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THREE TRIANGULAR NUMBERS CONTAINED IN GEOMETRIC PROGRESSION

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Abstract: In the present paper we prove that all three distinct triangular numbers in geometric progression and the positive integer solutions (x, y, z) of the equation $(x^2-1)(y^2-1) = (z^2-1)^2$, $1 < x < z < y$, $2 \not| xyz$ are one-to-one under the assumption that a conjecture on a system of diophantine equations holds.

Keywords: Triangular numbers, geometric progressions, Pell equations.

1. Introduction

The integers of the form $T_n = n(n+1)/2, n \in \mathbb{N}$, are called triangular numbers. In [7, D23], it is stated that Sierpinski asked the question of whether or not there exist four (distinct) triangular numbers in geometric progression. Szymiczek [12] conjectured that the answer is negative.

Recently M.Bennett [3] proved that there do not exist four distinct triangular numbers in geometric progression with the ratio being a positive integer. Chen and Fang [8] extended Bennett's result to the rational ratio and proved that there do not exist four distinct triangular numbers in geometric progression. By employing the theory of Pell's equations and a result of Y. Bilu, G. Hanrot and P.M. Voutier on primitive divisors of Lucas and Lehmer numbers, Yang and He [7] claimed that there is no geometric progression which contains four distinct triangular numbers. In their paper, they misunderstood the phrase "in geometric progression", and claimed that Bennett's proof is not complete and that they solved Sierpinski's problem completely. In fact, their proof is also under the assumption that the geometric progression has an integral common ratio. Fang [9], using only the Stӧrmer theorem on Pell's equation, showed that no geometric progression contains four distinct triangular numbers.

An old result of Gérardin [5] (see also [11]) implies that there are infinitely many such triples, the smallest of which is (T_1, T_3, T_8). In fact a simple calculation shows

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2000 Mathematics Subject Classification: 11D25, 11D41 that if $T_n = m^2$ then T_n, $T_{n+2m} = m(2n+3m+1)$, $T_{3n+4m+1} = (2n+3m+1)^2$ form a geometric progression.

The main purpose of the present paper is an attempt to determining all three such triangular numbers. To state the main theorem of the present paper, we first introduce a conjecture on a system of diophantine equations.

Conjecture 1.1. *Let $m > 1$ be a positive integer. Then the following system of equations*

$$\begin{cases} x^2 - (m^2-1)y^2 = 1, \\ z^2 - (m^2-1)y^r = 1 \end{cases} \tag{1}$$

has no positive integer solution (x, y, r) with $y > 1$ and $r > 2$.

Theorem 1.1. *If Conjecture 1.1 holds, then T_{n_1}, T_{n_2}, T_{n_3} is three distinct terms in a geometric progression if and only if $(x, y, z) = (2n_1+1, 2n_3+1, 2n_2+1)$ is a solution of the diophantine equation*

$$(x^2-1)(y^2-1) = (z^2-1)^2, \qquad 1 < x < z < y.$$

2. Some Lemmas

Suppose that D is a positive nonsquare integer, and let u_1 and v_1 be the smallest positive integers such that

$$u_1^2 - Dv_1^2 = 1.$$

We call $u_1 + v_1\sqrt{D}$ the fundamental solution to the Pell equation $x^2 - Dy^2 = 1$. Further we define sequences of integers $\{u_k\}$ and $\{v_k\}$ by

$$u_k + v_k\sqrt{D} = (u_1 + v_1\sqrt{D})^k, \qquad k \in \mathbb{Z}.$$

Lemma 2.1 (Störmer theorem [4]). *Let D be a positive nonsquare integer. If the Pell equation $x^2 - Dy^2 = 1$ has a positive integral solution (x_1, y_1), and every prime divisor of y_1 divides D, then $x_1 + y_1\sqrt{D}$ is the fundamental solution.*

Lemma 2.2 ([14]). *If $d = gcd(m, n)$, then $gcd(v_m, v_n) = v_d$.*

Theorem 2.1 ([1]). *If a, b and n are integers with $ab \neq 0$ and $n \geqslant 3$, then the equation*

$$|ax^n - by^n| = 1$$

has at most one solution in positive integers (x, y).

Lemma 2.3 ([10]). *If $2 \nmid v_1$, then v_k is even if and only if $k = 2h$.*

Lemma 2.4. *Let D be a positive non-square integer. Suppose that the diophantine equation $x^2 - Dy^2 = 1$ has two solutions $(x, y) = (m_1, a)$ and (m_2, b) satisfying $\gcd(a, b) = 1$, then $D = m^2 - 1$ for some positive integer m.*

Proof. Let $u_1 + v_1\sqrt{D}$ be the fundamental solution of the Pell equation $x^2 - Dy^2 = 1$, then $v_1|a$ and $v_1|b$. Hence our assumption $\gcd(a, b) = 1$ implies that $v_1 = 1$, and so $D = u_1^2 - 1$. The proof is complete. ■

Lemma 2.5 ([2]). *If D is a positive nonsquare integer and $n \geqslant 3$, then the equation $x^n = u_{2k}$, where x and k are positive integers, implies that*

$$D = 6083,\ n = 3,\ k = 1,\ x = 23.$$

Theorem 2.2 ([1]). *Let D be a positive nonsquare integer. Then there are at most two pairs of positive integers (x, y) such that*

$$x^2 - Dy^4 = 1.$$

If there are two such solutions, say (x_1, y_1) and (x_2, y_2) with $y_1 < y_2$, then $y_1^2 = v_1$ and $y_2^2 = v_2$, except if $D = 1785$ or $D = 16 \cdot 1785$, in which case $y_1^2 = v_1$ and $y_2^2 = v_4$.

The following result is a weak form of a result of Bennett [2]. However, Bennett [2] used a result which have not published now, so we provide the proof here, the idea of the proof is derived from [2].

Lemma 2.6. *Let n and D be fixed positive integers with $n \geqslant 3$ and D nonsquare. If $2|D$, then the equation*

$$x^2 - Dy^{2n} = 1 \tag{2}$$

has at most one solution in positive integer x and y.

Proof. If we have a solution in positive integers (x, y) to (2), we may write

$$x + y^n\sqrt{D} = u_k + v_k\sqrt{D} = (u_1 + v_1\sqrt{D})^k, \qquad k \in \mathbb{N}, \tag{3}$$

where $u_1 + v_1\sqrt{D}$ is the fundamental solution of the Pell equation $x^2 - Dy^2 = 1$.

We begin with the following observation. If the equation $x^2 - Dy^{2n} = 1$ has two positive solutions (x_1, y_1) and (x_2, y_2), then there exist positive integers x_1, x_2, y_1, y_2, k_1 and k_2 with

$$x_i + y_i^n\sqrt{D} = (u_1 + v_1\sqrt{D})^{k_i}, \qquad k_i = 1,\ 2, \tag{4}$$

and $k_1 < k_2$. We may further assume that k_1 is the smallest positive integer such that a relation of (3) holds with $k = k_1$. By Lemma 2.2, $v_{\gcd(k_1,\, k_2)} = \gcd(v_{k_1},\, v_{k_2}) = \gcd(y_1^n,\, y_2^n) = (\gcd(y_1,\, y_2))^n$, so we have that $k_1|k_2$. Therefore, by letting $D_1 = Dy_1^{2n}$, we have that the equation $x^2 - D_1y^{2n} = 1$ has two positive solution $(x, y) = (x_1, 1)$ and $(x_2, y_2/y_1)$. Hence without loss of generality, if the equation $x^2 - Dy^{2n} = 1$ has two positive solutions (x_1, y_1) and (x_2, y_2), we may assume that $D = m^2 - 1$ for a positive integer m. We will keep this assumption in the remaining arguments.

Let $u_1 + v_1\sqrt{D} = m + \sqrt{m^2-1}$ with odd m, $u_k + v_k\sqrt{D} = (u_1 + v_1\sqrt{D})^k$. If $y^n = v_p$ with p odd. It is readily verified that

$$\left(v_{\frac{p+1}{2}} - v_{\frac{p-1}{2}}\right)\left(v_{\frac{p+1}{2}} + v_{\frac{p-1}{2}}\right) = v_p, \tag{5}$$

and

$$\frac{m+1}{2}\left(v_{\frac{p+1}{2}} - v_{\frac{p-1}{2}}\right)^2 - \frac{m-1}{2}\left(v_{\frac{p+1}{2}} + v_{\frac{p-1}{2}}\right)^2 = 1. \tag{6}$$

It follows from the above two equalities that

$$\frac{x_1+1}{2}a^n - \frac{x_1-1}{2}b^n = 1.$$

Applying Theorem 2.1, we conclude that $a = b = 1$ and so $p = 1$.

If $y^n = v_{4k} = 2u_{2k}v_{2k}$, then it is easy to see that $u_{2k} = a^n$ for a positive integer a, by Lemma 2.5 we have that

$$D = 6083, \quad n = 3, \quad k = 1, \quad x = 23,$$

contradicting with $2|D$.

If $y^n = v_{2k}$ with k odd, then by Lemma 2.3 and our assumption we have $y^n = 2u_kv_k, 2 \nmid v_k$, $2 \nmid u_k$, which is impossible and we are done. ■

3. Proof of Theorem 1.1

Proof of theorem 1.1. Suppose that there is a geometric progression which contains three distinct triangular integers T_{n_1}, T_{n_2}, T_{n_3}. Let q be the common ratio. It is obvious that $q > 0$ and $q \neq 1$. Without loss of generality, we may assume that $0 < q = b/a < 1$ and $gcd(a,b) = 1$. Let $A = 8T_{n_1}$. Then

$$8T_{n_2} = Aq^{r_1}, \quad 8T_{n_3} = Aq^{r_2}, \qquad r_1,\, r_2 \in \mathbb{N}$$

Let $m_i = 2n_i + 1 (i = 1,\, 2,\, 3)$. Then

$$m_1^2 - 1 = A, \qquad m_2^2 - 1 = Aq^{r_1}, \qquad m_3^2 - 1 = Aq^{r_2}. \tag{7}$$

Since Aq^{r_2} is an integer and $gcd(a,b) = 1$, we have $a^{r_2}|A$. Let $A = a_0a^{r_2}$. From (7) we have

$$m_1^2 - a_0a^{r_2} = 1, \tag{8}$$

$$m_2^2 - a_0a^{r_2-r_1}b^{r_1} = 1, \tag{9}$$

and

$$m_3^2 - a_0b^{r_2} = 1. \tag{10}$$

We divide the remaining proof into three cases.

Case 1. $2|r_2$ and $r_2 > 2$. Write $r_2 = 2n, n \geqslant 2$. Then by (8), (10), Lemma 2.4 and our assumptions, we have

$$a_0 = m^2 - 1$$

for some odd positive integer m. If $n > 2$, then by (8), (10) we obtain that $(m_1, a^n), (m_3, b^n)$ are both positive integer solutions of the equation

$$x^2 - a_0 y^{2n} = 1,$$

which is impossible by Lemma 2.6. If $n = 2$, $a_0 \neq 16 \cdot 1785$, then by (8), (10) Theorem 2.2 we get that

$$b = 1, \qquad 2m = a^2,$$

which is not true since $2 \not| m$. If $n = 2$ and $a_0 = 16 \cdot 1785$, then $a = 26 \cdot 239$. It is easy to see that the equation

$$x^2 - 16 \cdot 1785(26 \cdot 239)^r = 1, \qquad 2 \not| r$$

has no solution in positive integers (x, r). Therefore Case 1 is impossible.

Case 2. $2 \not| r_2$. Then we have

$$\begin{cases} m_1^2 - a_0 a(a^{(r_2-1)/2})^2 = 1, \\ m_3^2 - a_0 b(b^{(r_2-1)/2})^2 = 1, \\ m_2^2 - a_0 a^{r_2-r_1} b^{r_1} = 1. \end{cases}$$

By Lemma 2.1, we see that $m_1 + a^{(r_2-1)/2}\sqrt{a_0 a}$ and $m_3 + b^{(r_2-1)/2}\sqrt{a_0 b}$ are the fundamental solutions of the equations $x^2 - a_0 a y^2 = 1$ and $x^2 - a_0 b y^2 = 1$, respectively. If $2|r_1$, then $m_2 + a^{(r_2-r_1-1)/2} b^{r_1/2}\sqrt{a_0 a}$ is a solution of $x^2 - a_0 a y^2 = 1$, and so $a^{(r_2-1)/2} | a^{(r_2-r_1-1)/2} b^{r_1/2}$, which is a contradiction since $gcd(a, b) = 1$ and $a > b \geqslant 1$. Hence $2 \not| r_1$ and $m_2 + a^{(r_2-r_1)/2} b^{(r_1-1)/2}\sqrt{a_0 b}$ is a solution of $x^2 - a_0 b y^2 = 1$. It follows that $b^{(r_2-1)/2} | a^{(r_2-r_1)/2} b^{(r_1-1)/2}$, and so $b = 1$, $a_0 = m_3^2 - 1$.

Let $r_2 - r_1 = 2n$, $n \geqslant 1$. If $n > 2$, then by (9), (10) we see that $(m_3, 1)$ and (m_2, a^n) are both positive integer solutions of the equation

$$x^2 - a_0 y^{2n} = 1,$$

contradicting Lemma 2.6. If $n = 2$, $D = a_0 = m_3^2 - 1 \neq 16 \cdot 1785$, then by $m_2^2 - (m_3^2 - 1)a^{2n} = 1$ and Theorem 2.2 , we have that

$$2m_3 = a^n,$$

which is impossible since $2 \not| m_3$. If $n = 2$ and $D = a_0 = 16 \cdot 1785$, then $a = 26 \cdot 239$. It is easy to see that the equation

$$x^2 - 16 \cdot 1785(26 \cdot 239)^r = 1, 2 \not| r$$

has no solution in positive integers (x, r). If $n = 1$, then we have to solve the following system of equations

$$\begin{cases} x^2 - (m^2 - 1)y^2 = 1, \\ z^2 - (m^2 - 1)y^r = 1. \end{cases}$$

By Conjecture 1.1, the above equation has no positive integer solutions (x, y, r) with $r \neq 2n$ and $r > 2$. Therefore Case 2 is also impossible.

Case 3. $r_2 = 2$. Then $r_1 = 1$ and we have

$$m_1^2 - 1 = (m^2-1)a^2, \qquad m_3^2 - 1 = (m^2-1)b^2, \qquad m_2^2 - 1 = (m^2-1)ab.$$

Obviously, $(m_1^2-1)(m_3^2-1) = (m_2^2-1)^2$, i.e. $(x, y, z) = (m_1, m_2, m_3)$ is a positive integer solution of the diophantine equation

$$(x^2-1)(z^2-1) = (y^2-1)^2, \qquad 2 \nmid xyz. \tag{11}$$

Conversely, we suppose that (x, y, z) is a solution of the equation (11). Then there is a positive integer D such that

$$x^2 - 1 = Da^2, \quad z^2 - 1 = Db^2, \quad y^2 - 1 = Dab, \quad \gcd(a, b) = 1.$$

It follows from Lemma 2.4 that $D = m^2 - 1$ for some positive integer, and so

$$x^2 - 1 = (m^2-1)a^2, \quad z^2 - 1 = (m^2-1)b^2, \quad y^2 - 1 = (m^2-1)ab, \quad \gcd(a, b) = 1.$$

Therefore $T_{(x-1)/2}$, $T_{(y-1)/2}$, $T_{(z-1)/2}$ are three triangular numbers in geometric progression with common ratio b/a. This completes the proof of Theorem 1.1. ■

4. An Open Problem

Let D and $r \geqslant 3$ be positive integers. Observe that the more general equation of the equation in Conjecture 1.1 is the following equation

$$x^2 - Dy^r = 1. \tag{12}$$

If $D = a^{2n}/4 - 1$, $r = 2n$, $n \geqslant 2$, where a is a positive even integer, then $(x, y) = (a^n/2, 1)$, $(a^{2n}/2 - 1, a)$ are two solutions to equation (12). Now the open problem is to solve the equation (12) completely. We conjecture that there are at most two positive integer solutions (x, y) to equation (12).

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