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# BILATERAL $q$-SERIES IDENTITIES AND RECIPROCAL FORMULAE

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**Abstract:** By splitting bilateral series into two unilateral series, we derive several reciprocal formulae from Ramanujan's ${}_1\psi_1$ and Bailey's ${}_6\psi_6$ series identities, which generalize the reciprocity theorems due to Ramanujan and Andrews (1981).

**Keywords:** Ramanujan's ${}_1\psi_1$-series identity, Bailey's well-poised ${}_6\psi_6$-series identity, reciprocity theorem.

For two indeterminate $x$ and $q$, the shifted-factorial of $x$ with base $q$ is defined by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}) \qquad \text{for } n \in \mathbb{N}.$$

When $|q| < 1$, we have two well-defined infinite products

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-q^k x) \qquad \text{and} \qquad (x;q)_n = (x;q)_\infty \,/\, (xq^n;q)_\infty\,.$$

The product and fraction of shifted factorials are abbreviated respectively to

$$\begin{aligned}[\,\alpha,\ \beta,\ \cdots,\ \gamma;q\,]_n &= (\alpha;q)_n\,(\beta;q)_n\cdots(\gamma;q)_n\,,\\ \left[\begin{matrix}\alpha, & \beta, & \cdots, & \gamma\\ A, & B, & \cdots, & C\end{matrix}\,\middle|\,q\right]_n &= \frac{(\alpha;q)_n\,(\beta;q)_n\cdots(\gamma;q)_n}{(A;q)_n\,(B;q)_n\cdots(C;q)_n}.\end{aligned}$$

Following Gasper and Rahman [14], the basic hypergeometric series is defined by

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$$
\begin{aligned}
{}_{1+r}\phi_s\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_s \end{matrix}\,\middle|\, q; z\right]
&= \sum_{n=0}^{\infty}\left\{(-1)^n q^{\binom{n}{2}}\right\}^{s-r}\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ q, & b_1, & \cdots, & b_s \end{matrix}\,\middle|\, q\right]_n z^n,\\
{}_r\psi_s\left[\begin{matrix} a_1, & a_2, & \cdots, & a_r \\ b_1, & b_2, & \cdots, & b_s \end{matrix}\,\middle|\, q; z\right]
&= \sum_{n=-\infty}^{+\infty}\left\{(-1)^n q^{\binom{n}{2}}\right\}^{s-r}\left[\begin{matrix} a_1, & a_2, & \cdots, & a_r \\ b_1, & b_2, & \cdots, & b_s \end{matrix}\,\middle|\, q\right]_n z^n;
\end{aligned}
$$

where the base $q$ will be restricted to $|q| < 1$ for nonterminating $q$-series.

In his lost notebook [16, P 40], Ramanujan recorded a beautiful reciprocity theorem, which may be reproduced equivalently as follows.

**Theorem 1.** *For two complex numbers $a,\ c \neq q^{-n}$ with $n \in \mathbb{N}$, then*

$$\rho(a,c) - \rho(c,a) = \Big(\frac{1}{a} - \frac{1}{c}\Big)\frac{[q, qa/c, qc/a; q]_\infty}{[qa, qc; q]_\infty}$$

*where*

$$\rho(a,c) := \Big(1 - \frac{1}{c}\Big)\sum_{n=0}^{\infty}\frac{(-a/c)^n}{(qa;q)_n} q^{\binom{n+1}{2}}.$$

The first published proof of this theorem is due to Andrews [3]. Other proofs can be found in [1, 8, 15, 17]. In particular, Berndt *et al* [6] gave three proofs, one of which being purely combinatorial. In the same paper, Andrews generalized Ramanujan's ${}_1\psi_1$-series identity to a four-free-variable formula, which may be stated in the following more symmetric form.

**Theorem 2 (Andrews [3, Theorem 6]).**

$$
\begin{aligned}
y\sum_{n=0}^{\infty}\frac{[q/ax, bdxy; q]_n}{[by, dy; q]_{n+1}}(ay)^n - x\sum_{n=0}^{\infty}\frac{[q/ay, bdxy; q]_n}{[bx, dx; q]_{n+1}}(ax)^n&\\
= (y-x)\left[\begin{matrix} q, & qx/y, & qy/x, & abxy, & adxy, & bdxy \\ ax, & ay, & bx, & by, & dx, & dy \end{matrix}\,\middle|\, q\right]_\infty&.
\end{aligned}
$$

Liu [19] rederived this relation by applying the $q$-exponential operator to Ramanujan's ${}_1\psi_1$-series identity. As conceived by Andrews [3] and Agarwal [2], the last relation could be deduced from a three-term relation [14, III-33] for ${}_3\phi_2$-series by Sears (1951). This is fulfilled recently by Kang [18], who established also multiparameter generalizations of the quintuple product identity.

The purpose of the present paper is to show reciprocal relations exclusively by writing bilateral basic hypergeometric series in terms of two unilateral ones. First, we shall review a couple of easier reciprocal relations through Ramanujan's ${}_1\psi_1$-series identity. Even though this has been an extensively beaten path (cf. [1, 6, 7, 15, 17]), it seems that the approach through Bailey's well-poised ${}_6\psi_6$-series identity has not been explored. Therefore our next natural task is to employ Bailey's identity on well-poised ${}_6\psi_6$-series to generalize Andrews' identity and derive a common extension of Jacobi's triple product identity and the quintuple product identity. Finally, the three-term relation expressing nonterminating well-poised ${}_8\phi_7$-series in terms of two balanced ${}_4\phi_3$-series will be used to reformulate our generalized reciprocal formula to another one. This last result not only extends again Andrews' theorem, but also remedies the failed attempt recently made by Zhang [20], who proved a false result via the $q$-exponential operator method.

## 1. Reciprocal Formulae from Ramanujan's ${}_1\psi_1$-series Identity

One of the fundamental identities in the theory of basic hypergeometric series is Ramanujan's identity of bilateral ${}_1\psi_1$-series (cf. [11] and [14, II-29]):

$$ {}_1\psi_1\left[\begin{matrix} a \\ c \end{matrix}\Big| q; z\right] = \left[\begin{matrix} q, & c/a, & az, & q/az \\ c, & q/a, & z, & c/az \end{matrix}\Big| q\right]_\infty, \qquad \text{where} \quad |c/a| < |z| < 1. \tag{1} $$

Splitting the bilateral ${}_1\psi_1$-series just displayed into two unilateral series

$$ {}_1\psi_1\left[\begin{matrix} a \\ c \end{matrix}\Big| q; z\right] = \frac{(1-a)z}{1-c}\,{}_2\phi_1\left[\begin{matrix} q, & qa \\ & qc \end{matrix}\Big| q; z\right] + {}_2\phi_1\left[\begin{matrix} q, & q/c \\ & q/a \end{matrix}\Big| q; \frac{c}{az}\right] \tag{2} $$

and then applying Heine's transformation formula [14, III-2]

$$ {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\Big| q; z\right] = \left[\begin{matrix} c/a, & az \\ c, & z \end{matrix}\Big| q\right]_\infty {}_2\phi_1\left[\begin{matrix} abz/c, & a \\ & az \end{matrix}\Big| q; \frac{c}{a}\right] \tag{3} $$

to the first ${}_2\phi_1$-series, we get the following relation

$$ \begin{aligned} &\left[\begin{matrix} q, & c/a, & az, & q/az \\ c, & q/a, & z, & c/az \end{matrix}\Big| q\right]_\infty \\ &\qquad = \frac{(1-a)z}{1-c}\frac{1-c}{1-z}\,{}_2\phi_1\left[\begin{matrix} q, & qaz/c \\ & qz \end{matrix}\Big| q; c\right] + {}_2\phi_1\left[\begin{matrix} q, & q/c \\ & q/a \end{matrix}\Big| q; \frac{c}{az}\right]. \end{aligned} $$

Replacing $a$, $c$, $z$ respectively by $1/a$, $cx$, $c$ and then making some simplification, we obtain the following reciprocity theorem.

**Theorem 3.** *For two complex numbers $a,\ c \neq q^{-n}$ with $n \in \mathbb{N}$ and $|ax|$, $|cx| < 1$, then*

$$ \lambda(a,c) - \lambda(c,a) = \Big(\frac{1}{a} - \frac{1}{c}\Big)\left[\begin{matrix} q, & qa/c, & qc/a, & acx \\ qa, & qc, & ax, & cx \end{matrix}\Big| q\right]_\infty $$

*where*

$$ \lambda(a,c) := \Big(1 - \frac{1}{c}\Big)\sum_{n=0}^{\infty} \frac{(q/cx;q)_n}{(qa;q)_n}(ax)^n. $$

Instead of Heine's transformation (3), applying Jackson's formula [14, III-4]

$$
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix} \,\middle|\, q; z\right] = \frac{(az;q)_\infty}{(z;q)_\infty}\,{}_2\phi_2\left[\begin{matrix} a, \ c/b \\ az, \ c \end{matrix} \,\middle|\, q; bz\right] \tag{4}
$$

to both ${}_2\phi_1$-series displayed in (2), Kang [18, §4] finds another relation

$$
\left[\begin{matrix} q, c/a, az, q/az \\ c, q/a, z, c/az \end{matrix} \,\middle|\, q\right]_\infty = \frac{(1-a)z}{(1-c)(1-z)}\,{}_2\phi_2\left[\begin{matrix} q, c/a \\ qc, qz \end{matrix} \,\middle|\, q; qaz\right] + \frac{1}{1-c/az}\,{}_2\phi_2\left[\begin{matrix} q, \ c/a \\ qc/az, q/a \end{matrix} \,\middle|\, q; \frac{q}{az}\right].
$$

Replacing $a$, $c$, $z$ respectively by $1/a$, $cx$, $c$ and then simplifying the result, we recover another generalization of Ramanujan's reciprocity theorem.

**Theorem 4 (Kang [18, Theorem 4.1]).** *For two complex numbers $a$, $c \neq q^{-n}$ with $n \in \mathbb{N}$ and $|ax|$, $|cx| < 1$, then*

$$
\mu(a,c) - \mu(c,a) = \Big(\frac{1}{a} - \frac{1}{c}\Big)\left[\begin{matrix} q, & qa/c, & qc/a, & acx \\ qa, & qc, & ax, & cx \end{matrix} \,\middle|\, q\right]_\infty
$$

*where*

$$
\mu(a,c) := \Big(1 - \frac{1}{c}\Big)\sum_{n=0}^{\infty} \frac{(acx;q)_n\,(-a/c)^n}{(qa;q)_n\,(ax;q)_{n+1}}\, q^{\binom{n+1}{2}}.
$$

When $x \to 0$, both Theorems 3 and 4 reduce to the reciprocity theorem of Ramanujan anticipated at the beginning of the paper.

## 2. Reciprocal Formulae from Bailey's ${}_6\psi_6$-series Identity

Among the classical hierarchy of basic hypergeometric series identities, the most important one perhaps is the very well-poised ${}_6\psi_6$-series identity discovered by Bailey [4] (see also [12] and [14, II-33]), which may be stated as

$$
{}_6\psi_6\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix} \,\middle|\, q; \frac{qa^2}{bcde}\right] \tag{5a}
$$

$$
= \left[\begin{matrix} q, qa, q/a, qa/bc, qa/bd, qa/be, qa/cd, qa/ce, qa/de \\ qa/b, qa/c, qa/d, qa/e, q/b, q/c, q/d, q/e, qa^2/bcde \end{matrix} \,\middle|\, q\right]_\infty \tag{5b}
$$

provided that $|qa^2/bcde| < 1$ for convergence.

Split the last bilateral ${}_6\psi_6$-series into two unilateral series

$$
{}_6\psi_6\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c & d, & e \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix}\,\middle|\, q;\frac{qa^2}{bcde}\right] \tag{6a}
$$

$$
=\sum_{n=0}^{\infty}\frac{1-aq^{2n}}{1-a}\left[\begin{matrix} b, & c & d, & e \\ qa/b, & qa/c, & qa/d, & qa/e \end{matrix}\,\middle|\, q\right]_n\Big(\frac{qa^2}{bcde}\Big)^n \tag{6b}
$$

$$
-\frac{q(1-q^2/a)(1-a/b)(1-a/c)(1-a/d)(1-a/e)}{a(1-a)(1-q/b)(1-q/c)(1-q/d)(1-q/e)} \tag{6c}
$$

$$
\times\sum_{n=0}^{\infty}\frac{1-q^{2n+2}/a}{1-q^2/a}\left[\begin{matrix} qb/a, & qc/a, & qd/a, & qe/a \\ q^2/b, & q^2/c, & q^2/d, & q^2/e \end{matrix}\,\middle|\, q\right]_n\Big(\frac{qa^2}{bcde}\Big)^n. \tag{6d}
$$

Equating (5b) with (6b-6c-6d) and then relabeling the parameters $a$, $b$, $c$, $d$, $e$ respectively with $qy/x$, $q/bx$, $q/cx$, $q/dx$, $q/ax$, we derive, after some routine simplification, the following general reciprocity theorem.

**Theorem 5 ($|abcdx^2y^2/q|<1$).**

$$
\begin{aligned}
&y\sum_{n=0}^{\infty}\big\{1-q^{2n+1}y/x\big\}\frac{[q/ax,q/bx,q/cx,q/dx;q]_n}{[ay,by,cy,dy;q]_{n+1}}(abcdx^2y^2/q)^n\\
&-x\sum_{n=0}^{\infty}\big\{1-q^{2n+1}x/y\big\}\frac{[q/ay,q/by,q/cy,q/dy;q]_n}{[ax,bx,cx,dx;q]_{n+1}}(abcdx^2y^2/q)^n\\
&=(y-x)\left[\begin{matrix} q,qy/x,qx/y,abxy,acxy,adxy,bcxy,bdxy,cdxy \\ ax,\ ay,\ bx,\ by,\ cx,\ cy,\ dx,\ dy,\ abcdx^2y^2/q \end{matrix}\,\middle|\, q\right]_\infty.
\end{aligned}
$$

In particular, specifying $a$, $c$ respectively by $\sqrt{q/xy}$, $-\sqrt{q/xy}$, we get from the last theorem the following reciprocity theorem.

**Proposition 6 (Reciprocal formula).**

$$
\begin{aligned}
&y\sum_{n=0}^{\infty}\frac{[q/bx,q/dx;q]_n}{[by,dy;q]_{n+1}}(-bdxy)^n-x\sum_{n=0}^{\infty}\frac{[q/by,q/dy;q]_n}{[bx,dx;q]_{n+1}}(-bdxy)^n\\
&\qquad=(y-x)\frac{(bdxy;q)_\infty\,[q^2,q^2y/x,q^2x/y,qb^2xy,qd^2xy;q^2]_\infty}{[bx,by,dx,dy,-bdxy;q]_\infty}.
\end{aligned}
$$

Letting $b\to 0$, $d\to 1$, $x\to b$, $y\to d$ further, we obtain the following reciprocal formula, which is quite different from those displayed in Theorems 1, 3 and 4.

**Corollary 7.** *For two complex numbers $b,\ d\neq q^{-n}$ with $n\in\mathbb{N}$, then*

$$
\varrho(b,d)-\varrho(d,b)=\Big(\frac{1}{b}-\frac{1}{d}\Big)\frac{[q^2,q^2b/d,q^2d/b,qbd;q^2]_\infty}{[qb,qd;q]_\infty}
$$

*where*

$$
\varrho(b,d):=\Big(1-\frac{1}{d}\Big)\sum_{n=0}^{\infty}q^{\binom{n+1}{2}}\frac{(q/d;q)_n}{(qb;q)_n}b^n.
$$

**Proof.** The last reciprocal formula can also be derived from the following transformation due to Andrews [3, Eq 3.27]:

$$\sum_{n=0}^{\infty}\frac{q^n(qbd;q^2)_n}{[qb,qd;q]_n}=\Big(1-\frac{1}{b}\Big)\sum_{n=0}^{\infty}q^{\binom{n+1}{2}}\frac{(q/b;q)_n}{(qd;q)_n}d^n+\frac{(qbd;q^2)_\infty}{b[qb,qd;q]_\infty}\sum_{n=0}^{\infty}q^{2\binom{n+1}{2}}(-d/b)^n.$$

In fact, writing

$$\varrho(b,d)=\sum_{n=0}^{\infty}\frac{q^n(qbd;q^2)_n}{[qb,qd;q]_n}-\frac{(qbd;q^2)_\infty}{d[qb,qd;q]_\infty}\sum_{n=0}^{\infty}q^{2\binom{n+1}{2}}(-b/d)^n$$

and then canceling the symmetric parts in common, we can reformulate the symmetric difference as follows:

$$\varrho(b,d)-\varrho(d,b)=\frac{(qbd;q^2)_\infty}{[qb,qd;q]_\infty}\times\Big\{\frac{1}{b}\sum_{n=0}^{\infty}q^{2\binom{n+1}{2}}(-d/b)^n-\frac{1}{d}\sum_{n=0}^{\infty}q^{2\binom{n+1}{2}}(-b/d)^n\Big\}.$$

Making the replacement $n\to -n-1$ for the second sum and then applying Jacobi's triple product identity, we can factorize the difference displayed in the last line into the following product:

$$\begin{aligned}\frac{1}{b}\sum_{n=0}^{\infty}q^{2\binom{n+1}{2}}(-d/b)^n+\frac{1}{b}\sum_{n=-\infty}^{-1}q^{2\binom{n+1}{2}}(-d/b)^n&\\ =\frac{1}{b}\sum_{n=-\infty}^{+\infty}q^{2\binom{n+1}{2}}(-d/b)^n=\Big\{\frac{1}{b}-\frac{1}{d}\Big\}&[q^2,q^2b/d,q^2d/b;q^2]_\infty.\end{aligned}$$

This confirms the reciprocal relation stated in Corollary 7. ∎

Letting $a\to 0$ in Theorem 5 recovers also the following reciprocal formula.

**Proposition 8 (Kang [18, Theorem 1.2]).**

$$\begin{aligned}&y\sum_{n=0}^{\infty}\{1-q^{2n+1}y/x\}\frac{[q/bx,q/cx,q/dx;q]_n}{[by,cy,dy;q]_{n+1}}q^{\binom{n}{2}}(-bcdxy^2)^n\\ -\,&x\sum_{n=0}^{\infty}\{1-q^{2n+1}x/y\}\frac{[q/by,q/cy,q/dy;q]_n}{[bx,cx,dx;q]_{n+1}}q^{\binom{n}{2}}(-bcdx^2y)^n\\ =\,&(y-x)\left[\begin{matrix}q, & qy/x, & qx/y, & bcxy, & bdxy, & cdxy\\ bx, & by, & cx, & cy, & dx, & dy\end{matrix}\,\Big|\,q\right]_\infty.\end{aligned}$$

As observed by Kang [18, §4], this proposition is equivalent to Andrews' reciprocity displayed in Theorem 2, which is justified by reformulating both sums in the proposition via the limiting case of Watson's transformation formula [14, III-17] from the terminating very well-poised ${}_8\phi_7$-series to the balanced ${}_4\phi_3$-series.

Therefore, Theorem 5 may be considered as a generalization of Theorem 2 due to Andrews.

Performing the replacements $y \to 1$, $a \to y$, $b \to b/xy$, $c \to c/xy$, $d \to d/xy$ and appealing to the following relation

$$(q/a;q)_n = (-1)^n q^{\binom{n+1}{2}} a^{-n} \left(q^{-n}a;q\right)_n$$

we derive from Theorem 5 the following common generalization of Jacobi's triple product identity and quintuple product identity with five free-parameters.

**Theorem 9** **(**$|bcd/qxy^2| < 1$**).**

$$\begin{aligned} &\sum_{n=0}^{\infty} \{1 - q^{2n+1}/x\} \frac{[q/xy, q^{-n}b/y, q^{-n}c/y, q^{-n}d/y; q]_n}{[y, b/xy, c/xy, d/xy; q]_{n+1}} q^{\frac{n(3n+1)}{2}} (-y/x)^n \\ &- x \sum_{n=0}^{\infty} \{1 - q^{2n+1}x\} \frac{[q/y, q^{-n}b/xy, q^{-n}c/xy, q^{-n}d/xy; q]_n}{[xy, b/y, c/y, d/y; q]_{n+1}} q^{\frac{n(3n+1)}{2}} (-x^2y)^n \\ &= \left[ \begin{matrix} q, & x, & q/x, & b, & c, & d, & bc/xy^2, & bd/xy^2, & cd/xy^2 \\ y, & xy, & b/y, & c/y, & d/y, & b/xy, & c/xy, & d/xy, & bcd/qxy^2 \end{matrix} \middle| q \right]_\infty . \end{aligned}$$

This theorem contains the following interesting special cases.

- Berndt *et al* [6, Theorem 3.1]: $b$, $c$, $d \to 0$.
- Bhargava *et al* [7]: $b$, $d \to 0$.
- Kang [18, Theorem 6.1]: $c \to 0$.

They have originally been discovered by different approaches.

In addition, two special cases are worth mentioning. First, letting $b$, $c$, $d \to 0$ in the last theorem, we recover the following limiting case of Bailey's ${}_6\psi_6$-series:

$$\sum_{n=-\infty}^{+\infty} \{1 - q^{2n+1}/x\} \frac{(q/xy;q)_n}{(qy;q)_n} q^{3\binom{n}{2}} (-q^2y/x)^n = \left[ \begin{matrix} q,\ x,\ q/x \\ qy,\ xy \end{matrix} \middle| q \right]_\infty . \tag{7}$$

Then, multiplying both sides of the formula in Theorem 9 by $1 - y$ and letting $y \to 1$ and $x \to 1/x$, we recover the well-known $q$-Dougall sum [14, II-20]:

$$\sum_{n=0}^{\infty} \{1 - xq^{2n+1}\} \frac{[qx, q^{-n}b, q^{-n}c, q^{-n}d; q]_n}{[q, qbx, qcx, qdx; q]_n} q^{3\binom{n}{2}} (-q^2x)^n \tag{8a}$$

$$= \left[ \begin{matrix} qx, & bcx, & bdx, & cdx \\ qbx, & qcx, & qdx, & bcdx/q \end{matrix} \middle| q \right]_\infty , \quad \text{where } |bcdx/q| < 1. \tag{8b}$$

The first identity extends both identities of triple and quintuple product (cf. [9] and [10, 13]), while the last one results in a generalization of Sylvester's identity. For the details, the readers can consult Kang [18, §6].

## 3. Further Reciprocal Relation

Recall the nonterminating three-term transformation formula [14, III-36]:

$$
{}_8\phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & f \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & qa/f \end{matrix}\,\Big|\, q; \frac{q^2a^2}{bcdef}\right] \tag{9a}
$$

$$
= \left[\begin{matrix} qa, qa/de, qa/df, qa/ef \\ qa/d, qa/e, qa/f, qa/def \end{matrix}\,\Big|\, q\right]_\infty {}_4\phi_3\left[\begin{matrix} qa/bc, d, e, f \\ qa/b, qa/c, def/a \end{matrix}\,\Big|\, q; q\right] \tag{9b}
$$

$$
+ \left[\begin{matrix} qa, qa/bc, d, e, f, q^2a^2/bdef, q^2a^2/cdef \\ qa/b, qa/c, qa/d, qa/e, qa/f, q^2a^2/bcdef, def/qa \end{matrix}\,\Big|\, q\right]_\infty \tag{9c}
$$

$$
\times\ {}_4\phi_3\left[\begin{matrix} qa/de, qa/df, qa/ef, q^2a^2/bcdef \\ q^2a^2/bdef, q^2a^2/cdef, q^2a/def \end{matrix}\,\Big|\, q; q\right]. \tag{9d}
$$

We utilize this relation to transform Theorem 5 to another reciprocity theorem.

Writing (6b) in terms of a ${}_8\phi_7$-series and then reformulating it through the last three-term transformation formula, we have

$$
\begin{aligned}
{}_8\phi_7&\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & a \end{matrix}\,\Big|\, q; \frac{qa^2}{bcde}\right] \\
&= \left[\begin{matrix} qa, qa/de, a/d, a/e \\ qa/d, qa/e, a, a/de \end{matrix}\,\Big|\, q\right]_\infty {}_4\phi_3\left[\begin{matrix} q, d, e, qa/bc \\ qa/b, qa/c, qde/a \end{matrix}\,\Big|\, q; q\right] \\
&+ \left[\begin{matrix} qa, qa/bc, d, e, q, qa^2/bde, qa^2/cde \\ qa/b, qa/c, qa/d, qa/e, a, qa^2/bcde, de/a \end{matrix}\,\Big|\, q\right]_\infty \\
&\times\ {}_3\phi_2\left[\begin{matrix} a/d, a/e, qa^2/bcde \\ qa^2/bde, qa^2/cde \end{matrix}\,\Big|\, q; q\right].
\end{aligned}
$$

Similarly, we can reformulate (6d) as the following expression:

$$
\begin{aligned}
{}_8\phi_7&\left[\begin{matrix} q^2/a, & q^2/\sqrt{a}, & -q^2/\sqrt{a}, & qb/a, & qc/a, & qd/a, & qe/a, & q \\ & q/\sqrt{a}, & -q/\sqrt{a}, & q^2/b, & q^2/c, & q^2/d, & q^2/e, & q^2/a \end{matrix}\,\Big|\, q; \frac{qa^2}{bcde}\right] \\
&= \left[\begin{matrix} q^3/a, qa/de, q/d, q/e \\ q^2/d, q^2/e, q^2/a, a/de \end{matrix}\,\Big|\, q\right]_\infty {}_4\phi_3\left[\begin{matrix} q, qd/a, qe/a, qa/bc \\ q^2/b, q^2/c, qde/a \end{matrix}\,\Big|\, q; q\right] \\
&+ \left[\begin{matrix} q^3/a, qa/bc, qd/a, qe/a, q, q^2a/bde, q^2a/cde \\ q^2/b, q^2/c, q^2/d, q^2/e, q^2/a, qa^2/bcde, de/a \end{matrix}\,\Big|\, q\right]_\infty \\
&\times\ {}_3\phi_2\left[\begin{matrix} q/d, q/e, qa^2/bcde \\ q^2a/bde, q^2a/cde \end{matrix}\,\Big|\, q; q\right].
\end{aligned}
$$

Substituting both three-term transformation formulae into (6a-6b-6c-6d) and then replacing $a$, $b$, $c$, $d$, $e$ respectively by $qy/x$, $q/bx$, $q/dx$, $q/ax$, $q/cx$, we find, after some simplification, the following equivalent form of Theorem 5.

**Theorem 10 (Reciprocal formula).**

$$
\begin{aligned}
&y\sum_{n=0}^{\infty}\frac{[q/ax,q/cx,bdxy;q]_n\,q^n}{(q^2/acxy;q)_n\,[by,dy;q]_{n+1}}-x\sum_{n=0}^{\infty}\frac{[q/ay,q/cy,bdxy;q]_n\,q^n}{(q^2/acxy;q)_n\,[bx,dx;q]_{n+1}}\\
&\quad=(y-x)\left[\begin{matrix} q,\ qy/x,\ qx/y,\ abxy,\ acxy/q,\ adxy,\ bcxy,\ bdxy,\ cdxy\\ ax,\ ay,\ bx,\ by,\ cx,\ cy,\ dx,\ dy,\ abcdx^2y^2/q\end{matrix}\,\bigg|\,q\right]_\infty\\
&\quad+\frac{acxy}{q}\left[\begin{matrix} q, & q/ax, & q/ay, & q/cx, & q/cy, & bdxy\\ bx, & by, & dx, & dy, & q^2/acxy, & abcdx^2y^2/q\end{matrix}\,\bigg|\,q\right]_\infty\\
&\quad\times\left\{y\left[\begin{matrix} bx,dx,abcxy^2,acdxy^2\\ ay,cy,q/ay,q/cy\end{matrix}\,\bigg|\,q\right]_\infty {}_3\phi_2\left[\begin{matrix} ay,cy,abcdx^2y^2/q\\ abcxy^2,acdxy^2\end{matrix}\,\bigg|\,q;q\right]\right.\\
&\quad\left.-x\left[\begin{matrix} by,dy,abcx^2y,acdx^2y\\ ax,cx,q/ax,q/cx\end{matrix}\,\bigg|\,q\right]_\infty {}_3\phi_2\left[\begin{matrix} ax,cx,abcdx^2y^2/q\\ abcx^2y,acdx^2y\end{matrix}\,\bigg|\,q;q\right]\right\}.
\end{aligned}
$$

When $c\to 0$, the last theorem reduces to Andrews' one displayed in Theorem 2. This theorem corrects the reciprocal formula due to Zhang [20], where the correcting terms displayed in the last three lines have been missing.

## References


[1] C. Adiga, N. Anitha, *On a reciprocity theorem of Ramanujan*, Tamsui Oxford J. Math. Sci. **22** (2006), 9–15.

[2] R.P. Agarwal, *On the paper "A 'lost' notebook of Ramanujan"*, Adv. Math. **53** (1984), 291–300.

[3] G.E. Andrews, *Ramanujan's "lost" notebook. I. Partial θ-functions*, Adv. Math. **41** (1981), 137–172.

[4] W.N. Bailey, *Series of hypergeometric type which are infinite in both directions*, Quart. J. Math. (Oxford) **7** (1936), 105–115.

[5] B.C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.

[6] B.C. Berndt, S.H. Chan, B.P. Yeap, A.J. Yee, *A reciprocity theorem for certain q-series found in Ramanujan's lost notebook*, Ramanujan J. **13** (2007), 27–37.

[7] S. Bhargava, C. Adiga, M.S. Mahadeva Naika, *Ramanujan's remarkable summation formula as a 2-parameter generalization of the quintuple product identity*, Tamkang J. Math. **33** (2002), 285–288.

[8] S. Bhargava, D.D. Somashekara, S.N. Fathima, *Some q-Gamma and q-Beta function identities deducible from the reciprocity theorem of Ramanujan*, Adv. Stud. Contemp. Math. (Kyungshang) **11** (2005), 227–234.

[9] L. Carlitz, M.V. Subbarao, *A simple proof of the quintuple product identity*, Proc. Amer. Math. Society **32**(1) (1972), 42–44.

[10] W. Chu, *Durfee rectangles and the Jacobi triple product identity*, Acta Math. Sinica [New Series] **9**(1) (1993), 24–26.

[11] W. Chu, *Abel's lemma on summation by parts and Ramanujan's ${}_1\psi_1$-series Identity*, Aequationes Mathematicae **72**(1/2) (2006), 172–176.

[12] W. Chu, *Bailey's very well-poised ${}_6\psi_6$-series identity*, Journal of Combinatorial Theory (Series A) **113**(6) (2006), 966–979.
[13] W. Chu, *Jacobi's triple product identity and the quintuple product identity*, Boll. Un. Mat. Ital. **B10**(8) (2007), 867–874.
[14] G. Gasper, M. Rahman, *Basic Hypergeometric Series* (2nd ed.), Cambridge University Press, Cambridge, 2004.
[15] T. Kim, D.D. Somashekara, S.N. Fathima, *On a generalization of Jacobi's triple product identity and its applications*, Adv. Stud. Contemp. Math. (Kyungshang) **9** (2004), 165–174.
[16] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.
[17] D.D. Somashekara, S.N. Fathima, *An interesting generalization of Jacobi's triple product identity* Far East J. Math. Sci. **9** (2003), 255–259.
[18] S.Y. Kang, *Generalizations of Ramanujan's reciprocity theorem and their applications*, J. London Math. Soc. (2) **75** (2007), 18–34.
[19] Z.-G. Liu, *Some operator identities and q-series transformation formulas*, Discrete Math. **265** (2003), 119–139
[20] Z.-Z. Zhang, *A note on an identity of Andrews* (with Erratum), Electron. J. Combin. **12** (2005), #N3.

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