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ON THE CONSTANT IN THE MERTENS PRODUCT FOR ARITHMETIC PROGRESSIONS. I. IDENTITIES

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Abstract: We prove new identities for the constant in the Mertens product over primes in the arithmetic progressions a mod q.

Keywords: Mertens product, primes in arithmetic progressions.

1. Introduction

Let a, q be integers with $(q,a)=1$ and denote by p a prime number. In 1974 Williams [9] proved that

$$P(x;q,a)=\prod_{\substack{p\leqslant x\\ p\equiv a \bmod q}}\Big(1-\frac{1}{p}\Big)=\frac{C(q,a)}{(\log x)^{1/\varphi(q)}}+\mathcal{O}\bigg(\frac{1}{(\log x)^{1/\varphi(q)+1}}\bigg) \tag{1}$$

as $x\to+\infty$, where $C(q,a)$ is real and positive and satisfies

$$C(q,a)^{\varphi(q)}=e^{-\gamma}\frac{q}{\varphi(q)}\prod_{\chi\neq\chi_0}\Big(\frac{K(1,\chi)}{L(1,\chi)}\Big)^{\overline{\chi}(a)},$$

where γ is the Euler constant, φ is the Euler totient function, $L(s,\chi)$ is the Dirichlet L-function associated to the Dirichlet character χ mod q and χ_0 is the principal character to the modulus q. The function K is defined by means of

$$K(s,\chi)=\sum_{n=1}^{+\infty}k_\chi(n)n^{-s},$$

where $k_\chi(n)$ is the completely multiplicative function whose value at primes is given by

$$k_\chi(p)=p\left(1-\Big(1-\frac{\chi(p)}{p}\Big)\Big(1-\frac{1}{p}\Big)^{-\chi(p)}\right).$$

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In our recent paper [4] we obtained a version of Williams's result stated in (1) which is uniform in the q aspect. In the same paper, as a by-product, we also obtained the following elementary expression for $C(q,a)$:

$$C(q,a)^{\varphi(q)} = e^{-\gamma} \prod_{p} \Bigl(1 - \frac{1}{p}\Bigr)^{\alpha(p;q,a)} \tag{2}$$

where $\alpha(p;q,a) = \varphi(q) - 1$ if $p \equiv a \bmod q$ and $\alpha(p;q,a) = -1$ otherwise. The infinite product is convergent, though not absolutely, by the Prime Number Theorem for Arithmetic Progressions. Actually, a slightly simpler proof of identity (2) than the one we gave in [4], sect. 6, can be obtained as follows, once the relevant limit is known to exist: taking logarithms in (1) and using the classical Mertens Theorem, we find

$$\begin{aligned}
\log C(q,a) &= \lim_{x\to+\infty} \Biggl\{ \log \prod_{\substack{p\leqslant x \\ p\equiv a \bmod q}} \Bigl(1 - \frac{1}{p}\Bigr) + \frac{1}{\varphi(q)} \log\log x \Biggr\} \\
&= -\frac{\gamma}{\varphi(q)} + \lim_{x\to+\infty} \Biggl\{ \log \prod_{\substack{p\leqslant x \\ p\equiv a \bmod q}} \Bigl(1 - \frac{1}{p}\Bigr) - \log \prod_{p\leqslant x} \Bigl(1 - \frac{1}{p}\Bigr)^{1/\varphi(q)} \Biggr\} \\
&= -\frac{\gamma}{\varphi(q)} + \frac{1}{\varphi(q)} \log \Biggl\{ \lim_{x\to+\infty} \prod_{p\leqslant x} \Bigl(1 - \frac{1}{p}\Bigr)^{\alpha(p;q,a)} \Biggr\}.
\end{aligned}$$

The product in (2) is very slowly convergent and it is difficult to compute an accurate numerical approximation to $C(q,a)$ from it. Our aim here to give a different form for the constant defined in (2): unfortunately, this form is not suitable for numerical computations, a problem we tackle in part II [5].

As a corollary of the identities proved in the first part of the paper, we derive the formulae that Uchiyama [8] gave in the case $q = 4$ and $a \in \{1, 3\}$, though Uchiyama's direct proof is obviously much simpler. We also derive the explicit expressions that Williams gave in Theorem 2 of [9] for $C(24,a)$ for every integer a such that $(24,a) = 1$ and the ones that Grosswald [3] obtained for $C(q,a)$ for $q \in \{4, 6, 8\}$ and every integer a coprime to q. We recall that, in Proposition 1 of [7], Moree gives formula (8) below when q is a prime number and $a = 1$.

The statement of these new formulae themselves is not simple, and reflects both the structure of the group $\mathbb{Z}_q^*$ and the properties of the residue class a. For this reason, we will not state a formal theorem here, but rather point to the various results as formula (8) or its alternative version (9) when $\mathbb{Z}_q^*$ is cyclic and $a = 1$, then (13) for general q and $a = 1$, and (15) in the most general case. We think our results will be clearer if we go through stages of increasing generality.

We may summarize our main result saying that, in the case $a = 1$, for each reduced residue class b we have to determine a positive integer t_b (actually, the order of b in the multiplicative group $\mathbb{Z}_q^*$) and we will then express $C(q,1)$ as a sort of Euler product where a prime p has the exponent $-t_p$. Collecting all residue classes of maximal order, we may reduce the number of factors needed, at the price of the computation of the power of a suitable value of the Riemann zeta function at an even integer. The important feature of (13) is that $t_b \geqslant 2$ for all $b \neq 1$, whereas the exponent of all primes in the Euler factors in (2) are -1.

The case $a \neq 1$ is genuinely more complicated: in fact, in general it will not be possible to give a simple closed form for the Euler factors, though they can be expressed by means of a rapidly convergent power series. On the other hand, the constant defined in (16) will arise: it is related to the so-called Meissel–Mertens constant; for its computation in the case $q = 1$ see §2.2 of Finch [2] and [6] for the general case.

We would like to thank Pieter Moree for providing us some references and Giuseppe Molteni for suggesting a simplification in the proof of Lemma 1.

2. Reduction to character sums

It turns out to be better to get rid of the prime factors of q at the outset: therefore, we let $c(q,a)$ be defined by means of

$$C(q,a)^{\varphi(q)} = e^{-\gamma}\frac{q}{\varphi(q)}\,c(q,a) \qquad \text{so that } c(q,a) = \prod_{p\nmid q}\Bigl(1-\frac{1}{p}\Bigr)^{\alpha(p;q,a)} \tag{3}$$

and let

$$c(x;q,a) = \prod_{\substack{p\leqslant x\\ p\nmid q}}\Bigl(1-\frac{1}{p}\Bigr)^{\alpha(p;q,a)}$$

denote its partial product. Our strategy is to express the product of $c(x;q,a)$ and partial products of powers of $L(1,\chi)$ as a complicated but quickly convergent product, where χ ranges over all non-principal Dirichlet characters modulo q. When necessary, we use the abbreviation

$$\Pi(q,a) = \prod_{\substack{\chi \bmod q\\ \chi\neq\chi_0}} L(1,\chi)^{-\overline{\chi}(a)}. \tag{4}$$

By orthogonality, we have

$$\begin{aligned} c(x;q,a) &\prod_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\prod_{p\leqslant x}\Bigl(1-\frac{\chi(p)}{p}\Bigr)^{-\overline{\chi}(a)} \\ &= \prod_{\substack{p\leqslant x\\ p\nmid q}}\prod_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\biggl\{\Bigl(1-\frac{1}{p}\Bigr)^{\overline{\chi}(a)\chi(p)}\Bigl(1-\frac{\chi(p)}{p}\Bigr)^{-\overline{\chi}(a)}\biggr\}. \end{aligned} \tag{5}$$

Using the Taylor series expansion of $\log(1-t)$ we see that

$$\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}} \log\biggl\{\Bigl(1-\frac{1}{p}\Bigr)^{\overline{\chi}(a)\chi(p)}\Bigl(1-\frac{\chi(p)}{p}\Bigr)^{-\overline{\chi}(a)}\biggr\} = \sum_{m\geqslant 2}\frac{1}{mp^m}\,S_m(p;q,a), \tag{6}$$

say, where $S_m(p;q,a)$ is the character sum defined by

$$S_m(p;q,a) = \sum_{\chi \bmod q} \overline{\chi}(a)\bigl(\chi^m(p) - \chi(p)\bigr). \tag{7}$$

It is this character sum that reflects the structure of $\mathbb{Z}_q^*$ and the properties of the element a. We will prove below in (10) and (11) that either $p \equiv a \bmod q$, or $S_m(p;q,a)$ vanishes unless m belongs to a suitable arithmetic progression modulo a divisor of $\varphi(q)$. The simplest case, not surprisingly, is when $\mathbb{Z}_q^*$ is cyclic and $a = 1$.

3. The character sum S_m in the simplest case

We notice that, obviously, $S_m(1;q,a) = 0$, and we may assume that $p \not\equiv 1 \bmod q$. For the time being, we also assume that $a = 1$. Let t_p denote the order of p in the multiplicative group $\mathbb{Z}_q^*$, that is, the smallest positive integer k such that $p^k \equiv 1 \bmod q$, and notice that $t_p \geqslant 2$ since $p \not\equiv 1 \bmod q$. It is then quite easy to see that

$$\sum_{\chi \bmod q} \chi^m(p) = \sum_{\chi \bmod q} \chi(p^m) = \begin{cases} \varphi(q) & \text{if } t_p \mid m \\ 0 & \text{otherwise.} \end{cases}$$

Hence, using orthogonality and the Taylor series for $\log(1-t)$ again, we have

$$\sum_{m \geqslant 2} \frac{1}{mp^m} S_m(p;q,1) = \sum_{n \geqslant 1} \frac{\varphi(q)}{nt_p p^{nt_p}} = \log\Bigl(1 - \frac{1}{p^{t_p}}\Bigr)^{-\varphi(q)/t_p}.$$

We classify primes according to their residue class $b \bmod q$, and notice that t_p depends only on b, if $p \equiv b \bmod q$. Substituting into (5) and letting $x \to +\infty$, we see that

$$c(q,1) = \Pi(q,1) \prod_{b \in \mathbb{Z}_q^* \setminus \{1\}} \; \prod_{p \equiv b \bmod q} \Bigl(1 - \frac{1}{p^{t_b}}\Bigr)^{-\varphi(q)/t_b}. \tag{8}$$

We notice that the quantity $\Pi(q,1)$ is connected to the Dedekind zeta function of the q-th cyclotomic field $K = \mathbb{Q}(\zeta_q)$ by means of the relation

$$\Pi(q,1)^{-1} = \operatorname*{Res}_{s=1} \zeta_K(s) \prod_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \prod_{p \mid q} \Bigl(1 - \frac{\chi_f(p)}{p}\Bigr),$$

where χ_f denotes the primitive character that induces χ and f is its conductor.

Assume that $\mathbb{Z}_q^*$ is cyclic. Relation (8) is our first formula, and it is worth noticing that a slightly better form, from the point of view of the explicit computation of $C(q,1)$, can be given grouping the contribution of primes of maximal order. More specifically, we may rewrite (8) as

$$
\begin{aligned}
c(q,1) &= \Pi(q,1) \prod_{\substack{n\mid\varphi(q)\\ n>1}} \prod_{\substack{p\\ t_p=n}} \Bigl(1-\frac{1}{p^n}\Bigr)^{-\varphi(q)/n} \\
&= \Pi(q,1) \prod_{\substack{p\\ t_p=\varphi(q)}} \Bigl(1-\frac{1}{p^{\varphi(q)}}\Bigr)^{-1} \prod_{\substack{n\mid\varphi(q)\\ 1<n<\varphi(q)}} \prod_{\substack{p\\ t_p=n}} \Bigl(1-\frac{1}{p^n}\Bigr)^{-\varphi(q)/n} \\
&= \Pi(q,1) \prod_{p\nmid q} \Bigl(1-\frac{1}{p^{\varphi(q)}}\Bigr)^{-1} \prod_{p\equiv 1 \bmod q} \Bigl(1-\frac{1}{p^{\varphi(q)}}\Bigr) \\
&\quad\times \prod_{\substack{n\mid\varphi(q)\\ 1<n<\varphi(q)}} \prod_{\substack{p\\ t_p=n}} \Bigl\{\Bigl(1-\frac{1}{p^n}\Bigr)^{-\varphi(q)/n}\Bigl(1-\frac{1}{p^{\varphi(q)}}\Bigr)\Bigr\} \\
&= \zeta(\varphi(q))\,\Pi(q,1) \prod_{p\mid q} \Bigl(1-\frac{1}{p^{\varphi(q)}}\Bigr) \prod_{p\equiv 1 \bmod q} \Bigl(1-\frac{1}{p^{\varphi(q)}}\Bigr) \\
&\quad\times \prod_{\substack{n\mid\varphi(q)\\ 1<n<\varphi(q)}} \prod_{\substack{p\\ t_p=n}} \Bigl\{\Bigl(1-\frac{1}{p^n}\Bigr)^{-\varphi(q)/n}\Bigl(1-\frac{1}{p^{\varphi(q)}}\Bigr)\Bigr\}. \qquad (9)
\end{aligned}
$$

The value of $\zeta(\varphi(q))$ is easily computed, at least when q is comparatively small, by means of the Bernoulli numbers, since $\varphi(q)$ is even for $q \geqslant 3$. It is also worth noticing that the exponents of the prime numbers in the last product are all at least 2, though they are usually much larger. Uchiyama's formula (17) for $C(4,1)$ in [8] is the case $q = 4$ of the above expression: there is only one non-principal character χ modulo 4, and $L(1,\chi) = \pi/4$. The last product is empty, and $\zeta(2) = \pi^2/6$. The formula for $C(4,3)$ is easily deduced from this and the classical Mertens Theorem since $C(4,3)C(4,1) = 2e^{-\gamma}$. Grosswald's formula (18) for $C(6,1)$ in [3] is another special case of (3) and (9) since $\varphi(6) = 2$, there is only one non-principal character χ modulo 6, and $L(1,\chi) = \pi/(2\sqrt{3})$. By the Mertens Theorem we have $C(6,5)C(6,1) = 3e^{-\gamma}$ and the formula for $C(6,5)$ can be easily deduced. Our formula (9) also contains Moree's [7] which is the special case where q is prime.

4. The sum S_m in the general case

Lemma 1. *Let S_m be the character sum defined in* (7)*. If $a \equiv b \bmod q$ then there exists a positive integer t_a dividing $\varphi(q)$ such that*

$$
S_m(a;q,a) = \begin{cases} -\varphi(q) & \text{if } m \not\equiv 1 \bmod t_a, \\ 0 & \text{otherwise.} \end{cases} \qquad (10)
$$

In this case, t_a is precisely the order of a in the group $\mathbb{Z}_q^*$, so that $t_a \geqslant 2$ unless $a = b = 1$. If $a \not\equiv b \bmod q$ then either the equation $b^y \equiv a \bmod q$ has no solution

$y \in \mathbb{N}$, or there exist a positive integer t_b dividing $\varphi(q)$ and an integer s_b such that $1 \leqslant s_b \leqslant t_b$ and

$$S_m(b; q, a) = \begin{cases} \varphi(q) & \text{if } b^y \equiv a \bmod q \text{ has a solution, and } m \equiv s_b \bmod t_b, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

In particular, if $a = 1$ then $S_m(1; q, 1) = 0$, while, if $b \not\equiv 1 \bmod q$, then there is an integer $t_b \geqslant 2$ such that $S_m(b; q, 1) = \varphi(q)$ if $m \equiv 0 \bmod t_b$ and is 0 otherwise.

Proof. The complete multiplicativity of the Dirichlet characters implies that

$$S_m(b; q, a) = \sum_{\chi \bmod q} \bigl(\chi(b^m a^{-1}) - \chi(b a^{-1})\bigr)$$

If $a \equiv b \equiv 1 \bmod q$ then $S_m(a; q, a) = 0$. If $a \equiv b \not\equiv 1 \bmod q$ then the equation $a^y \equiv a \bmod q$ has the solution $y \equiv 1 \bmod t_a$, where t_a is the order of a in $\mathbb{Z}_q^*$. Hence, $S_m(a; q, a) = 0$ if $m \equiv 1 \bmod t_a$ and $S_m(a; q, a) = -\varphi(q)$ otherwise, by orthogonality.

If $a \not\equiv b \bmod q$ and the equation $b^y \equiv a \bmod q$ has no solution, then $S_m(b, q, a) = 0$ by orthogonality. If the equation above has the solution $y \equiv s_b \bmod t_b$ (where $t_b \geqslant 2$ is a suitable divisor of $\varphi(q)$ and s_b is an integer with $1 \leqslant s_b \leqslant t_b$) then $S_m(b; q, a) = \varphi(q)$ if $m \equiv s_b \bmod t_b$ and is 0 otherwise. ■

It is quite clear from the Lemma above that the case $a = 1$ is indeed much simpler than the general one. We see that we have proved that (8) holds also in the general case, where t_b is the divisor of $\varphi(q)$ such that the solution of the equation $b^y \equiv 1 \bmod q$ is the class $0 \bmod t_b$, that is, the order of b in $\mathbb{Z}_q^*$. The computations that lead to (9) are still essentially valid, with one modification. We recall the definition of the Carmichael λ function: if $q = 2^\alpha p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then

$$\lambda(q) = \begin{cases} \operatorname{lcm}\{\varphi(2^\alpha), \varphi(p_1^{\alpha_1}), \dots, \varphi(p_k^{\alpha_k})\} & \text{if } \alpha \leqslant 2; \\ \operatorname{lcm}\{2^{\alpha-2}, \varphi(p_1^{\alpha_1}), \dots, \varphi(p_k^{\alpha_k})\} & \text{if } \alpha \geqslant 3. \end{cases} \tag{12}$$

In other words, $\lambda(q)$ is the highest order of the elements of the group $\mathbb{Z}_q^*$. Let $\mathcal{A}(q) = \{b \in \mathbb{Z}_q^* \setminus \{1\} \colon b^k \equiv 1 \bmod q \text{ for some positive } k < \lambda(q)\}$ denote the set of elements of $\mathbb{Z}_q^* \setminus \{1\}$ whose order is not maximal. Arguing as in the proof of (9), that is, grouping the contribution of the primes of maximal order, we obtain the following Theorem, which generalizes Williams's [9] and Grosswald's [3] formulae for $C(24, 1)$ and $C(8, 1)$ respectively and contains (9) as a special case.

Theorem 1. *For all integers $q \geqslant 3$ the value of the constant $c(q, 1)$ is given by*

$$\begin{aligned} c(q,1) &= \zeta(\lambda(q))^{\varphi(q)/\lambda(q)} \Pi(q,1) \\ &\quad \times \prod_{p \mid q} \Bigl(1 - \frac{1}{p^{\lambda(q)}}\Bigr)^{\varphi(q)/\lambda(q)} \prod_{p \equiv 1 \bmod q} \Bigl(1 - \frac{1}{p^{\lambda(q)}}\Bigr)^{\varphi(q)/\lambda(q)} \\ &\quad \times \prod_{b \in \mathcal{A}(q)} \prod_{p \equiv b \bmod q} \Biggl\{ \Bigl(1 - \frac{1}{p^{t_b}}\Bigr)^{-\varphi(q)/t_b} \Bigl(1 - \frac{1}{p^{\lambda(q)}}\Bigr)^{\varphi(q)/\lambda(q)} \Biggr\} \end{aligned} \tag{13}$$

where $\Pi(q,a)$ *is defined in* (4), λ *is the Carmichael lambda function defined in* (12), t_b *denotes the order of* b *in the group* $\mathbb{Z}_q^*$ *and* $\mathcal{A}(q) = \{b \in \mathbb{Z}_q^* \setminus \{1\} \colon t_b < \lambda(q)\}$.

5. The general formula

In this section we assume that $a \not\equiv 1 \bmod q$, and we let $\mathcal{B}(q,a)$ denote the set $\{b \in \mathbb{Z}_q^* \setminus \{1,a\} \colon$ the equation $b^y \equiv a \bmod q$ has a solution$\}$. For the elements of this set, we implicitly define the integers t_b and s_b as in the proof of Lemma 1. In order to state the formula corresponding to (13) in general, we need to introduce the function $f_{t,s}$ which is defined, for positive integers t and s with $1 \leqslant s \leqslant t$ and real x with $|x| < 1$, by means of the relation

$$f_{t,s}(x) = \sum_{\substack{n \geqslant 1 \\ n \equiv s \bmod t}} \frac{x^n}{n} = \int_0^x \frac{u^{s-1}}{1-u^t} \, \mathrm{d}u. \tag{14}$$

The rightmost equality is proved computing the derivative of the function $f_{t,s}$ and then summing the ensuing geometric progression. Notice that, when $s = t$, a closed form for the integral can be easily given in terms of the logarithmic function, as we did above: indeed, $f_{t,t}(x) = -t^{-1} \log(1 - x^t)$. In general, the Taylor series for $f_{t,s}$ is fairly quickly convergent since we will compute it at $x = p^{-1}$.

Lemma 1 above amounts to saying that, given integers q and a and a reduced residue class $b \bmod \varphi(q)$, either the equation $b^y \equiv a \bmod q$ does not have a solution, or its solution is a congruence class $s_b \bmod t_b$, where $t_b \mid \varphi(q)$ and we may assume that $1 \leqslant s_b \leqslant t_b$ and that $t_b \geqslant 2$, since $t_b = 1$ if and only if $a = b = 1$. Moreover, unless $a = b$, the congruence class s_b will not be $1 \bmod q$. Therefore, classifying primes according to their residue class $b \bmod q$ again, and substituting either (10) or (11) into (6), we see that the corresponding factor in the product (5) is 1 if $b = 1$ or $b \notin \mathcal{B}(q,a) \cup \{a\}$, and is

$$\prod_{p \equiv b \bmod q} \exp\Bigl(\varphi(q) f_{t_b,s_b}(p^{-1})\Bigr)$$

if $b \in \mathcal{B}(q,a)$. Recall that for $b \equiv a \bmod q$ we have $s_a = 1$: hence, for primes $p \equiv a \bmod q$ we have a factor

$$\begin{aligned}
&\prod_{p \equiv a \bmod q} \exp\Bigl(-\varphi(q) \sum_{\substack{m \geqslant 2 \\ m \not\equiv 1 \bmod t_a}} \frac{1}{mp^m}\Bigr) \\
&\quad = \prod_{p \equiv a \bmod q} \exp\Bigl(\varphi(q) \sum_{\substack{m \geqslant 2 \\ m \equiv 1 \bmod t_a}} \frac{1}{mp^m} - \varphi(q) \sum_{m \geqslant 2} \frac{1}{mp^m}\Bigr) \\
&\quad = \prod_{p \equiv a \bmod q} \exp\Bigl(\varphi(q) f_{t_a,1}\Bigl(\frac{1}{p}\Bigr) - \varphi(q) \sum_{m \geqslant 1} \frac{1}{mp^m}\Bigr) \\
&\quad = \exp\Bigl(\varphi(q) \sum_{p \equiv a \bmod q} \Bigl(\frac{1}{p} + \log\Bigl(1 - \frac{1}{p}\Bigr)\Bigr)\Bigr) \exp\Bigl(\varphi(q) \sum_{p \equiv a \bmod q} \Bigl(f_{t_a,1}\Bigl(\frac{1}{p}\Bigr) - \frac{1}{p}\Bigr)\Bigr).
\end{aligned}$$

Finally, collecting all identities, we see that we have proved the following result.

Theorem 2. *For all integers $q \geqslant 3$ and all integers a such that $(q,a)=1$ and $a \not\equiv 1 \bmod q$, the value of the constant $c(q,a)$ is given by*

$$c(q,a) = \Pi(q,a) \prod_{b\in\mathcal{B}(q,a)} \prod_{p\equiv b \bmod q} \exp\Bigl(\varphi(q) f_{t_b,s_b}\Bigl(\frac{1}{p}\Bigr)\Bigr)$$

$$\times \exp(\varphi(q)B(q,a)) \exp\Bigl(\varphi(q) \sum_{p\equiv a \bmod q} \Bigl(f_{t_a,1}\Bigl(\frac{1}{p}\Bigr) - \frac{1}{p}\Bigr)\Bigr), \tag{15}$$

where $\mathcal{B}(q,a) = \{b \in \mathbb{Z}_q^* \setminus \{1,a\}\colon$ the equation $b^y \equiv a \bmod q$ has a solution$\}$,

$$B(q,a) = \sum_{p\equiv a \bmod q} \Bigl(\frac{1}{p} + \log\Bigl(1-\frac{1}{p}\Bigr)\Bigr), \tag{16}$$

$\Pi(q,a)$ is defined in (4) and $f_{t,s}$ is defined in (14).

For the special cases $q \in \{4,\ 6,\ 8,\ 24\}$ and $(q,a)=1$ with $a \not\equiv 1 \bmod q$, equation (15) collapses to the formulae given by Uchiyama [8], Williams [9] and Grosswald [3].

For some special values of t and s it is possible to compute a closed form for $f_{t,s}$ as in the previous sections, and give a more explicit result: the following section contains some examples.

6. Explicit values

Using (3), (9) and (13), we can compute explicitly a few values of the constant $C(q,a)$. For the evaluation of $L(1,\chi)$ needed to determine $\Pi(q,1)$ we refer to Corollary 10.3.2 and Proposition 10.3.5 of Cohen [1]. The value (17) is due to Uchiyama while the values (18) and (19) are due to Grosswald. Notice that $\mathcal{A}(8) = \varnothing$ and that $\Pi(8,1) = 32\pi^{-2}(\log(3+2\sqrt{2}))^{-1}$.

$$C(4,1)^2 = \pi\ e^{-\gamma} \prod_{p\equiv 1 \bmod 4} \Bigl(1-\frac{1}{p^2}\Bigr) \tag{17}$$

$$C(6,1)^2 = \frac{2\pi\sqrt{3}}{3}\ e^{-\gamma} \prod_{p\equiv 1 \bmod 6} \Bigl(1-\frac{1}{p^2}\Bigr) \tag{18}$$

$$\begin{aligned} C(8,1)^4 &= \frac{1}{32}\ \pi^4\ e^{-\gamma}\ \Pi(8,1) \prod_{p\equiv 1 \bmod 8} \Bigl(1-\frac{1}{p^2}\Bigr)^2 \\ &= \frac{\pi^2 e^{-\gamma}}{\log(3+2\sqrt{2})} \prod_{p\equiv 1 \bmod 8} \Bigl(1-\frac{1}{p^2}\Bigr)^2. \end{aligned} \tag{19}$$

For $q=24$ we recover the value given on page 357 of Williams [9]: using (3) and (13) we get

$$C(24,1)^8 = \frac{2\pi^4 e^{-\gamma}}{9\log(2+\sqrt{3})\log(1+\sqrt{2})\log(5+2\sqrt{6})} \prod_{p\equiv 1 \bmod 24} \Bigl(1-\frac{1}{p^2}\Bigr)^4$$

since $\varphi(24)=8$, $\lambda(24)=2$, $\mathcal{A}(24)=\varnothing$ and

$$\Pi(24,1)=\frac{486}{\pi^4\log(2+\sqrt{3})\log(1+\sqrt{2})\log(5+2\sqrt{6})}.$$

For the more complicated case $q=15$ we get

$$\begin{aligned}C(15,1)^8 &= \frac{15}{8}\,\frac{\pi^8}{90^2}\,\frac{3328^2}{3375^2}\;e^{-\gamma}\;\Pi(15,1)\\ &\quad\times\prod_{p\equiv 1 \bmod 15}\Bigl(1-\frac{1}{p^4}\Bigr)^2\prod_{\substack{b\in\{4,11,14\}\\ p\equiv b \bmod 15}}\Bigl(\frac{1+p^{-2}}{1-p^{-2}}\Bigr)^2\end{aligned}$$

where we have used the fact that $\mathcal{A}(15)=\{4,\,11,\,14\}$ and the values $\zeta(4)=\pi^4/90$ and $\lambda(15)=4$. A fairly lengthy computation reveals that

$$\begin{aligned}\Pi(15,1)^{-1} &= \frac{2^9\,\pi^4}{3\cdot 15^4}\Bigl(\log^2\Bigl(\frac{\sin(\pi/15)}{\sin(4\pi/15)}\Bigr)\\ &\qquad+\log^2\Bigl(\frac{\sin(7\pi/15)}{\sin(2\pi/15)}\Bigr)\Bigr)\log\Bigl(\frac{1+\sqrt{5}}{2}\Bigr).\end{aligned}$$

This can be made more explicit using suitable trigonometrical identities and the value for $\sin(\pi/15)$,

Using (9) we can compute $C(5,1)$:

$$\begin{aligned}C(5,1)^4 &= \frac{5}{4}\,\frac{\pi^4}{90}\;e^{-\gamma}\;\Pi(5,1)\,\frac{624}{625}\prod_{p\equiv 1 \bmod 5}\Bigl(1-\frac{1}{p^4}\Bigr)\prod_{p\equiv 4 \bmod 5}\Bigl(1-\frac{1}{p^2}\Bigr)^{-2}\Bigl(1-\frac{1}{p^4}\Bigr)\\ &= \frac{13\sqrt{5}\;\pi^2\;e^{-\gamma}}{150\;\log((1+\sqrt{5})/2)}\prod_{p\equiv 1 \bmod 5}\Bigl(1-\frac{1}{p^4}\Bigr)\prod_{p\equiv 4 \bmod 5}\Bigl(\frac{1+p^{-2}}{1-p^{-2}}\Bigr)\end{aligned}$$

where we have used the value $\Pi(5,1)=25\sqrt{5}\pi^{-2}(4\log((1+\sqrt{5})/2))^{-1}$ and the fact that $\mathcal{A}(5)=\{4\}$.

The last examples are for $q=5$ and $a\in\{2,\,3,\,4\}$. A short computation shows that

$$\begin{aligned}f_{4,1}(x) &= \frac{1}{4}\log\Bigl(\frac{1+x}{1-x}\Bigr)+\frac{1}{2}\arctan(x)\\ f_{4,2}(x) &= \frac{1}{4}\log\Bigl(\frac{1+x^2}{1-x^2}\Bigr)\\ f_{4,3}(x) &= \frac{1}{4}\log\Bigl(\frac{1+x}{1-x}\Bigr)-\frac{1}{2}\arctan(x).\end{aligned}$$

Furthermore, using the fact that $\mathbb{Z}_5^*$ is generated by 2, we see that $\mathcal{B}(5,2)=\{3\}$ (with $s_3=3$ and $t_3=4$), $\mathcal{B}(5,3)=\{2\}$ (with $s_2=3$ and $t_2=4$) and $\mathcal{B}(5,4)=\{2,\,3\}$ (with $s_2=s_3=2$ and $t_2=t_3=4$). These results show that

$$
\begin{aligned}
c(5,2) &= \Pi(5,2)\ \exp(4B(5,2)) \\
&\qquad \times \prod_{p\equiv 3 \bmod 5} \exp\Bigl(4f_{4,3}\Bigl(\frac{1}{p}\Bigr)\Bigr) \exp\Bigl(4 \sum_{p\equiv 2 \bmod 5} \Bigl(f_{4,1}\Bigl(\frac{1}{p}\Bigr) - \frac{1}{p}\Bigr)\Bigr) \\
c(5,3) &= \Pi(5,3)\ \exp(4B(5,3)) \\
&\qquad \times \prod_{p\equiv 2 \bmod 5} \exp\Bigl(4f_{4,3}\Bigl(\frac{1}{p}\Bigr)\Bigr) \exp\Bigl(4 \sum_{p\equiv 3 \bmod 5} \Bigl(f_{4,1}\Bigl(\frac{1}{p}\Bigr) - \frac{1}{p}\Bigr)\Bigr) \\
c(5,4) &= \Pi(5,4)\ \exp(4B(5,4)) \\
&\qquad \times \prod_{\substack{b\in\{2,3\} \\ p\equiv b \bmod 5}} \exp\Bigl(4f_{4,2}\Bigl(\frac{1}{p}\Bigr)\Bigr) \exp\Bigl(4 \sum_{p\equiv 4 \bmod 5} \Bigl(f_{2,1}\Bigl(\frac{1}{p}\Bigr) - \frac{1}{p}\Bigr)\Bigr),
\end{aligned}
$$

and the corresponding values for the constants $C(5,a)$ can be found using (3). In the case of $C(5,4)$ we can be slightly more explicit since $f_{2,1}(x) = 1/2\log((1+x)/(1-x))$, so that

$$
c(5,4) = \Pi(5,4)\ \exp(4B(5,4)) \prod_{\substack{b\in\{2,3\} \\ p\equiv b \bmod 5}} \Bigl(\frac{1+p^{-2}}{1-p^{-2}}\Bigr) \prod_{p\equiv 4 \bmod 5} \Bigl(\frac{1+p^{-1}}{1-p^{-1}}\Bigr)^2 e^{-4/p}.
$$

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