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ON CERTAIN GENERALIZED MODULAR FORMS

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Abstract: The main result of this note is a characterization of those generalized modular functions of weight zero on $\Gamma_0(N)$ that have empty divisor, in terms of the growth of the exponents in their q-product expansion.

Keywords: Generalized modular function, q-product expansion, divisor

1. Introduction and statement of results

For $N \in \mathbf{N}$ let $\Gamma_0(N)$ be the usual Hecke congruence subgroup of level N consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 := SL_2(\mathbf{Z})$ with $N|c$.

Let f be a generalized modular form (GMF) of integral weight k on $\Gamma_0(N)$, i.e., f is a holomorphic function on the complex upper half-plane $\mathcal{H}$ which satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi\begin{pmatrix} a & b \\ c & d \end{pmatrix}(cz+d)^k f(z) \qquad \left(\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\right)$$

for some (not necessarily unitary) character $\chi : \Gamma_0(N) \to \mathbf{C}^*$, and which is meromorphic at the cusps. We will also require that $\chi(\gamma) = 1$ for every parabolic $\gamma \in \Gamma_0(N)$ of trace 2.

For more details we refer to [3], where a general study of GMF's was initiated and where a GMF in the above sense was called a PGMF (P for parabolic).

We note that at the cusp infinity such an f has an expansion

$$f(z) = \sum_{n \geqslant h} a(n) q^n \qquad (0 < |q| < \epsilon)$$

where $q = e^{2\pi i z} (z \in \mathcal{H})$, $h \in \mathbf{Z}$ and $\epsilon > 0$.

Contrary to the classical situation where χ is unitary, there exist non-constant GMF's f of weight zero with $div(f) = \emptyset$ whenever the genus of $\Gamma_0(N)$ is at least

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one. Indeed, according to a fundamental result of [3] such f correspond to cusp forms of weight 2 and trivial character, by taking logarithmic derivatives.

Like any complex valued meromorphic function on $\mathcal{H}$ which has period 1, is meromorphic at infinity and does not vanish identically, f has an infinite product expansion

$$f(z) = cq^h \prod_{n\geqslant 1}(1-q^n)^{c(n)}. \tag{1}$$

Here c is a non-zero constant, h is the order of f at infinity and the q-exponents $c(n)\,(n \in \mathbf{N})$ are uniquely determined complex numbers. The infinite product in (1) is convergent in a small neighborhood of $q = 0$ [1,2]. As usual we understand that complex powers are determined by the principal branch of the complex logarithm.

The main result of this note is a characterization of those GMF's of weight zero on $\Gamma_0(N)$ that have empty divisors, in terms of the growth of the exponents $c(n)$.

Theorem. *Let $f \neq 0$ be a GMF of weight zero on $\Gamma_0(N)$. Then $div(f) = \emptyset$ if and only if*

$$c(n) \ll_\epsilon n^{-\frac{1}{2}+\epsilon} \qquad (\epsilon > 0).$$

As a straightforward consequence we obtain

Corollary 1. *Let f be a non-constant GMF of weight zero on $\Gamma_0(N)$ with $div(f) = \emptyset$. Then the $c(n)(n \in \mathbf{N})$ take infinitely many different values.*

The result of Corollary 1 generalizes the main result of [5] where for $N \geqslant 11$ squarefree examples of GMF's f of weight zero on $\Gamma_0(N)$ with empty divisors were constructed such that the $c(n)$ take infinitely many different values. Note that in the Theorem in [5] it is merely stated that $div(f) \subset \mathbf{P}^1(\mathbf{Q})$ for those f, but the proof together with [3, Thm. 2 and Supplement] indeed reveals that $div(f) = \emptyset$.

If f has algebraic Fourier coefficients, then in fact one can sharpen the result of Corollary 1 and prove that the $c(p)$ where p runs over primes only already take infinitely many different values, cf. [7].

Recall that the cusps of $\Gamma_0(N)$ are represented by the numbers $\frac{a}{c}$ where c runs over positive divisors of N, and for given c, a runs through integers with $1 \leqslant a \leqslant N$, $(a,N) = 1$ that are inequivalent modulo $(c, \frac{N}{c})$.

According to [6], we say that a non-zero GMF f of weight k on $\Gamma_0(N)$ satisfies condition (C) if for each $c|N$, the order $ord_{\frac{a}{c}} f$ is independent of a. For example, if N is squarefree condition (C) is always satisfied.

If

$$\Delta(z) = q\prod_{n\geqslant 1}(1-q^n)^{24} \qquad (z \in \mathcal{H})$$

is the discriminant function of weight 12 on Γ_1, then a meromorphic modular form of type

$$\prod_{t|N} \Delta(tz)^{n_t}$$

with integers n_t will be called a Δ-product. (Thus a Δ-product is the 24th power of what usually is called an η-product.) Note that the exponents of a Δ-product take only finitely many different values.

Corollary 2. *Let $f \neq 0$ be a GMF of integral weight k on $\Gamma_0(N)$ and suppose that f satisfies condition (C). Then $div(f) \subset \mathbf{P}^1(\mathbf{Q})$ if and only if*

$$c(n) = \frac{1}{M} d(n) + \mathcal{O}_\epsilon(n^{-\frac{1}{2}+\epsilon}) \qquad (\epsilon > 0)$$

where M is a non-zero integer and the $d(n)(n \in \mathbf{N})$ are the exponents of a Δ-product of weight kM on $\Gamma_0(N)$.

2. Proof of Theorem

We let

$$\theta = \frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}$$

be Ramanujan's θ-operator and set

$$g := \frac{\theta f}{f}.$$

Then g is a meromorphic modular form of weight 2 on $\Gamma_0(N)$ with trivial character, holomorphic at the cusps, and g is a cusp form if and only if f has empty divisor [3]. If $b(n)(n \in \mathbf{N})$ are the Fourier coefficients of g, then the identity

$$b(n) = \begin{cases} h, & \text{if } n = 0 \\ -\sum_{d|n} dc(d), & \text{if } n \geqslant 1 \end{cases} \tag{2}$$

holds [1,2]. Now suppose that $div(f) = \emptyset$. Then by Deligne's estimate

$$b(n) \ll_\epsilon n^{\frac{1}{2}+\epsilon} \qquad (\epsilon > 0).$$

Inverting the second formula in (2) we find

$$c(n) = -\frac{1}{n}\sum_{d|n} \mu\left(\frac{n}{d}\right) b(d) \qquad (n \geqslant 1)$$

and hence

$$c(n) \ll_\epsilon \frac{1}{n}\sum_{d|n} d^{\frac{1}{2}+\epsilon} \ll_\epsilon \frac{1}{n}\cdot n^{\frac{1}{2}+\epsilon}\sigma_0(n) \ll_\epsilon n^{-\frac{1}{2}+2\epsilon}.$$

Now we give the proof in the other direction which is a bit more involved. Suppose that

$$c(n) \ll_\epsilon n^{-\frac{1}{2}+\epsilon} \qquad (\epsilon > 0). \tag{3}$$

Then from (2) we see that the Fourier series of g converges on $\mathcal{H}$, so g is holomorphic on $\mathcal{H}$. Also from (2) and (3) we infer as above that

$$b(n) \ll_\epsilon \sum_{d|n} d^{\frac{1}{2}+\epsilon} \ll_\epsilon n^{\frac{1}{2}+2\epsilon} \qquad (\epsilon > 0).$$

Therefore it will be sufficient to show the following

Proposition. *Let g be a holomorphic modular form of weight 2 on $\Gamma_0(N)$ with trivial character and suppose that its Fourier coefficients $b(n)(n \geqslant 1)$ satisfy*

$$b(n) \ll_\epsilon n^{\frac{1}{2}+\epsilon} \qquad (\epsilon > 0). \tag{4}$$

Then g is a cusp form.

Proof. The space $\mathcal{M}_2(N)$ of holomorphic modular forms of weight 2 on $\Gamma_0(N)$ splits up into a direct sum

$$\mathcal{M}_2(N) = \mathcal{E}_2(N) \oplus S_2(N)$$

where $\mathcal{E}_2(N)$ is the subspace generated by Eisenstein series and $S_2(N)$ is the subspace of cusp forms. Since by Deligne's estimate the Fourier coefficients of cusp forms satisfy (4), we only have to show that if g is in $\mathcal{E}_2(N)$ and g satisfies (4), then $g = 0$.

We let

$$E_2(z) = 1 - 24 \sum_{n \geqslant 1} \sigma_1(n) q^n \qquad (z \in \mathcal{H})$$

be the nearly holomorphic Eisenstein series of weight 2 on Γ_1. For each $t|N$, we define

$$E_{2,t} := E_2 - tE_2|V_t, \tag{5}$$

where V_t is the operator given on functions $h : \mathcal{H} \to \mathbf{C}$ by $(h|V_t)(z) := h(tz)$. Then $E_{2,t}$ is in $M_2(t)$.

If N is squarefree, our claim is easy to see, since in this case as is well-known a basis for $\mathcal{E}_2(N)$ is given by

$$\{E_{2,t} \mid t|N, t > 1\},$$

and one can use induction on the number of prime factors of N, together with $\sigma_1(n) \gg n$ and choosing n in an appropriate and obvious way.

Now let N be arbitrary. One has

$$\dim \mathcal{E}_2(N) = \sigma_\infty(N) - 1$$

where

$$\sigma_\infty(N) = \sum_{t|N} \phi((t, \frac{N}{t}))$$

is the number of cusps of $\Gamma_0(N)$. A basis for $\mathcal{E}_2(N)$ can be constructed as follows, for details we (partly) refer to [8, sect. 4.7].

If χ is a primitive Dirichlet character modulo M with $M > 1$, we put

$$E_{2,\chi}(z) := \sum_{n \geqslant 1} \left(\sum_{d|n} \chi\left(\frac{n}{d}\right) \overline{\chi}(d) d \right) q^n. \tag{6}$$

Then $E_{2,\chi}$ is in $\mathcal{M}_2(M^2)$. Note that the Hecke L-function attached to $E_{2,\chi}$ is

$$L(s,\chi)L(s-1,\overline{\chi}),$$

where $L(s,\chi)$ is the Dirichlet L-function attached to χ.

We have

$$\mathcal{E}_2(N) = \left(\bigoplus_{\chi \ primitive \ mod M, M^2|N, M>1} \mathcal{E}_2^{\chi}(N) \right) \oplus \mathcal{E}_2^{\chi_0}(N) \tag{7}$$

where χ runs over all primitive Dirichlet characters modulo M with $M^2|N, M > 1$ and where

$$\mathcal{E}_2^{\chi}(N) := \bigoplus_{t|\frac{N}{M^2}} \mathbf{C} E_{2,\chi}|V_t,$$

$$\mathcal{E}_2^{\chi_0}(N) := \bigoplus_{t|N, t>1} \mathbf{C} E_{2,t}$$

and $E_{2,t}$ is defined by (5).

If $\mathcal{H}_N$ is the Hecke algebra generated by all Hecke operators T_m with $m \geqslant 1$, $(m,N) = 1$, then each direct summand on the right-hand side of (7) is an eigenspace of $\mathcal{H}_N$, and different eigenspaces have different Hecke characters. Hence for each of these eigenspaces we can find $T \in \mathcal{H}_N$ that acts on this eigenspace by multiplication with a non-zero scalar and annihilates all the other eigenspaces.

Now observe that if g satisfies (4), so does $g|T$ for any $T \in \mathcal{H}_N$, as immediately follows form the well-known action of the T_m on Fourier coefficients.

Hence it is sufficient to take any g satisfying (4) in one of the eigenspaces and to show that $g = 0$.

If a function $g \in \mathcal{E}_2^{\chi_0}(N)$ satisfies (4), then one can argue in a similar way as above to deduce that $g = 0$.

Now let χ be a primitive Dirichlet character modulo M, where $M > 1$ and $M^2|N$ and suppose that the Fourier coefficients of

$$g = \sum_{t|K} \lambda_t E_{2,\chi}|V_t \qquad (\lambda_t \in \mathbf{C})$$

satisfy (4), where we have abbreviated $K := \frac{N}{M^2}$. The arguing is similar as above, but for the reader's convenience we give the details here. By (6) we have

$$\sum_{t|K} \lambda_t \left(\sum_{d|\frac{n}{t}} \chi\left(\frac{n}{td}\right) \overline{\chi}(d) d \right) \ll_\epsilon n^{\frac{1}{2}+\epsilon} \qquad (\epsilon > 0). \tag{8}$$

To prove that $\lambda_t = 0$ for all $t|K$ we use induction on the number $r \geqslant 0$ of prime factors of t, counted with multiplicities. At the r-th step we will show that $\lambda_t = 0$ for all $t|K$ where t has r prime factors.

If $r = 0$, i.e. $t = 1$ we choose $n = p$ a prime with $p \equiv 1 \pmod{N}$. Then from (8) we obtain immediately

$$\lambda_t(1+p) \ll_\epsilon p^{\frac{1}{2}+\epsilon} \qquad (\epsilon > 0).$$

Invoking Dirichlet's Prime Number Theorem and letting p going to infinity, we obtain $\lambda_1 = 0$.

Now suppose that $r \geqslant 1$ and $\lambda_{\tilde{t}} = 0$ had already been shown for all divisors $\tilde{t}$ of K with at most $r-1$ prime factors. Suppose that $t = p_1 \dots p_r$ and take n of the form $n = p_1 \dots p_r \cdot p$, where p is a prime with $p \equiv 1 \pmod{N}$. Then by the induction hypothesis the left-hand side of (8) is equal to

$$\lambda_t\big(1+p\big) \ll_\epsilon p^{\frac{1}{2}+\epsilon} \qquad (\epsilon > 0),$$

hence with p going to infinity we obtain $\lambda_t = 0$. ■

3. Proof of Corollaries

The proof of Corollary 1 is immediate. Indeed, if f is a GMF of weight zero on $\Gamma_0(N)$ with $div(f) = \emptyset$ and the $c(n)$ take only finitely many values, then by the Theorem we must have $c(n) = 0$ for $n \gg 1$. By (2) therefore the $b(n)$ are bounded, hence the Rankin-Selberg zeta function attached to g converges for $Re(s) > 1$. However, the latter has a pole at $s = 2$ with residue (up to a universal constant) equal to the Petersson scalar product $\langle g, g\rangle$. Hence $g = 0$ and so f is constant, a contradiction.

To prove Corollary 2, we proceed as in [4] for N squarefree resp. as in [6] for arbitrary N. Suppose that $div(f) \subset \mathbf{P}^1(\mathbf{Q})$. Then under the condition (C) there exists a non-zero integer M and a Δ-product F of weight kM on $\Gamma_0(N)$ such that $\frac{f^M}{F}$ is a GMF of weight zero on $\Gamma_0(N)$ with empty divisor. Hence our assertion follows from the Theorem.

Conversely, suppose that

$$c(n) = \frac{1}{M} d(n) + \mathcal{O}_\epsilon(n^{-\frac{1}{2}+\epsilon}) \qquad (\epsilon > 0)$$

where the $d(n)$ are the exponents of a Δ-product F of weight kM on $\Gamma_0(N)$. Then

$$G := \frac{f^M}{F}$$

is a GMF of weight zero on $\Gamma_0(N)$ with n-th q-exponents bounded by $n^{-\frac{1}{2}+\epsilon}$ $(\epsilon > 0)$, hence by the Theorem $div(G) = \emptyset$. Since the divisor of F is supported at the cusps, the same must be true for f.

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