac{1}{32}. \tag{23}$$
Let $\eta \in \{\pm 1, \pm\varrho, \pm\varrho^2\}$ be the algebraic integer such that
$$\left|\bar{\omega}^{1/6} - \eta\frac{s+ri}{s-ri}\right| = \min_{0\leqslant k\leqslant 5}\left|\bar{\omega}^{1/6} - e^{k\pi i/3}\frac{s+ri}{s-ri}\right|.$$
By (23), we may assume that
$$\left|\bar{\omega}^{1/6} - \eta\frac{s+ri}{s-ri}\right| \leqslant 0.01. \tag{24}$$
In fact, one can see that
$$\left|\bar{\omega}^{1/6} - \eta\frac{s+ri}{s-ri}\right| < \left(\frac{1}{32}\right)^{\frac{1}{6}} < 0.57.$$
Since
$$\begin{aligned}\left|\bar{\omega} - \left(\frac{s+ri}{s-ri}\right)^6\right| &= \left|\bar{\omega}^{1/6} - \eta\frac{s+ri}{s-ri}\right| \times \left|\bar{\omega}^{1/6} - \eta\frac{s+ri}{s-ri} + 2\eta\frac{s+ri}{s-ri}\right| \\ &\times \left|\bar{\omega}^{1/6} - \eta\frac{s+ri}{s-ri} + (1+\varrho)\eta\frac{s+ri}{s-ri}\right| \\ &\times \left|\bar{\omega}^{1/6} - \eta\frac{s+ri}{s-ri} + (1-\varrho)\eta\frac{s+ri}{s-ri}\right| \\ &\times \left|\bar{\omega}^{1/6} - \eta\frac{s+ri}{s-ri} + (1+\varrho^2)\eta\frac{s+ri}{s-ri}\right| \\ &\times \left|\bar{\omega}^{1/6} - \eta\frac{s+ri}{s-ri} + (1-\varrho^2)\eta\frac{s+ri}{s-ri}\right|,\end{aligned}$$
it follows that
$$\begin{aligned}\left|\bar{\omega} - \left(\frac{s+ri}{s-ri}\right)^6\right| &\geqslant (2-0.57)(\sqrt{3}-0.57)^2(1-0.57)^2\left|\bar{\omega}^{1/6} - \eta\frac{s+ri}{s-ri}\right| \\ &\geqslant 0.35\left|\bar{\omega}^{1/6} - \eta\frac{s+ri}{s-ri}\right|,\end{aligned}$$

and so
$$\left|\bar{\omega}^{1/6}-\eta\frac{s+ri}{s-ri}\right|<\frac{1}{32}\times\frac{1}{0.35}<0.09.$$
We apply the above process one more time and we get
$$\left|\bar{\omega}^{1/6}-\eta\frac{s+ri}{s-ri}\right|<0.01.$$
Thus we have
$$\begin{aligned}\left|\bar{\omega}-\left(\frac{s+ri}{s-ri}\right)^6\right| &\geqslant (2-0.01)(\sqrt{3}-0.01)^2(1-0.01)^2\left|\bar{\omega}^{1/6}-\eta\frac{s+ri}{s-ri}\right|\\ &\geqslant 5.78\left|\bar{\omega}^{1/6}-\eta\frac{s+ri}{s-ri}\right| \qquad (25)\end{aligned}$$
and
$$\left|\bar{\omega}^{1/6}-\eta\frac{s+ri}{s-ri}\right|<\frac{1}{2.89Y^3}. \qquad (26)$$
We see that each integer solution $(X,Y)\neq(1,1)$ to $X^2-(1+b^2)Y^6=-b^2$ is related to one of sixth root of unity $\eta\in\{\pm1,\pm\varrho,\pm\varrho^2\}$ as one can see by (26).

Lemma 3.3. *Let the assumptions be as in Lemma 3.1. Suppose that (X_1, Y_1) and (X_2, Y_2) are two solutions in coprime positive integers to $X^2-(1+b^2)Y^6=-b^2$ and $Y_2>Y_1>1$. If (X_1,Y_1) and (X_2,Y_2) are related to $\pm\eta$, for a fixed η, then*
$$Y_2>8Y_1^5.$$

Proof. We know from Lemma 3.1 that there exist integers r_1,s_1,r_2,s_2 such that
$$\pm X_j\pm bi=(1+bi)(s_j\pm r_ji)^6,\qquad Y_j=r_j^2+s_j^2,\qquad (j=1,2). \qquad (27)$$
Also, inequality (26) implies
$$\left|\bar{\omega}^{1/6}-\eta_j\frac{s_j+r_ji}{s_j-r_ji}\right|<\frac{1}{2.89Y_j^3},\qquad (j=1,2).$$
Thus we have the following inequality
$$\begin{aligned}\left|\eta_1\frac{s_1+r_1i}{s_1-r_1i}-\eta_2\frac{s_2+r_2i}{s_2-r_2i}\right| &\leqslant \left|\bar{\omega}^{1/6}-\eta_1\frac{s_1+r_1i}{s_1-r_1i}\right|+\left|\bar{\omega}^{1/6}-\eta_2\frac{s_2+r_2i}{s_2-r_2i}\right|\\ &\leqslant \frac{1}{2.89Y_1^3}+\frac{1}{2.89Y_2^3}. \qquad (28)\end{aligned}$$
Since the two solutions (X_1,Y_1) and (X_2,Y_2) are related to $\pm\eta$, then $\eta_1/\eta_2=\pm1$. If
$$\eta_1\frac{s_1+r_1i}{s_1-r_1i}=\eta_2\frac{s_2+r_2i}{s_2-r_2i},$$

then

$$\eta_1 \frac{(s_1 + r_1 i)^2}{Y_1} = \eta_2 \frac{(s_2 + r_2 i)^2}{Y_2}.$$

This implies that

$$\frac{(s_1 + r_1 i)^6}{Y_1^3} = \pm \frac{(s_2 + r_2 i)^6}{Y_2^3}.$$

By (27), we have

$$(\pm X_1 \pm bi)Y_2^3 = \pm(\pm X_2 \pm bi)Y_1^3.$$

Identifying the real parts and the imaginary parts, we get a contradiction. Therefore, we obtain

$$|\eta_1| \left| \frac{s_1 + r_1 i}{s_1 - r_1 i} - \eta_2/\eta_1 \frac{s_2 + r_2 i}{s_2 - r_2 i} \right| = \left| \frac{s_1 + r_1 i}{s_1 - r_1 i} \pm \frac{s_2 + r_2 i}{s_2 - r_2 i} \right| \geqslant \frac{2}{\sqrt{Y_1 Y_2}}. \tag{29}$$

By inequalities (28) and (29), we obtain

$$\frac{2}{\sqrt{Y_1 Y_2}} < \frac{1}{2.39 Y_1^3} + \frac{1}{2.39 Y_2^3} < \frac{2}{2.39 Y_1^3}$$

Then we deduce that

$$Y_2 > 8 Y_1^5.$$

This completes the proof of Lemma 3.3. ∎

4. Proof of the main theorem

As we know, the sixth roots of unity are three pairs, that ± 1, $\pm\rho$ and $\pm\rho^2$, with $\rho = \frac{1+\sqrt{-3}}{2}$. As discussed in Section 3, any solution $(X, Y) \neq (1, 1)$ of equation (9) is related to a $\eta \in \{\pm 1, \pm\rho, \pm\rho^2\}$. We will prove that only one solution is related to a pair $\pm\eta$.

Indeed, suppose that (X_1, Y_1) and (X_2, Y_2) are two coprime positive integer solutions to $X^2 - (1 + b^2)Y^6 = -b^2, Y_2 > Y_1 > 1$, and both related to a same $\pm\eta$, where $b = p^m$. By Lemma 3.1, there exist integers r_1, s_1, r_2, s_2 such that

$$\pm X_j \pm bi = (1 + bi)(s_j \pm r_j i)^6, \qquad Y_j^2 = r_j^2 + s_j^2, \qquad (j = 1, 2).$$

We will assume that

$$X_1 \pm bi = (1 + bi)(s_1 + r_1 i)^6, \qquad X_2 \pm bi = (1 + bi)(s_2 + r_2 i)^6,$$

as the argument for the other cases are identical. Then we have

$$(1 + bi)(s_j + r_j i)^6 - (1 - bi)(s_j - r_j i)^6 = \pm 2bi, \qquad (j = 1, 2).$$

Since $X_1 \pm bi = (1 + bi)(s_1 + r_1 i)^6$, we obtain

$$(X_1 \pm bi)(s_1 - r_1 i)^6 (s_2 + r_2 i)^6 - (X_1 \mp bi)(s_1 + r_1 i)^6 (s_2 - r_2 i)^6 = \pm 2b Y_1^6 i.$$

Define x, y by

$$x+yi=(s_1-r_1i)(s_2+r_2i).$$

It follows that we obtain a new Thue equation

$$\left|(X_1\pm bi)(x+yi)^6-(X_1\mp bi)(x-yi)^6\right|=2bY_1^6. \tag{30}$$

Let $\omega=\frac{X_1\pm bi}{X_1\mp bi}$ and $\eta'\in\{\pm1,\pm\rho,\pm\rho^2\}$ be the algebraic integer such that

$$\left|\omega^{1/6}-\eta'\frac{x-yi}{x+yi}\right|=\min_{0\leqslant k\leqslant 5}\left|\omega^{1/6}-e^{k\pi i/3}\frac{x-yi}{x+yi}\right|.$$

Using an argument similar to inequality (25) and from (30), we have

$$\frac{2b}{\sqrt{1+b^2}Y_2^3}=\frac{2bY_1^6}{|X_1\mp bi||x+yi|^6}=\left|\omega-\left(\frac{x-yi}{x+yi}\right)^6\right|>5.78\left|\omega^{1/6}-\eta'\frac{x-yi}{x+yi}\right|. \tag{31}$$

Since $Y_2^3>4b^4\geqslant 64$, then we deduce that

$$\left|\omega^{1/6}-\eta'\frac{x-yi}{x+yi}\right|<0.01.$$

Let $\bar{\omega}=\frac{1-bi}{1+bi}$. By (24), we get

$$\left|\eta_1\frac{s_1+r_1i}{s_1-r_1i}-\eta_2\frac{s_2+r_2i}{s_2-r_2i}\right|<\left|\bar{\omega}^{1/6}-\eta_1\frac{s_1+r_1i}{s_1-r_1i}\right|+\left|\bar{\omega}^{1/6}-\eta_2\frac{s_2+r_2i}{s_2-r_2i}\right|<0.02,$$

and so

$$\left|\frac{\eta_1}{\eta_2}\frac{x-yi}{x+yi}-1\right|=\left|\frac{\eta_1}{\eta_2}\frac{s_1+r_1i}{s_1-r_1i}\frac{s_2-r_2i}{s_2+r_2i}\right|<0.02.$$

From the definition of ω, we know that $\omega=\frac{X_1\pm bi}{X_1\mp bi}$ and $\omega^{1/n}=e^{i\varphi/n}$. So the fact $0<|\varphi|<\pi/3$ implies $0<|\varphi/6|<\pi/18$. Therefore, we have

$$|\omega^{1/6}-1|=|-1+\cos(\varphi/6)+i\sin(\varphi/6)|=\sqrt{2-2\cos(\varphi/6)}<0.18.$$

Thus we get

$$\left|\omega^{1/6}-\frac{\eta_1}{\eta_2}\frac{x-yi}{x+yi}\right|\leqslant|\omega^{1/6}-1|+\left|\frac{\eta_1}{\eta_2}\frac{x-yi}{x+yi}-1\right|<0.2.$$

If $\eta'\neq\frac{\eta_1}{\eta_2}$, then one has

$$\begin{aligned}\left|\omega^{1/6}-\eta'\frac{x-yi}{x+yi}\right|&=\left|\omega^{1/6}-\frac{\eta_1}{\eta_2}\frac{x-yi}{x+yi}+\left(\frac{\eta_1}{\eta_2}-\eta'\right)\frac{x-yi}{x+yi}\right|\\&\geqslant\left|\frac{\eta_1}{\eta_2}-\eta'\right|-0.2=0.8.\end{aligned}$$

This contradicts the fact the left side less that 0.01. Then we have

$$\eta' = \frac{\eta_1}{\eta_2} = \pm 1.$$

Now, we will apply Theorem 2.1. Let $A = X_1, B = \pm b$. It is easy to see that $A > \sqrt{3}|B|$, $2e^{3.09}(\sqrt{A^2+B^2} - A)/27 < 1$. Thus we put

$$\varepsilon_1 = \sqrt{X_1^2 + b^2} + X_1, \quad \varepsilon_2 = \sqrt{X_1^2 + b^2} - X_1, \quad w_1 = e^{2.56}\varepsilon_1/8, \quad w_2 = 2e^{3.09}\varepsilon_2/27.$$

We may take $q = x + yi$, $p = \eta'(x - yi)$ and choose

$$C_1 = 1, \qquad f = \frac{b}{X_1}.$$

In fact, by Lemma 3.3 and the equality $X_1^2 - (1+b^2)Y_1^6 = -b^2$, one can verify that

$$\frac{|qB|}{2|A|} = \frac{|\pm b(x+yi)|}{2X_1} = \frac{b\sqrt{Y_1Y_2}}{2X_1} > \frac{b\sqrt{8Y_1^5Y_1}}{2\sqrt{1+b^2}Y_1^3} > 1 = C_1$$

and

$$\begin{aligned}\frac{|B|}{2C_1|A|}\frac{(w_1 - w_2)}{(w_1 - 1)} &< \frac{b}{2X_1} \cdot \frac{1.62(\varepsilon_1 - \varepsilon_2)}{1.61\varepsilon_1 - 1} \\ &< \frac{b}{2X_1} \cdot \frac{1.62 \cdot 2X_1}{1.61(\sqrt{X_1^2 + b^2} + X_1) - 1} < \frac{b}{X_1} = f.\end{aligned}$$

Therefore, by Theorem 2.1 we have

$$\left|\omega^{1/6} - \eta' \frac{x - yi}{x + yi}\right| = |\omega^{1/6} - p/q| > \frac{1 - w_2}{C|q|^{1+\lambda}}. \tag{32}$$

It is easy to see that

$$w_1 = e^{2.56}\varepsilon_1/8 = e^{2.56}/8 \cdot (\sqrt{X_1^2 + b^2} + X_1) < (e^{2.56}/8) \cdot 2.1X_1 < 6.81X_1. \tag{33}$$

It implies that

$$C = 1.3w_1(w_1 - w_2)|f|^\lambda < 1.3w_1^2|f|^\lambda < 43.38X_1^2 f^\lambda. \tag{34}$$

From Lemma 3.2 and $X_1^2 - (1+b^2)Y_1^6 = -b^2$ we have

$$X_1^2 - b^2Y_1^6 = Y_1^6 - b^2 > (4b^4)^2 - b^2 > 0,$$

and so $X_1 > bY_1^3 > 4b^5$. Then we get

$$w_2 = 2e^{3.09}\varepsilon_2/27 < 1.63\varepsilon_2 = \frac{1.63b^2}{\sqrt{X_1^2 + b^2} + X_1} < \frac{1.63b^2}{2X_1} < \frac{1.63}{8b^3} \leqslant 0.03. \tag{35}$$

Combining (32), (34), and (35), we obtain

$$|\omega^{1/6}-p/q| > \frac{0.97}{43.38X_1^2 f^{\lambda}|q|^{1+\lambda}}. \tag{36}$$

By the definition of λ and as $X_1 > 4b^5$, we have

$$\begin{aligned}\lambda = \frac{|\log w_1|}{|\log w_2|} = \frac{\log w_1}{-\log w_2} &< \frac{\log(6.81X_1)}{-\log\frac{1.63b^2}{2X_1}} < \frac{\log X_1+1.92}{\log X_1 - 2\log b + 0.2}\\ &< \frac{\log X_1+1.92}{3/5\log X_1+0.2} = \frac{5}{3}\frac{\log X_1+1.92}{\log X_1+1/3} < \frac{5}{3}\left(1+\frac{1.58}{\log X_1+0.34}\right) < 3.\end{aligned}$$

Notice that $|fq|>1$. From (31), (36), and the upper bound of λ, we get

$$\frac{0.97}{43.38X_1^2 f^3|q|^4} < \frac{2b}{5.78Y_2^3}.$$

Using $f=b/X_1$ and $|q|=\sqrt{Y_1Y_2}$, we obtain

$$X_1Y_2 < 15.48b^4Y_1^2.$$

Since $Y_2 > 8Y_1^5$, then

$$X_1Y_1^3 < 1.94b^4.$$

But this and $Y_1^3 > 4b^4$ have a contradiction. This completes the proof of Theorem 1.1.

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Address: Department of Mathematics, ABa Teacher's College, Wenchuan, Sichuan 623000, P.R. China;
Alain Togbé, Department of Mathematics, Purdue University North Central, 1401 S. U.S. 421 Westville, IN 46391 USA;
School of Mathematics, South China Normal University, Guangzhou, 510631 China, P.R. China.

E-mail: bhe@live.cn, atogbe@pnc.edu, mcsypz@mail.sysu.edu.cn

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