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A NOTE ON DENSITY MODULO 1 OF CERTAIN SETS OF SUMS

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Abstract: Let $a_1 > a_2 > 1$ and $b_1 > b_2 > 1$ be two distinct pairs of multiplicatively independent integers. If $b_1 > a_1$ and $a_2 > b_2$ or $b_1 < a_1$ and $a_2 < b_2$ then we prove that for every ξ_1, ξ_2, with at least one ξ_i irrational, there exists $q \in \mathbb{N}$ such that for any sequence of real numbers r_m the set of sums

$$\{a_1^m a_2^n q\xi_1 + b_1^m b_2^n q\xi_2 + r_m : m, n \in \mathbb{N}\},$$

is dense modulo 1. The sets with algebraic numbers a_i, b_i are also considered.

Keywords: Density modulo 1, topological dynamics, multiplicatively independent algebraic numbers.

1. Introduction and main results

In 1967 Furstenberg proved the following

Theorem 1.1 (Furstenberg, [2]). *If $p, q > 1$ are multiplicatively independent integers (i.e., $\log p / \log q$ is irrational) then for every irrational ξ the set*

$$\{p^m q^n \xi : m, n \in \mathbb{N}\} \tag{1.1}$$

is dense modulo 1.

The following two theorems proved by Kra in [3] generalize Furstenberg's theorem.

Theorem 1.2 ([3, Theorem 1.2]). *Suppose that the pairs $p_i, q_i \in \mathbb{N}$ are multiplicatively independent with $1 < p_i < q_i$ for $i = 1, \dots, k$, $k \in \mathbb{N}$, $(p_i, q_i) \neq (p_j, q_j)$ for $i \neq j$, and $p_1 \leqslant p_2 \leqslant \dots \leqslant p_k$. Then for distinct $\xi_1, \dots, \xi_k \in [0, 1]$ with at least one $\xi_i \notin \mathbb{Q}$ the set*

$$\left\{ \sum_{i=1}^{k} p_i^m q_i^n \xi_i : m, n \in \mathbb{N} \right\}$$

is dense modulo 1.

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Theorem 1.3 ([3, Corollary 2.2]). *Let $p, q > 1$ be multiplicatively independent integers and let r_m be any sequence of real numbers. Then for any irrational ξ,*

$$\{p^n q^m \xi + r_m : n, m \in \mathbb{N}\}$$

is dense modulo 1.

For some generalizations of Theorem 1.2 and Theorem 1.3 to the case of algebraic numbers see [5, 6] and [7], respectively.

The aim of this note is to prove the following result, which can be considered as a kind of a mixture of Theorem 1.2 and Theorem 1.3.

Theorem 1.4. *Let $a_1 > a_2 > 1$ and $b_1 > b_2 > 1$ be two pairs of multiplicatively independent integers. Suppose that*

$$a_1 < b_1 \qquad \text{and} \qquad a_2 > b_2. \tag{1.2}$$

Then, for any real numbers ξ_1, ξ_2 with at least one ξ_i irrational, there exists $q \in \mathbb{N}$ such that for any sequence of real numbers r_m, the set

$$\{a_1^m a_2^n q\xi_1 + b_1^m b_2^n q\xi_2 + r_m : m, n \in \mathbb{N}\} \tag{1.3}$$

is dense modulo 1.

Remark 1.1. It is clear that we can consider sets of the form (1.3) with not necessarily all of a_i, b_i positive. In fact, using squares of the original parameters we have a subset of (1.3).

In the proof of Theorem 1.4 we use topological dynamics methods from [3] extended to our setting.

It is a natural question to ask what happens if we replace integers a_i, b_i by algebraic numbers. It turns out that using results from [5] and [6] we can extend Theorem 1.4 to the case of algebraic integers and algebraic numbers of degree 2, respectively.

Theorem 1.5. *Let λ_1, μ_1 and λ_2, μ_2 be two distinct pairs of multiplicatively independent real algebraic integers of degree 2. Assume that*

(i) *$|\lambda_i|, |\mu_1| > 1$, $i = 1, 2$, and the absolute values of their conjugates, $\tilde{\lambda}_i, \tilde{\mu}_i$ are also greater than 1.*
(ii) *$\mu_i = g_i(\lambda_i)$, for some $g_i \in \mathbb{Z}[x]$, $i = 1, 2$.*
(iii) *In each pair λ_i, μ_i there is at least one element with the property that for every $n \in \mathbb{N}$, its n-th power is irrational.*
(iv) *There exist $k, l, k', l' \in \mathbb{N}$ such that*

$$\min\{|\lambda_2|^k|\mu_2|^l, |\tilde{\lambda}_2|^k|\tilde{\mu}_2|^l\} > \max\{|\lambda_1|^k|\mu_1|^l, |\tilde{\lambda}_1|^k|\tilde{\mu}_1|^l\}$$

and

$$\min\{|\lambda_1|^{k'}|\mu_1|^{l'}, |\tilde{\lambda}_1|^{k'}|\tilde{\mu}_1|^{l'}\} > \max\{|\lambda_2|^{k'}|\mu_2|^{l'}, |\tilde{\lambda}_2|^{k'}|\tilde{\mu}_2|^{l'}\}.$$

Then for any real numbers ξ_1, ξ_2 *with at least one* $\xi_i \neq 0$, *there exists a natural number* q *such that for any real sequence* r_m *the set*

$$\{\lambda_1^n \mu_1^m q\xi_1 + \lambda_2^n \mu_2^m q\xi_2 + r_m : n, m \in \mathbb{N}\}$$

is dense modulo 1.

In order to prove Theorem 1.5 we generalize the proof of Theorem 1.4 to higher dimension. Namely, the idea of the proof is to construct, using the companion matrices associated with λ_i's, an appropriate semigroup M of endomorphisms of the 4-dimensional torus $\mathbb{T}^4 = \mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}^2/\mathbb{Z}^2$. Then we have to chose a point α in $\mathbb{T}^4$ such that in the coordinates of the orbit $M\alpha$ we can recognize the expression we are interested in.

In the next result we do not need to assume that a_i, b_i are algebraic integers.

Theorem 1.6. *Let* λ_1, μ_1 *and* λ_2, μ_2 *be two distinct pairs of multiplicatively independent algebraic numbers of degree 2. Assume that*

(i) $|\lambda_i|, |\mu_1| > 1$, $i = 1, 2$, *and the absolute values of their conjugates,* $\tilde{\lambda}_i, \tilde{\mu}_i$ *are also greater than 1.*

(ii) $\mu_i = g_i(\lambda_i)$, *for some* $g_i \in \mathbb{Q}[x]$, $i = 1, 2$.

(iii) *At least one element in each pair* λ_i, μ_i *has all non-negative powers irrational.*

Let $S = \{\infty, p_1, p_2, \ldots, p_s\}$, *where for* $k = 1, \ldots, s$, $p_k \geqslant 2$ *are the primes appearing in the denominators of coefficients of* $g_1, g_2 \in \mathbb{Q}[x]$, *and the minimal polynomials* $P_{\lambda_1}, P_{\lambda_2} \in \mathbb{Q}[x]$ *of* λ_1 *and* λ_2, *respectively.*

Assume further that

(iv) *there exist* $k, l, k', l' \in \mathbb{N}$ *such that*

$$\min_{p \in S}(\min\{|\lambda_2|_p^k |\mu_2|_p^l, |\tilde{\lambda}_2|_p^k |\tilde{\mu}_2|_p^l\}) > \max_{p \in S}(\max\{|\lambda_1|_p^k |\mu_1|_p^l, |\tilde{\lambda}_1|_p^k |\tilde{\mu}_1|_p^l\})$$

and

$$\min_{p \in S}(\min\{|\lambda_1|_p^{k'} |\mu_1|_p^{l'}, |\tilde{\lambda}_1|_p^{k'} |\tilde{\mu}_1|_p^{l'}\}) > \max_{p \in S}(\max\{|\lambda_2|_p^{k'} |\mu_2|_p^{l'}, |\tilde{\lambda}_2|_p^{k'} |\tilde{\mu}_2|_p^{l'}\}),$$

where $|\cdot|_p$ *is the* p*-adic norm, whereas* $|\cdot|_\infty$ *stands for the usual absolute value, and*

$$\min\{|\lambda_i|_p, |\mu_i|_p, |\tilde{\lambda}_i|_p, |\tilde{\mu}_i|_p : i = 1, 2,\ p \in S\} > 1.$$

Then for any pair of real numbers ξ_1, ξ_2, *with at least one* ξ_i *non-zero, there exists a natural number* q *such that for any sequence of real numbers* r_m *the set*

$$\{\lambda_1^n \mu_1^m q\xi_1 + \lambda_2^n \mu_2^m q\xi_2 + r_m : n, m \in \mathbb{N}\}$$

is dense modulo 1.

We shall omit the proof of Theorem 1.6 as it goes along the lines of the proof of Theorem 1.5. The difference is that instead of the dynamical system on $\mathbb{T}^2 \times \mathbb{T}^2$ one would have to consider a similar system on the product of appropriate solenoids as in [6].

2. Proof of Theorem 1.4

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-dimensional torus. Consider a semigroup $S = \langle s_1, s_2 \rangle \subset \mathrm{End}(\mathbb{T}^2)$ of toral endomorphisms generated by the following two matrices:

$$s_1 = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}.$$

Let $\xi = (\xi_1, \xi_2) + \mathbb{Z}^2 \in \mathbb{T}^2$ and denote by F the closure of the orbit of the point ξ under the action of the semigroup S :

$$F = \overline{S\xi}.$$

Clearly, F is closed and S-invariant subset of $\mathbb{T}^2$.

Lemma 2.1. *The set F is infinite.*

Proof. By the assumption one of ξ_i's is irrational. Suppose that ξ_1 (ξ_2, resp.) is irrational. Then, by Theorem 1.1, for every $x \in \mathbb{T}$ ($y \in \mathbb{T}$, resp.) there are subsequences n_k and $m_k \subset \mathbb{N}$ such that $a_1^{n_k} a_2^{m_k} \xi_1 \to x$ ($b_1^{n_k} b_2^{m_k} \xi_2 \to y$, resp.) as $k \to \infty$. Since $\mathbb{T}$ is compact it follows that there exists $y \in \mathbb{T}$ ($x \in \mathbb{T}$, resp.) such that $(x, y) \in F$. Hence F is infinite. ■

By [3, Corollary 3.2] it follows that F contains a non-isolated rational point $r = p/q$, $q \in \mathbb{N}$, $p \in \mathbb{Z}^2$. Define

$$F' = qF.$$

Then $(0, 0) \in F'$, and we have the following.

Lemma 2.2. *The set F' contains at least one of the following sets*

$$\begin{aligned} T_1 &= \mathbb{T} \times \{0\}, \\ T_2 &= \{0\} \times \mathbb{T}. \end{aligned} \tag{2.1}$$

Proof. It follows from [3, Lemma 3.4] since (1.2) implies that the condition (3) of [3, Lemma 3.4] can not hold. ■

Proof of Theorem 1.4. We extend the proof of [3, Lemma 2.1] to our setting. Consider the set

$$\mathcal{O} = \overline{\{s_1^k q\xi : k \in \mathbb{N}\}}.$$

We consider the space $\mathcal{C}_{\mathbb{T}^2}$ of all closed subsets of $\mathbb{T}^2$ with the Hausdorff metric d_H, defined as

$$d_H(A, B) = \max\{\max_{x \in A} d(x, B), \max_{x \in B} d(x, A)\},$$

where $d(x, B) = \min_{y \in B} d(x, y)$ is the distance of x from the set B. The space $(\mathcal{C}_{\mathbb{T}^2}, d_H)$ is a compact metric space.

Let

$$\mathcal{G} := \overline{\{s_2^l \mathcal{O} : l \in \mathbb{N}\}} \subset \mathcal{C}_{\mathbb{T}^2}.$$

Since the set $\mathcal{O}$ is s_1-invariant, it follows that every element (set) $G \in \mathcal{G}$ is also s_1-invariant. Define,

$$\mathcal{T} = \bigcup_{G \in \mathcal{G}} G \subset \mathbb{T}^2.$$

By definition $F' \subset \mathcal{T}$. Hence, by Lemma 2.2, $\mathcal{T}$ contains at least one of the sets T_1, T_2. Assume that

$$T_1 \subset \mathcal{T}.$$

(The proof for T_2 contained in $\mathcal{T}$ is the same.)

There exists $t_1 \in T_1$ such that the orbit $\{s_1^n t_1 : n \in \mathbb{N}\}$ is dense in $\mathbb{T} = T_1$, i.e.,

$$\overline{\{s_1^n t_1 : n \in \mathbb{N}\}} = T_1. \tag{2.2}$$

Clearly, $t_1 \in G$ for some $G \in \mathcal{G}$. By definition of $\mathcal{G}$, there is a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$G = \lim_k s_2^{n_k} \mathcal{O}, \tag{2.3}$$

and the limit is taken in the Hausdorff metric d_H. Since $t_1 \in G$ and G is s_1-invariant, we get $G \supset \overline{\{s_1^n t_1 : n \in \mathbb{N}\}}$. Hence, by (2.2),

$$G \supset T_1. \tag{2.4}$$

From (2.3) and (2.4) it follows that for every $\varepsilon > 0$ there is an $l \in \mathbb{N}$ such that $s_2^l \mathcal{O}$ is ε-dense in T_1.

Let $v_l = (0, r_l) + \mathbb{Z}^2 \in \mathbb{T}^2$. Since

$$s_2^l \mathcal{O} + v_l \tag{2.5}$$

is a translate of an ε-dense set in T_1, it is also ε-dense in T_1. Now, taking the sum of the first and the second coordinate of the set (2.5), we get 2ε-dense subset of the 1-dimensional torus

$$(s_2^l \mathcal{O} + v_l)_1 + (s_2^l \mathcal{O} + v_l)_2 \subset \mathbb{T}.$$

Comparing the above set with expression (1.3) the theorem follows. ■

3. Proof of Theorem 1.5

Let $\nu > 1$ be a real algebraic integer of degree 2 with minimal (monic) polynomial $P_\nu \in \mathbb{Z}[x]$, $P_\nu(x) = x^2 + c_1 x + c_0$. A *companion matrix of P_ν or ν* is the matrix of the form

$$\sigma_\nu = \begin{pmatrix} 0 & 1 \\ -c_0 & -c_1 \end{pmatrix}.$$

We associate with λ_i, the companion matrices $\sigma_i = \sigma_{\lambda_i}$ and with μ_i we associate matrices $\tau_i = g_i(\sigma_i)$. For $i = 1, 2$, we denote by $\Sigma_i = \langle \sigma_i, \tau_i \rangle$ the semigroups generated by σ_i and τ_i. We put $M_\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$ and $M_\tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$. Let $M = \langle M_\sigma, M_\tau \rangle$ be the semigroup of endomorphisms of $\mathbb{T}^2 \times \mathbb{T}^2$ generated by the matrices M_σ and M_τ.

Consider the orbit $M\alpha$ of the point $\alpha = (\alpha_1, \alpha_2)$ under the action of M. Taking as α_1 and α_2 the common eigenvectors of the semigroups Σ_1 and Σ_2, respectively, $\alpha_1 = \xi_1(1, \lambda_1)$ and $\alpha_2 = \xi_2(1, \lambda_2)$, we get

$$M\alpha = \{(\lambda_1^n \mu_1^m \xi_1, \lambda_1^{n+1} \mu_1^m \xi_1, \lambda_2^n \mu_2^m \xi_2, \lambda_2^{n+1} \mu_2^m \xi_2) : n, m \in \mathbb{N}\}.$$

Let

$$X_\alpha = \overline{M\alpha}\ .$$

It is clear that the set X_α is closed and M-invariant.

Definition 1 ([1]). *We say that the semigroup Σ of continuous endomorphisms of a d-dimensional torus $\mathbb{T}^d$ has the ID-property (or simply that Σ is an ID-semigroup) if the only infinite closed Σ-invariant subset of $\mathbb{T}^d$ is $\mathbb{T}^d$ itself.*

Lemma 3.1. *The set X_α is infinite.*

Proof. It follows from [1, Theorem 2.1] that the semigroup $\Sigma_1 = \langle \sigma_1, \tau_1 \rangle$ is an ID-semigroup. Therefore, since α_1 is not a rational point the orbit $\Sigma_1 \alpha_1$ is dense in $\mathbb{T}^2$. Hence, we obtain that for every $x \in \mathbb{T}^2$ there exist sequences $\{n_k\}$ and $\{m_k\}$, tending to infinity, such that $\sigma_1^{n_k} \tau_1^{m_k} \alpha_1 \to x$, as $k \to \infty$. Since $\mathbb{T}^2$ is compact, we can assume, choosing a subsequence, that $\sigma_2^{n_k} \tau_2^{m_k} \alpha_2 \to y$, for some $y \in \mathbb{T}^2$. Therefore, for every $x \in \mathbb{T}^2$ there exists $y \in \mathbb{T}^2$ so that $(x, y) \in X_\alpha$. ■

Theorem 3.2 ([5, Proposition 5.7]). *With the assumptions of Theorem 1.5 if $(0,0) \in X_\alpha$ then one of the following holds:*

(1) *The point $(0,0)$ is isolated in X_α.*
(2) *The set X_α contains the whole $\mathbb{T}^2 \times \{0\}$ or $\{0\} \times \mathbb{T}^2$.*

Remark 3.1. Actually, instead of X_α we may take in Theorem 3.2 an arbitrary closed, infinite, M_σ- and M_τ-invariant subset of $\mathbb{T}^2 \times \mathbb{T}^2$ containing $(0,0)$. The proof remains the same.

By [5, Lemma 4.3] X_α contains a rational point p/q, $p \in \mathbb{Z}^2$ and $q \in \mathbb{N}$. Let $\tilde{X}_\alpha = qX_\alpha$. Then $\tilde{X}_\alpha$ contains zero and by Remark 3.1 contains either $T_1 := \mathbb{T}^2 \times \{0\}$ or $T_2 := \{0\} \times \mathbb{T}^2$.

Proof of Theorem 1.5. We proceed as in the proof of Theorem 1.4. Instead of $\mathbb{T} \times \mathbb{T}$ we consider $\mathbb{T}^2 \times \mathbb{T}^2$. Instead of matrices s_1, s_2 we consider M_σ, M_τ. To prove the existence of $t_1 \in T_1 = \mathbb{T}^2 \times \{0\}$ such that $\overline{\{M_\sigma^n t_1 : n \in \mathbb{N}\}} = T_1$ we use the fact that the restriction of M_σ to the 2-dimensional sub-torus T_1 is σ_1, and σ_1 is a hyperbolic toral endomorphism (i.e., without eigenvalues of absolute value 1). Hence, σ_1 is ergodic and the existence of t_1 follows (see [4, Proposition 2.6, p. 104]). ■

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