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ON THE COMPOSITION OF A CERTAIN ARITHMETIC FUNCTION

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Abstract: Let $S(n)$ be the function which associates for each positive integer n the smallest positive integer k such that $n \mid k!$. In this note, we look at various inequalities involving the composition of the function $S(n)$ with other standard arithmetic functions such as the Euler function and the sum of divisors function. We also look at the values of $S(F_n)$ and $S(L_n)$, where F_n and L_n are the nth Fibonacci and Lucas numbers, respectively.

Keywords: Arithmetic functions connected with factorials, maximal orders of compositions of arithmetic functions, the largest prime factor of an integer, Fibonacci numbers.

1. Introduction

Let $S(n)$ be the function that associates to each positive integer n the smallest positive integer k such that $n \mid k!$. This function has received a lot of interest. For example, in 1918 A. J. Kempner [7] used the prime factorization of n to give the first algorithm for computing $S(n)$. A recent paper of Sondow [17] relates $S(n)$ with the measure of irrationality of e. Erdős [6] proposed to show that the set of n for which $S(n)$ is not the largest prime factor of n has asymptotic density zero. This was confirmed by Kastanas. For more on this story, the reader is referred to the beginning of Section 3.2.

In this paper, we look at the values of $S(f(n))$, where $f(n)$ is some other arithmetic function such as the Euler function of n and the sum of divisors function of n. We also investigate the values of $S(F_n)$ and $S(L_n)$, where F_n and L_n are the nth Fibonacci and Lucas numbers, respectively. Several results on compositions of arithmetic functions appear in [14], [15] and [16].

The rest of this paper, is organized as follows. In Section 2, we gather some properties of the function $S(n)$ that will be used in subsequent sections. In Section 3, we study the composition of $S(n)$ with several arithmetic functions. In Section 4, we study the values of $S(F_n)$ and $S(L_n)$, where $F = (F_n)_{n\geqslant 1}$ and $L = (L_n)_{n\geqslant 1}$ are the Fibonacci and Lucas numbers given by $F_1 = 1$, $F_2 = 1$,

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$L_1 = 1$, $L_2 = 3$ and $F_{n+2} = F_{n+1} + F_n$, $L_{n+2} = L_{n+1} + L_n$ for all $n \geqslant 1$. Section 5 is concerned with various limits.

Our notations are standard. We write $\phi(n)$ and $\sigma(n)$ for the Euler function and the sum of divisors function of n, respectively. We also write $\omega(n)$, $\Omega(n)$, $d(n)$, $\gamma(n)$ and $P(n)$ for the number of prime divisors of n, the number of prime power divisors (> 1) of n, the total number of divisors of n, the product of the distinct primes dividing n, and the largest prime factor of n, respectively. We make the convention that $P(1) = 1$. We use $\log x$ for the natural logarithm of x. We use the notations O, o as well as $\ll$, $\gg$ and $\asymp$ with their usual meanings. Recall that the notations $A = O(B)$, $A \ll B$ and $B \gg A$ are all equivalent to the fact that there exists a constant c such that $|A| < cB$. The notation $\asymp$ means that both $A \ll B$ and $B \ll A$ hold, while $A = o(B)$ means that $A/B \to 0$.

2. Estimates for $S(n)$

In the first lemma, we gather various inequalities concerning the function $S(n)$.

Lemma 1.

(i) *$S(a) \leqslant a$ holds for all positive integers a with equality if and only if $a = 4$, or a is prime;*

(ii) *If $n = p_1^{a_1} \cdots p_k^{a_k}$, where $p_1 < p_2 < \cdots < p_k$ are primes and $a_1, \ldots, a_k$ are positive integers, then*

$$S(n) = \max\{S(p_i^{a_i}) : i = 1, \ldots, k\}.$$

(iii) *$S(a) = P(a)$ holds for all squarefree positive integers a;*

(iv) *$S(ab) \leqslant S(a) + S(b)$ holds for all positive integers a and b;*

(v) *$S(ab) \leqslant aS(b)$ holds for all positive integers a and b with equality if and only if $a = 1$, or $(a, b) = (2, 2)$, $(4, 1)$, or a is prime and $b = 1$;*

(vi) *If $a \mid b$, then $S(b)/b \leqslant S(a)/a$;*

(vii) *$S(ab) \leqslant S(a)S(b)$ holds for all positive integers a and b;*

(viii)

$$S(a_1 \cdots a_k) \leqslant \min\{S(a_1) + \cdots + S(a_k),\ S(a_1) \cdots S(a_k)\}$$

holds for all positive integers $a_1, \ldots, a_k$.

Proof. Part (i) follows from the obvious fact that $a \mid a!$. The equality is clear if a is prime or $a = 4$, and it is easy to see that $a \mid (a-1)!$ holds for all composite $a \geqslant 6$.

Part (ii) is known (see [1], for example), and it is easy to prove. Indeed, it is clear that $S(p_i^{a_i}) \leqslant S(n)$. Conversely, if m is such that $p_i^{a_i} \mid m!$ holds for all $i = 1, \ldots, k$, then certainly $n \mid m!$. Taking m to be equal to $\max\{S(p_i^{a_i}) : i = 1, \ldots, k\}$ in the above argument, we get the desired equality.

Part (iii) is an immediate consequence of (i) and (ii).

Part (iv) follows from the fact that if $a \mid m!$ and $b \mid n!$, then

$$ab \mid m!n! \mid (m+n)!,$$

where the last divisibility relation follows from the fact that the binomial coefficient $\binom{m+n}{m}$ is an integer.

For part (v), observe that both sides of the claimed inequality are equal when $a = 1$. When $b = 1$, the inequality becomes $S(a) \leqslant a$, which holds by (i) with equality if and only if $a = 4$, or a is prime. Finally, when $a \geqslant 2$ and $b \geqslant 2$, then

$$S(ab) \leqslant S(a) + S(b) \leqslant a + S(b) \leqslant aS(b),$$

with equality if and only if $(a, b) = (2, 2)$.

Part (vi) follows by writing $b = ac$ and observing that, by (v), we have $S(b) = S(ac) \leqslant cS(a) = bS(a)/a$, which is equivalent to the claimed inequality.

Part (vii) is obvious when one of a or b is 1, and follows from (iv) and the known inequality $u + v \leqslant uv$ when both $u \geqslant 2$ and $v \geqslant 2$ applied with the pair $(u, v) = (S(a), S(b))$ when both $a \geqslant 2$ and $b \geqslant 2$.

Finally, part (viii) follows by induction on $k \geqslant 2$ having (iv) and (vii) as the induction bases. ■

In the next result, we collect various upper bounds for $S(n)$ in terms of other arithmetic functions of n.

Proposition 1.

(i) *$S(n) \leqslant P(n)n/\gamma(n)$;*

(ii) *If n is squarefull; i.e., $p^2 \mid n$ for all prime factors p of n, then $S(n) \leqslant 2P(n)n/\gamma(n)^2$;*

(iii) *$S(n) \leqslant P(n)\Omega(n)$;*

(iv) *If $n \geqslant 6$ is even, then $S(n) \leqslant n/2$;*

(v) *If $n > 1$ is odd, then $S(n) \leqslant P(n) \log n/(\log P(n))$;*

(vi) *$S(n) \leqslant \phi(n)$ holds for all positive integers $n \geqslant 8$ except when n is a prime or twice times a prime, with equality if and only if $n = 8,\ 9,\ 18$.*

Proof. Part (i) follows by writing $n = \gamma(n)(n/\gamma(n))$ and applying (v) of Lemma 1 with $a = (n/\gamma(n))$ and $b = \gamma(n)$ to conclude that

$$S(n) \leqslant \left(\frac{n}{\gamma(n)}\right) S(\gamma(n)) = \frac{P(n)n}{\gamma(n)}.$$

For the last equality above, we used (iii) of Lemma 1 together with the observation that $\gamma(n)$ is a squarefree number having the same largest prime factor as n does.

For part (ii), we use the same argument. Since n is squarefull, we get that $\gamma(n)^2 \mid n$. Thus, applying again (v) of Lemma 1, we first get

$$S(n) \leqslant S(\gamma(n))^2)\frac{n}{\gamma(n)^2}.$$

Next, since by (iv) of Lemma 1 we have $S(\gamma(n)^2) \leqslant 2S(\gamma(n)) = 2P(n)$, the desired inequality follows.

For (iii), write $n = p_1^{a_1} \cdots p_k^{a_k}$ with primes $p_1 < \cdots < p_k$ and positive integers $a_1, \ldots, a_k$ and observe that, by (viii) and (i) of Lemma 1, we have

$$S(n) \leqslant a_1 S(p_1) + \cdots + a_k S(p_k) \leqslant (a_1 + \cdots + a_k) p_k = P(n)\Omega(n).$$

For part (iv), observe first that if $n = 2^a$ is a power of 2, then by (viii) of Lemma 1 we have $S(n) = S(2^a) \leqslant 2a$. For $a \geqslant 4$, we have that $2a \leqslant 2^{a-1} = n/2$ and the desired equality holds. Since $S(8) = 4 = 8/2$, the desired inequality holds for $a = 3$ as well. When n is not a power of 2 but is even, then $\gamma(n) \geqslant 2P(n)$, and the desired inequality follows from (i) above.

For (v), we use (ii) and (viii) of Lemma 1 to conclude that $S(n) = S(p^a) \leqslant ap$ for some prime power divisor p^a of n. Since $p^a \leqslant n$, we have that $a \leqslant \log n / \log p$. Thus,

$$S(n) \leqslant \frac{p}{\log p} \log n \leqslant \frac{P(n)}{\log P(n)} \log n,$$

where we used the fact that $p \leqslant P(n)$ and the function $p \mapsto p/\log p$ is increasing as p runs through odd primes.

Let us now prove (vi). Write $n = p_1^{a_1} \cdots p_k^{a_k}$, where again $p_1 < \cdots < p_k$ are primes and $a_1, \ldots, a_k$ are positive integers. Assume first that $\omega(n) \geqslant 2$. Then, by (i), we have

$$S(n) \leqslant \frac{n}{p_1 \cdots p_{k-1}} \leqslant \frac{n}{p_1 p_2^{k-2}}. \tag{1}$$

However,

$$\frac{\phi(n)}{n} = \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) \geqslant \left(\frac{p_1 - 1}{p_1}\right) \left(\frac{p_2 - 1}{p_2}\right)^{k-1}. \tag{2}$$

Observe that the inequality

$$\left(\frac{p_1 - 1}{p_1}\right) \left(\frac{p_2 - 1}{p_2}\right)^{k-1} > \frac{1}{p_1 p_2^{k-2}}$$

is equivalent to $(p_1 - 1)(p_2 - 1)^{k-1} \geqslant p_2$, which holds for all $k \geqslant 2$ and primes $p_1 < p_2$ unless $k = 2$ and $p_1 = 2$. Thus, it remains to study the cases when $n = p^a$ and $n = 2^a p^b$.

Assume first that $n = p^a$. We may assume that $a \geqslant 2$, otherwise n is prime and the inequality fails anyway. We have that

$$S(n) = S(p^a) \leqslant ap$$

by (viii) and (i) of Lemma 1. Since $\phi(n) = p^{a-1}(p-1)$, it remains to check that

$$a < p^{a-2}(p-1). \tag{3}$$

Since $(p-1)p^{a-2} \geqslant \max\{2^{a-2}, p-1\}$, the above inequality (3) holds for all $a \geqslant 5$ (regardless of p) and all $p \geqslant 7$ (regardless of the value of $a \geqslant 2$). Checking the remaining values of n, we get that inequality (3) holds except when $n \in \{4, 8, 9, 16\}$.

Now we verify that $S(n) \leqslant \phi(n)$ holds when $n \neq 4$ in the above set with equality if and only if $n = 8,\ 9$.

Assume next that $n = 2^a p^b$. We may assume that not both a and b are 1, since in this case the desired inequality does not hold. Note that

$$\phi(n) = 2^{a-1}(p-1)p^{b-1}.$$

If both $a > 1$ and $b > 1$, then $\phi(n) > \max\{2^a, p^b\} \geqslant \max\{S(2^a), S(p^b)\} = S(n)$. If $a = 1$, then $b > 1$, and we get $\phi(n) = (p-1)p^{b-1}$ and $S(n) = S(2p^b) = S(p^b) \leqslant bp$. From the above arguments, the inequality $(p-1)p^{b-1} > bp$ holds for all $p \geqslant 3$ and $b \geqslant 2$ except for $(b,p) = (2,3)$, which leads to $n = 18$, for which $S(18) = 6 = \phi(18)$. Finally, if $b = 1$ and $a > 1$, then the inequalities

$$\phi(n) = 2^{a-1}(p-1) \geqslant \max\{2^a, p\} \geqslant \max\{S(2^a), S(p)\} = S(2^a p) = S(n)$$

hold with both being equalities if and only if $a = 2$ and $p = 3$. This completes the proof of (vi). ■

We conclude this section with some lower bounds on $S(n)$.

Proposition 2.

(i) *We have*

$$S(n) \geqslant P(n) \geqslant \frac{n}{\phi(n)} \geqslant \frac{\sigma(n)}{n}.$$

(ii) *If $n > 1$ is odd, then*

$$P(n) > \frac{\sigma(n)}{\phi(n)}.$$

Proof. We first prove (i). The left most inequality follows from Lemma 1 and the last one well-known. For the middle one, assume that $n > 1$ and write it again as $n = p_1^{a_1} \cdots p_k^{a_k}$, where $2 \leqslant p_1 < p_2 < \cdots < p_k = P(n)$ are primes and $a_1, \ldots, a_k$ are positive integers. Clearly, $P(k) \geqslant k+1$, therefore

$$\frac{n}{\phi(n)} = \prod_{i=1}^{k} \frac{p_i}{p_i - 1} \leqslant \prod_{i=1}^{k} \frac{i+1}{i} = \frac{2}{1} \cdot \frac{3}{2} \cdots \frac{k+1}{k} = k+1 \leqslant P(n).$$

Next, we prove (ii). For this, we note that

$$\begin{aligned} \frac{\sigma(n)}{\phi(n)} &= \prod_{i=1}^{k} \frac{1}{1 - 1/p_i} \left(1 + \frac{1}{p_i} + \cdots + \frac{1}{p_i^{a_i}}\right) \\ &< \prod_{i=1}^{k} \frac{1}{(1 - 1/p_i)^2} = \prod_{i=1}^{k} \left(\frac{p_i}{p_i - 1}\right)^2. \end{aligned}$$

Thus, it remains to prove that

$$\prod_{i=1}^{k} \frac{p_i}{p_i - 1} < \sqrt{p_k}.$$

When $k = 1$, this becomes $\sqrt{p_1} < p_1 - 1$, or $p_1^2 - 3p_1 + 1 > 0$, which holds because $p_1 \geqslant 3$. Assuming it to be true by induction for some $k \geqslant 1$, we then have that

$$\prod_{i=1}^{k+1} \frac{p_i}{p_i - 1} < \frac{p_{k+1}\sqrt{p_k}}{p_{k+1} - 1}.$$

Imposing that the last expression above is $< \sqrt{p_{k+1}}$, we get the inequality

$$\sqrt{p_k p_{k+1}} < p_{k+1} - 1,$$

which is equivalent to $p_{k+1}(p_{k+1} - 2 - p_k) + 1 > 0$, which in turn holds true because $p_{k+1} \geqslant p_k + 2$. This completes the proof of (ii). ■

We note the following corollary to (i) of Proposition 2 above.

Corollary 3.

(i) $\sigma(n!) < n \cdot n!$ *for* $n > 1$*;*
(ii) $\phi(n!) \geqslant (n-1)!$ *for* $n > 1$ *and the equality holds only when* $n = 2, 3$.

Proof. For the inequalities in both (i) and (ii), apply (i) of Proposition 2 above to $n!$ upon noting that $S(n!) = n$. For the equalities in (ii), observe that the proof of (i) of Proposition 2 was based on the fact that $p_j \geqslant j + 1$ holds for all $j = 1, \ldots, k$, where p_j was the jth prime factor of n. When $k \geqslant 3$, then the above inequality with $j = k$ becomes a strict inequality showing that in fact the inequality $S(n) > n/\phi(n)$ holds whenever $P(n) \geqslant 5$. Applying this to $n!$, we get that $\phi(n!) > (n-1)!$ holds for all $n \geqslant 5$. For the remaining cases, the given inequality is checked by hand. ■

We remark that the Diophantine equations $\phi(n!) = m!$ and $\sigma(n!) = m!$ were solved in [12]. Observe that Corollary 3 above shows that $n! < \sigma(n!) < (n+1)!$ (for $n > 1$) and $(n-1)! < \phi(n!) < n!$ (for $n > 4$), so that these equations do not have positive integer solutions (n, m) when $n > 1$, and $n > 4$, respectively.

3. On the composition of $S(n)$ with $\phi(n)$ and $\sigma(n)$

3.1. Maximal orders

In this section, we look at large values of the composition of $S(n)$ with other arithmetic functions such as $\sigma(n)$ and $\phi(n)$.

Theorem 4.

(i) *The inequality*

$$\sigma(S(n)) \leqslant n + 1 \tag{4}$$

holds for all $n \neq 4,\ 9$ *with equality if and only if* n *is prime.*

(ii) *The inequality*

$$S(\sigma(n)) \leqslant 2n - \omega(n) \tag{5}$$

holds for all $n > 1$ *with equality if and only if* $2n - 1$ *is a Mersenne prime.*

Proof. We start with (i). Assume that $n > 1$. It follows easily from the results of Lemma 1 that $S(n)$ can be written as $S(n) = bp$, where p is prime, $b \leqslant a$ and $p^a \| n$. If $a = 1$, then $b = 1$, so $\sigma(S(n)) = \sigma(p) = p + 1 \leqslant n + 1$, with equality only for $n = p$.

Assume now that $a > 1$, so that also $b > 1$. Then

$$\sigma(S(n)) \leqslant \sigma(b)\sigma(p) = \sigma(b)(p+1).$$

It remains to show that $(p+1)\sigma(b) \leqslant p^b + 1$, as $p^b + 1 \leqslant p^a + 1 \leqslant n + 1$. Observe that the inequality

$$\sigma(b) \leqslant \frac{p^b+1}{p+1}$$

holds true provided that the inequality

$$\sigma(b) \leqslant \frac{2^b+1}{3} \tag{6}$$

holds true, because the function $p \mapsto (p^b+1)/(p+1)$ is increasing for $p \geqslant 2$. One checks that inequality (6) holds true for $b = 5,\ 6$. Since

$$\sigma(b) < b\left(1 + \frac{1}{2} + \cdots + \frac{1}{b}\right) < b\left(1 + \int_1^b \frac{dt}{t}\right) = b(1 + \log b),$$

we conclude that inequality (6) holds whenever $2^b + 1 \geqslant 3(\log b + 1)$. This last inequality holds when $b \geqslant 7$. Thus, we may assume $b \leqslant 4$, therefore $\sigma(b) \leqslant 7$. Thus, it is now enough to prove that

$$7 < \frac{p^b+1}{p+1}.$$

Since $b \geqslant 2$, it is enough that $7 < (p^2+1)/(p+1)$, and this last inequality holds for all $p \geqslant 11$. Thus, we only need to deal with the case when $b \leqslant 4$ and $p \leqslant 7$, so $S(n) = bp \leqslant 28$.

Using Mathematica, we checked that the desired inequality holds for all divisors of 28!, except when n is prime, or $n = 4,\ 9$, which finishes the proof of (i).

For (ii), we assume that $n > 1$ and write it as $n = \prod_{i=1}^k p_i^{a_i}$. Then

$$\sigma(n) = \prod_{i=1}^k \frac{p_i^{a_i+1}-1}{p_i-1}.$$

We shall use the inequality

$$\frac{p^{a+1}-1}{p-1} \leqslant 2p^a - 1,$$

which is equivalent to $(p^a-1)(p-2)\geqslant 2$, so it holds for all prime powers p^a with equality when $p=2$. Hence, by Lemma 1 (viii) and (i), we have

$$S(\sigma(n))\leqslant\sum_{i=1}^{k}S\left(\frac{p_i^{a_i+1}-1}{p_i-1}\right)\leqslant\sum_{i=1}^{k}\frac{p_i^{a_i+1}-1}{p_i-1}\leqslant\sum_{i=1}^{k}(2p_i^{a_i}-1)\leqslant 2n-\omega(n).$$

It can be seen easily that the equality $S(n)=2n-\omega(n)$ holds if and only if $k=1$, $p_1=2$, and $(p_1^{a_1+1}-1)/(p_1-1)$ is a prime. Hence the equality holds if and only if $2n-1$ is Mersenne prime. This completes the proof of (ii). ∎

Recall that the Dedekind function $\psi(n)$ is the multiplicative function whose value of the prime power p^a is $\psi(p^a)=p^{a-1}(p+1)$. We next record some inequalities for the compositions of $S(n)$, $\psi(n)$ and $\phi(n)$.

Proposition 5.

(i) *The inequality $S(\psi(n))<n$ holds for all $n\geqslant 4$;*
(ii) *The inequality $S(\psi(n))\leqslant\phi(\psi(n))$ holds for all positive integers $n\neq 2,3,4$, or n not a prime of the form $q=2p-1$ with p also prime;*
(iii) *The inequality $S(\phi(n))\leqslant\phi(\phi(n))$ holds for all positive integers $n\neq 3,4,5,6,8,10,12$, or n not a prime of the form $q=2p+1$ with p also prime. Equality occurs only when $n=1,\ 2,\ 15,\ 16,\ 20,\ 24,\ 30$;*
(iv) *The inequality $S(\sigma(n))<\phi(\sigma(n))$ holds whenever n is not a prime power;*
(v) *The inequality $S(\phi(n))\leqslant(n-1)/2$ holds for all $n\geqslant 9$ with equality if and only if n is a prime of the form $q=2p+1$.*

Proof. For part (i), assume first that n is coprime to 6. Observing that $\psi(n)=(n/\gamma(n))\psi(\gamma(n))$, it follows, by (v) and (viii) of Lemma 1, that

$$S(\psi(n))=S\left(\frac{n}{\gamma(n)}\psi(\gamma(n))\right)\leqslant\frac{n}{\gamma(n)}S(\psi(\gamma(n)))\leqslant\frac{n}{\gamma(n)}\prod_{p\mid n}S(p+1).\tag{7}$$

Since n is coprime to 6, we get that $S(p+1)\leqslant(p+1)/2<p$ for all primes p dividing n (see (iii) of Proposition 1), which takes care of (i). If n is not coprime to 6 but either $4\mid n$ or $9\mid n$, then letting q be one of the primes 2 or 3 such that $q^2\mid n$, we get, by a similar argument, that

$$S(\psi(n))=S\left(\frac{n}{q\gamma(n)}q(q+1)\prod_{\substack{p\mid n\\ p\neq q}}(p+1)\right)\leqslant\frac{n}{q\gamma(n)}S(q(q+1))\prod_{\substack{p\mid n\\ p\neq q}}S(p+1).$$

Since $q(q+1)\geqslant 6$ is even, it follows, by (iii) of Proposition 1, that $S(q(q+1))\leqslant q(q+1)/2$. Thus,

$$S(\psi(n))\leqslant\frac{q+1}{2}\,\frac{n}{\gamma(n)}\prod_{\substack{p\mid n\\ p\neq q}}S(p+1).$$

We now have again that $S(p+1) \leqslant (p+1)/2 < p$ holds for all primes $p \geqslant 5$ by (iii) of Proposition 1. Thus,

$$S(\psi(n)) \leqslant \frac{q+1}{2} \frac{n}{\prod_{\substack{p \leqslant 3 \\ p \mid n}} p} S(m),$$

where $m = 1$ if q is the only prime factor of n below 5, and $m = 6/q + 1$ if $6 \mid n$. Since $(q+1)/2 < q$, it suffices to deal with the case when $m = 6/q + 1$. In this case, if $q = 2$, we get

$$S(\psi(n)) \leqslant \frac{3}{2} \frac{n}{6} S(4) = n,$$

while if $q = 3$, we then get

$$S(\psi(n)) \leqslant \frac{4}{2} \frac{n}{6} S(3) = n.$$

As for equalities, tracing back our argument, we see that at (7) we used the fact that

$$S(\prod_{p \mid n} (p+1)) \leqslant \sum_{p \mid n} S(p+1) \leqslant \prod_{p \mid n} S(p+1),$$

and since $S(p+1) \geqslant 3$ for all primes p, we never get equality if $\omega(n) \geqslant 2$. Since also $S(p+1) < p$ for $p \geqslant 5$, it follows that it suffices to assume that $n = q^a$, where $q = 2, 3$. If $a \geqslant 3$, then we used

$$S(\psi(n)) = S(q^{a-1}(q+1)) \leqslant q^{a-2} S(q(q+1)),$$

but the above inequality is, in fact, a strict inequality. Thus, it remains to check that inequality (i) is strict for $n \in \{4, 9\}$, and this can be done by hand.

Assume last that $n \geqslant 4$, $\gcd(n, 6) > 1$, but that n is neither divisible by 4 nor by 9. Writing $p_1 < p_2 < \cdots < p_k$ for all the distinct prime factors of n, we have that $p_1 \in \{2, 3\}$, that $k \geqslant 2$, and that $p_2 \geqslant 5$. Again by (v) and (viii) of Lemma 1, we have that

$$S(\psi(n)) = S\left(\frac{n}{\gamma(n)} \psi(\gamma(n))\right) \leqslant \frac{n}{\gamma(n)} S(\psi(\gamma(n))) \leqslant \frac{n}{\gamma(n)} (p_k + 1), \tag{8}$$

where we used the obvious fact that

$$\psi(\gamma(n)) = (p_1 + 1) \cdots (p_k + 1) \mid (p_k + 1)!,$$

therefore $S(\gamma(n)) \leqslant p_k + 1$. Since obviously $p_k + 1 < 2p_k \leqslant \gamma(n)$, the desired inequality follows easily from (8). This completes the proof of (i).

For part (ii), we apply Proposition 1 (vi). We remark that $\psi(n) = 4$ only for $n = 3$, and $\psi(n)$ is a prime only for $n = 2$ because $\psi(n)$ is even for $n \geqslant 3$. Finally, let us look at those n such that $\psi(n) = 2p$ with p an odd prime. Write $n = 2^a m$, with $a \geqslant 0$ and m odd. If a is positive, then $3 \mid \psi(n)$, therefore $p = 3$ and $a = 2$

and $m = 1$ giving $n = 4$. If $a = 0$, then n cannot have two distinct prime factors since otherwise $\psi(n)$ would be a multiple of 4. Thus, $n = q^a$ with an odd prime q, therefore $\psi(n) = q^{a-1}(q+1) = 2p$. Since $q + 1 > 2$ is even, it follows that $a = 1$ and $q + 1 = 2p$.

For part (iii), we apply again Proposition 1 (vi). It remains to study the exceptions and the cases when equality might occur. Observe that if $n \geqslant 3$, then $\varphi(n) \geqslant 2$. Furthermore, since $\phi(n)$ is even, it follows that $\varphi(n)$ prime if and only if $\varphi(n) = 2$ and this happens only for $n = 3, 4, 6$. Furthermore, $\varphi(n) = 4$ only for $n = 5, 8, 10, 12$. Thus, we only need to study the cases when $\varphi(n) = 2p$, with p an odd prime. As in the case of (ii) above, it is easy to see that this is possible only when or $n = 9$ or $n = q$, where q is a prime of the form $q = 2p + 1$. The equality occurs only when $\phi(n) = 1,\ 8,\ 9$, giving the values of n appearing in (iii) above.

For part (iv), we apply again Proposition 1 (vi). Since $\omega(n) \geqslant 2$, it follows that $\sigma(n) \geqslant \sigma(pq) = (p+1)(q+1) > 9$, where p and q are two distinct primes dividing n. Proposition 1 (v) tells us that we also have to study the cases when $\sigma(n) = p$, or $\sigma(n) = 2p$, where p is an odd prime. However, if $p^a \| n$ and $q^b \| n$ are two prime powers exactly dividing n, then $\sigma(p^a)\sigma(q^b) \mid \sigma(n)$, and both numbers $\sigma(p^a) \geqslant p + 1 \geqslant 3$ and $\sigma(q^b) \geqslant q + 1 \geqslant 3$ exceed 2. This shows that $\sigma(n)$ cannot be a prime or twice times a prime for such values of n.

For part (v), observe that, by (iii), we have that $S(\phi(n)) \leqslant \phi(\phi(n))$ with the exceptions appearing at (ii). Since $n \geqslant 9$, we have that $\phi(n)$ is even, so that $\phi(\phi(n)) \leqslant \phi(n)/2 \leqslant (n-1)/2$. Thus, the only possible exceptions to the desired inequality are among $n = 10,\ 12$, or n a prime of the form $q = 2p + 1$ with p also prime. One checks that none of these values is in fact exceptional, and that when $n = q = 2p + 1$, we have in fact equality. ■

We conclude this section by remarking that

$$S(d(n)) \leqslant \omega(n) + \Omega(n).$$

Indeed, for $n = p_1^{a_1} \cdots p_k^{a_k}$, where $p_1 < \cdots < p_k$ are primes and $a_1, \ldots, a_k$ are positive integers, we have that $d(n) = (a_1 + 1) \cdots (a_k + 1)$, so by Lemma 1 (iv) and (i), we have that

$$S(d(n)) \leqslant \sum_{i=1}^{k} S(a_i + 1) \leqslant \sum_{i=1}^{k} (a_i + 1) = \Omega(n) + \omega(n).$$

3.2. Normal orders

Here, we look at what $S(\sigma(n))$ and $S(\phi(n))$ "normally"; i.e., for most integers, are. We recall that it was a problem of Erdős to prove that $S(n) = P(n)$ holds for most positive integers n and to find the counting function of the exceptional set. Work on this problem has an amusing history which we now recall. In 1999, K. Ford [9] showed that the number of $n \leqslant x$ such that $S(n) \neq P(n)$, denoted by $N(x)$, satisfies

$$N(x) = x \exp\left(-(\sqrt{2} + o(1))(\log x \log\log x)^{1/2}\right) \tag{9}$$

as $x \to \infty$. In fact, he proposed a more precise formula giving an asymptotic for $N(x)$ as

$$N(x) \sim \frac{\sqrt{\pi}(1+\log 2)}{2^{3/4}}(\log x \log\log x)^{3/4} x^{1-1/u_0}\rho(u_0) \tag{10}$$

as $x \to \infty$, where u_0 is defined implicitly by

$$\log x = u_0(x^{1/u_0^2} - 1),$$

and $\rho(u)$ is the Dickman – de Bruijn function defined as

$$\rho(u) = 1 \qquad (0 \leqslant u \leqslant 1), \qquad \rho(u) = 1 - \int_1^u \frac{\rho(v-1)}{v} dv \qquad (u > 1).$$

It turns out that the constant $1+\log 2$ in (10) is incorrect and the correct constant should be 2. This was worked out in 2005 by Ivić in [11]. However, the formula (9) is correct. Meanwhile, in 2003, De Koninck and Doyon [8] published a paper in which they proved a formula of the same shape as (9) with $\sqrt{2}$ replaced by 2, which ended up being incorrect.

Here, we show that $S(\sigma(n))$ and $S(\phi(n))$ are, perhaps unsurprisingly, $P(\sigma(n))$ and $P(\phi(n))$, respectively, for most positive integers n and give some bounds on the size of the exceptional set. Our bounds are not sharp and they can probably be improved, but we do not have matching lower bounds for them.

Theorem 6. *The number $M(x)$ of positive integers $n \leqslant x$ such that either $S(\sigma(n))$ or $S(\phi(n))$ is not equal to $P(\sigma(n))$ or $P(\phi(n))$, respectively, satisfies*

$$M(x) \leqslant x \exp\left(-(\log x)^{1/8}\right)$$

for $x > x_0$.

Proof. We deal only with the function $\sigma(n)$, since for $\phi(n)$ the argument is even easier. Let $\varepsilon \in (0,1)$ be fixed, x be large and $y,\ z,\ w$ be parameters depending on x satisfying

$$y = \exp((\log x)^{7/8}), \qquad z = \exp((\log x)^{3/4}), \qquad w = \frac{(1+\varepsilon)}{\log 2}(\log x)^{1/8}$$

as $x \to \infty$.

Let $\mathcal{M}(x)$ be the set of $n \leqslant x$ such that $S(\sigma(n)) \neq P(\sigma(n))$. We start by discarding some elements $n \in \mathcal{M}(x)$ as follows.

Write $n = p_1^{a_1} \cdots p_k^{a_k}$, where $p_1 < \cdots < p_k$ are primes and $a_1, \ldots, a_k$ are positive integers.

We let $\mathcal{M}_1(x)$ be the subset of $n \leqslant x$ such that $\Omega(n) \geqslant w$. Lemma 13 in [13] shows that

$$\begin{aligned} \#\mathcal{M}_1(x) &\ll \frac{wx\log x}{2^w} \ll \frac{x(\log x)^2}{2^w} = \frac{x}{\exp((\log 2 + o(1))w)} \\ &= \frac{x}{\exp((1+\varepsilon+o(1))(\log x)^{1/8})} \end{aligned} \tag{11}$$

as $x \to \infty$.

From now on, we look at positive integers $n \leqslant x$ which are not in $\mathcal{M}_1(x)$. For them, $k = \omega(n) \leqslant \Omega(n) < w$. Next, we let $\mathcal{M}_2(x)$ be the set of $n \leqslant x$ such that there is $i \in \{1, \ldots, k\}$ with $\Omega((p_i^{a_i+1} - 1)/(p_i - 1)) \geqslant w$. Assume that $n \in \mathcal{M}_2(x)$. Write $n = p_i^{a_i} m$. Then

$$\frac{p_i^{a_i+1} - 1}{p_i - 1} = \frac{\sigma(n)}{\sigma(m)} \ll \frac{x \log\log x}{m},$$

where we used the maximal order $\sigma(n) \ll x \log\log x$ of $\sigma(n)$ in the interval $[1, x]$. Since the number $(p_i^{a_i+1} - 1)/(p_i - 1)$ has $\geqslant w$ prime factors counting repetitions, by the same Lemma 13 in [13], we infer that the number of possibilities for it once m is fixed is

$$\ll \frac{wx \log x \log\log x}{2^w m} \ll \frac{x(\log x)^3}{2^w m}.$$

Given the number $(p_i^{a_i+1} - 1)/(p_i - 1)$ of size $\ll x \log\log x$, we conclude that $a_i = O(\log x)$. But once both a_i and $(p_i^{a_i+1} - 1)/(p_i - 1)$ are fixed, then p_i is also uniquely determined. This argument shows that for a fixed m, the number of $n = p_i^{a_i} m$ such that $n \leqslant x$ and $\Omega((p_i^{a_i+1} - 1)/(p_i - 1)) \geqslant w$ is

$$\ll \frac{x(\log x)^4}{2^w m}.$$

Summing up over possible values for m, we get that

$$\begin{aligned} \#\mathcal{M}_2(x) &\ll \frac{x(\log x)^4}{2^w} \sum_{m \leqslant x} \frac{1}{m} \ll \frac{x(\log x)^5}{2^w} = \frac{x}{\exp((\log 2 + o(1))w)} \\ &= \frac{x}{\exp((1 + \varepsilon + o(1))(\log x)^{1/8})} \end{aligned} \tag{12}$$

as $x \to \infty$.

From now on, we work with integers $n \leqslant x$ not in $\cup_{i=1}^{2} \mathcal{M}_i(x)$. Note that for them,

$$\Omega(\sigma(n)) = \sum_{i=1}^{k} \Omega\left(\frac{p_i^{a_i+1} - 1}{p_i - 1}\right) \leqslant kw \leqslant w^2, \tag{13}$$

therefore

$$d(\sigma(n)) \leqslant 2^{\Omega(\sigma(n))} \leqslant 2^{w^2}. \tag{14}$$

Next, we discard the set of positive integers $n \leqslant x$ having $P(n) \leqslant y$ denoted by $\mathcal{M}_3(x)$. The numbers in $\mathcal{M}_3(x)$ are called *y-smooth numbers*. The counting function of y-smooth numbers below x is denoted by $\Psi(x, y)$. From known results concerning the counting function of smooth numbers, we have that in our ranges of y versus x the estimate

$$\#\mathcal{M}_3(x) = \Psi(x, y) \leqslant \frac{x}{\exp((1 + o(1))u \log u)}, \qquad \text{where } u = \frac{\log x}{\log y} \tag{15}$$

holds as $x \to \infty$ (see [4], for example). Hence,

$$\#\mathcal{M}_3(x) \leqslant \frac{x}{\exp((1/8 + o(1))(\log x)^{1/8} \log\log x)} \tag{16}$$

as $x \to \infty$, which is acceptable for us.

Next we discard the set of $n \leqslant x$ not in $\mathcal{M}_3(x)$ such that $P^2(n) \mid n$. Let $\mathcal{M}_4(x)$ be the set of such positive integers. For each integer $n \in \mathcal{M}_4(x)$, there is a prime $p > y$ (namely, $p = P(n)$), such that $p^2 \mid n$. For a fixed prime p, the number of $n \leqslant x$ which are multiples of p^2 is $\lfloor x/p^2 \rfloor \leqslant x/p^2$. Summing this up over all the possible values of the prime $p > y$, we get that

$$\begin{aligned}\#\mathcal{M}_4(x) &\leqslant x \sum_{y \leqslant p \leqslant x^{1/2}} \frac{1}{p^2} \leqslant x \sum_{y \leqslant m} \frac{1}{m^2} \ll x \int_y^\infty \frac{dt}{t^2} \\ &\ll \frac{x}{y} = \frac{x}{\exp((1+o(1))\log y)} = \frac{x}{\exp((1+o(1))(\log x)^{7/8})}\end{aligned} \tag{17}$$

as $x \to \infty$.

From now on, we assume that $n \leqslant x$ is not in $\cup_{i=1}^4 \mathcal{M}_i(x)$. In particular, we can write $n = Pm$, where $P = P(n) > y$ and $P > P(m)$.

We next discard the set of numbers $n \leqslant x$ of the above form such that $P(P+1) \leqslant z$. Let $\mathcal{M}_5(x)$ be the set of such numbers. Let $P > y$ be such a fixed prime. The number of $n \leqslant x$ such that $P \mid n$ is $\leqslant \lfloor x/P \rfloor \ll x/(P+1)$. Summing up over all the values of $P+1$, which are positive integers in $[y+1, x+1]$ that are z-smooth, we get that

$$\mathcal{M}_5(x) \leqslant x \sum_{\substack{y+1 \leqslant \ell \leqslant x+1 \\ P(\ell) \leqslant z}} \frac{1}{\ell}.$$

For the last sum above, we use the Abel summation formula to get that

$$\begin{aligned}\mathcal{M}_5(x) &= x \sum_{\substack{y+1 \leqslant \ell \leqslant x+1 \\ P(\ell) \leqslant z}} \frac{1}{\ell} \leqslant x \int_{y+1}^{x+1} \frac{d\Psi(t,z)}{t} \\ &= \frac{x\Psi(t,z)}{t}\Big|_{t=y+1}^{t=x+1} + x \int_{y+1}^{x+1} \frac{\Psi(t,z)}{t^2} dt.\end{aligned}$$

Putting $v = \log y / \log z$, one gets immediately, via estimate (15) for $\Psi(t,z)$, that

$$\begin{aligned}\#\mathcal{M}_5(x) &\leqslant \frac{x}{\exp((1+o(1))v \log v)} \int_{y+1}^{x+1} \frac{dt}{t} \\ &\leqslant \frac{x \log(x+1)}{\exp((1+o(1))v \log v)} = \frac{x}{\exp((1+o(1))v \log v))} \\ &= \frac{x}{\exp((1/8+o(1))(\log x)^{1/8} \log\log x)}\end{aligned} \tag{18}$$

as $x \to \infty$.

From now on, we assume that $n \leqslant x$ is not in $\cup_{i=1}^{5} \mathcal{M}_i(x)$. We write $P(n)+1 = Q\ell$, where $Q = P(P(n)+1)$. Clearly, $Q > z$. Let $\mathcal{M}_6(x)$ be the set of such positive integers $n \leqslant x$ such that $S(P+1) \neq Q$. In this case, either $Q^2 \mid P+1$, or $Q \| P+1$ but there is some prime power $p^b \mid P+1$ such that $S(p^b) > Q$. In this last case, $b > 1$ and $bp \geqslant S(p^b) > Q \geqslant z$. Furthermore, since $p^b \leqslant P+1 \leqslant x+1$, we have that $b \leqslant c_1 \log(x+1)$, where we can take $c_1 = 1/\log 2$. Thus, $p > z/(c_1 \log(x+1))$. Note also that $p^2 \mid P+1$. In conclusion, P has the property that there exists a prime $p > z/(c_1 \log(x+1))$ with $p^2 \mid P+1$. Note further that $p^2 \neq P+1$, since if $p^2 = P+1$, then $P = p^2 - 1 = (p-1)(p+1)$, but this last number is not prime for large x (the last number above is prime only for $p = 2$, but for us $p > z/(c_1 \log(x+1))$, and this last expression exceeds 2 for large values of x). Fix m and p. Then $P+1 \leqslant x/m+1 \leqslant 2x/m$. The number of numbers P such that $P+1$ is divisible by p^2 without being equal to p^2 (even by ignoring the fact that such P are primes) is at most $2x/(mp^2)$. Summing the above inequality up over all $m \leqslant x$ and $p \geqslant z/(c_1 \log(x+1))$, we get that

$$\begin{aligned} \#\mathcal{M}_6(x) &\leqslant 2x \left(\sum_{m \leqslant x} \frac{1}{m} \right) \left(\sum_{z/(c_1 \log(x+1)) < p} \frac{1}{p^2} \right) \ll \frac{x \log x}{z/(c_1 \log(x+1))} \\ &\ll \frac{x (\log x)^2}{z} = \frac{x}{\exp((1+o(1)) \log z)} = \frac{x}{\exp((1+o(1))(\log x)^{3/4})}, \quad (19) \end{aligned}$$

where in the above inequalities we used the same argument as the one used at inequality (17) to deal with the sum of the reciprocals of the squares of primes exceeding $z/(c_1 \log(x+1))$.

From now on, we work with numbers $n \leqslant x$ not in $\cup_{i=1}^{6} \mathcal{M}_i(x)$. Note that for them $n = Pm$, $P > P(m)$, $\sigma(n) = (P+1)\sigma(m)$, $P+1 = Q\ell$, $Q = P(P+1) > P(\ell)$ and $S(P+1) = Q$. Clearly, $S(\sigma(n)) \geqslant Q$ and it is not equal to Q, since otherwise it would follow easily that $Q = P(\sigma(n))$ contradicting our assumption that $n \in \mathcal{M}(x)$. Thus, $S(\sigma(n)) > Q$. We now distinguish two cases according to whether $S(\sigma(m)) = P(m)$ or not.

Let $\mathcal{M}_7(x)$ be the set of such n with $S(\sigma(m)) = P(\sigma(m))$. Let $R = P(\sigma(m))$. It follows that $R \| \sigma(m)$. Since $n \in \mathcal{M}(x)$, it follows that there exists a prime power $p^b \mid \sigma(n)$ such that $S(\sigma(n)) = S(p^b) > \max\{Q, R\}$. Put $b = b_1 + b_2$, where b_1 and b_2 are integers such that $p^{b_1} \| (P+1)$ and $p^{b_2} \| \sigma(m)$. Then $bp \geqslant S(p^b) \geqslant Q > z$, therefore by an argument used previously, we have that $b < c_1 \log(x+1)$ and $p > Q/b > z/(c_1 \log(x+1))$, and this last bound exceeds $c_1 \log(x+1) > b$ for large x. Thus, $p > b$, therefore $S(p^b) = pb$. It is now clear that both b_1 and b_2 are positive, since if, say $b_1 = 0$, then $p^b \mid \sigma(m)$, therefore $S(\sigma(m)) \geqslant S(p^b) = pb > R = P(\sigma(m))$, which is a contradiction. Thus, p divides both $P+1$ and $\sigma(m)$, and $p > z/(c_1 \log(x+1))$. Fix P and p. Then,

$$m \leqslant \frac{x}{P}$$

is a positive integer such that $\sigma(m) \equiv 0 \pmod{p}$. Theorem 1 in [2] shows that the number of such m is

$$\ll \frac{x}{pP} \min\left\{\log(x/P)^{2/3}, (\log p)^2(\log\log(x/P))^{5/3}\right\} \ll \frac{x(\log x)^3}{pP}.$$

Keeping p fixed and summing up the above inequality over all primes $P \leqslant x$ with $P+1 \equiv 0 \pmod p$, it follows that the number of such possibilities for n when p is fixed is

$$\begin{aligned} &\ll \frac{x(\log x)^3}{p} \sum_{\substack{P\leqslant x \\ P\equiv -1 \pmod p}} \frac{1}{P} \leqslant \frac{x(\log x)^3}{p} \sum_{1\leqslant\lambda\leqslant x/p} \frac{1}{-1+p\lambda} \\ &\ll \frac{x(\log x)^3}{p^2} \sum_{1\leqslant\lambda\leqslant x} \frac{1}{\lambda} \ll \frac{x(\log x)^4}{p^2}. \end{aligned}$$

Finally, summing up the above inequality over all possible values for $p \geqslant z/(c_1\log(x+1))$, we get that

$$\begin{aligned} \#\mathcal{M}_7(x) &\ll x(\log x)^4 \sum_{z/(c_1\log(x+1))<p} \frac{1}{p^2} \ll \frac{x(\log x)^4}{z/(c_1\log(x+1))} \ll \frac{x(\log x)^5}{z} \\ &= \frac{x}{\exp((1+o(1))\log z)} = \frac{x}{\exp((1+o(1))(\log x)^{3/4})} \end{aligned} \tag{20}$$

as $x \to \infty$.

Finally, assume that $n \in \mathcal{M}_8(x) = \mathcal{M}(x)\backslash\left(\cup_{i=1}^7 \mathcal{M}_i(x)\right)$. Then $S(\sigma(m)) > \max\{Q, P(\sigma(m))\}$. In particular, $\sigma(m) > z$ is a positive integer whose S function is not its largest prime factor. Thus, writing $\mathcal{N} = \{n : S(n) \neq P(n)\}$, we get that $\sigma(m) \in \mathcal{N}$. Observe that

$$(P+1)\sigma(m) = \sigma(n) \ll x\log\log x,$$

therefore for some positive constant c_2 we have

$$P+1 \leqslant \frac{c_2 x\log\log x}{\sigma(m)}. \tag{21}$$

Observe also that $\sigma(m) \in \mathcal{N}(c_2 x\log\log x)$, where for a positive real number t we write $\mathcal{N}(t) = \mathcal{N} \cap [1,t]$. Given $\sigma(m)$, let us see in how many ways can we recover m. Well,

$$\sigma(m) = \prod_{i=1}^{k-1} \frac{p_i^{a_i+1}-1}{p_i-1}.$$

Given $\sigma(m)$, the number $(p_i^{a_i+1}-1)/(p_i-1)$ can be chosen in $d(\sigma(m))$ ways, and once $(p_i^{a_i+1}-1)/(p_i-1)$ is chosen, then a_i can be chosen in at most $c_1\log x$ ways, after which p_i is uniquely determined. Thus, $p_i^{a_i}$ can be chosen in at most

$$c_1(\log x)d(\sigma(m)) \leqslant c_1(\log x)2^{w^2} = \exp((\log 2 + o(1))w^2)$$

ways by (14). Thus, since $k \leqslant w$, it follows that $m = p_1^{a_1} \cdots p_{k-1}^{a_k - 1}$ can be chosen in at most $\exp((\log 2 + o(1))w^3)$ ways once $\sigma(m)$ is known. This argument shows that the number of possibilities for $n \in \mathcal{M}_8$ is

$$\ll x \log\log x \exp((\log 2 + o(1))w^3) \sum_{\substack{z<\sigma<c_2 x \log\log x \\ \sigma \in \mathcal{N}}} \frac{1}{\sigma}.$$

For the last sum above, we use the Abel summation formula together with Ford's estimate (10) for $N(t) = \#\mathcal{N}(t)$ to get that

$$\begin{aligned} \sum_{\substack{z<\sigma<c_2 x \log\log x \\ \sigma \in \mathcal{N}}} \frac{1}{\sigma} &= \int_z^{c_2 x \log\log x} \frac{dN(t)}{t} \\ &\leqslant \frac{1}{\exp((\sqrt{2} + o(1))(\log z \log\log z)^{1/2})} \int_z^{c_2 x \log\log x} \frac{dt}{t} \\ &\ll \frac{x \log x}{\exp((\sqrt{2} + o(1))(\log z \log\log z)^{1/2})}. \end{aligned}$$

Thus, we get that

$$\begin{aligned} \#\mathcal{M}_8(x) &\leqslant \frac{x(\log x)^2}{\exp((\sqrt{2} + o(1))(\log z \log\log z)^{1/2} - (\log 2 + o(1))w^3)} \\ &= \frac{x}{\exp((\sqrt{3/2} + o(1))(\log x)^{3/8}(\log\log x)^{1/2})} \end{aligned} \tag{22}$$

as $x \to \infty$. Since the sets $\mathcal{M}_i(x)$ for $i \in \{1, \ldots, 8\}$ cover $\mathcal{M}(x)$, the conclusion of the theorem for the function $\sigma(n)$ follows from the inequalities (11), (12), (16), (17), (18), (19), (20) and (22).

A close analysis of our arguments shows that the specific ingredients of the proof for the case of the function $\sigma(n)$ were on one hand the fact that given $\sigma(p^a) = (p^{a+1} - 1)/(p - 1)$ on the scale of $O(x)$, then there are at most $O(\log x)$ possibilities for p^a, while on the other hand Theorem 1 from [2] concerning an upper bound for the number of $n \leqslant x$ such that $p \mid \sigma(n)$ which is uniform in p. Since both these results also hold for $\sigma(n)$ replaced by $\phi(n)$ (and, in fact, their proofs are even easier), the theorem follows for $\phi(n)$ as well. ■

Remark and an Open Problem. It is likely that a more careful analysis of our arguments will show that for some positive constants c_0 and κ, the inequality

$$\#\mathcal{M}(x) \leqslant \frac{x}{\exp((c_0 + o(1))(\log x)^{1/8}(\log\log x)^{\kappa})}$$

holds as $x \to \infty$. We did not make any efforts in this direction as we do not have a matching lower bound. We live as a challenge to the reader to find an asymptotic for $\#\mathcal{M}(x)$ as $x \to \infty$, or at least constants positive c_0, α and β such that the estimate

$$\#\mathcal{M}(x) = \frac{x}{\exp((c_0 + o(1))(\log x)^{\alpha}(\log\log x)^{\beta})}$$

holds as $x \to \infty$.

4. On the S function of Fibonacci and Lucas numbers

Recall that a Lucas sequence is a sequence of integers of the form

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{for } n = 0,\ 1\ \ldots,$$

where α and β are the roots of the quadratic equation $x^2 - rx - s = 0$. We make the convention that $|\alpha| \geqslant |\beta|$. Here, it is assumed that r and s are coprime nonzero integers, that $\Delta = r^2 + 4s \neq 0$, and that α/β is not a root of 1. Classical examples are the Fibonacci sequence $(F_n)_{n\geqslant 0}$ obtained for $r = s = 1$, as well as the sequence of Mersenne numbers $M_n = 2^n - 1$ for which $r = 3,\ s = -2$. Given a Lucas sequence $(u_n)_{n\geqslant 0}$, its companion sequence is the sequence of general term

$$v_n = \alpha^n + \beta^n \qquad \text{for all } n = 0,\ 1,\ \ldots.$$

When $u_n = F_n$, the companion sequence $(v_n)_{n\geqslant 0}$ is the sequence of Lucas numbers $(L_n)_{n\geqslant 0}$ given by $L_0 = 2,\ L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geqslant 0$. When $u_n = M_n$, its companion sequence is the sequence of general term $v_n = 2^n + 1$ for $n = 0,\ 1,\ \ldots$.

The quantity $P(u_n)$ has received considerable interest. For example, building upon some results of C. L. Stewart [18] and [19], Ford, Luca and Shparlinski showed recently [10] that

$$\sum_{n\geqslant 1} \frac{(\log n)^\alpha}{P(2^n - 1)}$$

is convergent for all fixed $\alpha < 1/2$. They cannot improve on the upper bound 1/2 on the exponent α.

Here, we prove a few results on $S(F_n)$ and $S(L_n)$.

Theorem 7.

(i) *If $n > 12$, then each of $S(F_n)$ and $S(L_n)$ is of the form $S(p^a) = ap$ for some prime power p^a such that $a < p$;*

(ii) *If $n > 1$, then $S(F_n) \neq S(L_n)$.*

Proof. By the Primitive Divisor Theorem for Lucas sequences (see, for example, [3]), if $n > 12$, then F_n has a primitive prime factor. That is, there exists a prime $p \mid F_n$ such that $p \nmid F_m$ for all $1 \leqslant m \leqslant n - 1$. The same holds for L_n. Furthermore, if p is a primitive prime for either F_n or L_n, then $p \equiv \pm 1 \pmod{n}$. Thus, both inequalities $P(F_n) \geqslant n - 1$ and $P(L_n) \geqslant n - 1$ hold, so, both inequalities $S(F_n) \geqslant n - 1$ and $S(L_n) \geqslant n - 1$ hold as well.

Next, we certainly have that $S(F_n) = S(p^a)$, where p^a is some prime power dividing F_n. It is known that if $p \mid F_n$, then n is a multiple of the *order of apparition* $z(p)$ of p in the Fibonacci sequence. Recall that this is the smallest positive integer k such that $p \mid F_k$. Let $\mu_p(m)$ be the exponent at which the prime

p appears in the factorization of the positive integer m. Then it is known that if we put $e_p = \mu_p(F_{z(p)})$, we have

$$\mu_p(F_n) = e_p + \mu_p(n/z(p)) \leqslant e_p + \frac{\log n}{\log p}. \tag{23}$$

Furthermore, $z(5) = 5$, but if $p \neq 5$, then $p \equiv \pm 1 \pmod{z(p)}$. In particular, $z(p) \leqslant p + 1$ holds for all primes p. Thus,

$$p^{e_p} \leqslant F_{z(p)} \leqslant F_{p+1} \leqslant \alpha^p,$$

where $\alpha = (1 + \sqrt{5})/2$ in this case. Thus,

$$e_p \leqslant \frac{(p+1)\log \alpha}{\log p}.$$

In conclusion, we get that

$$a \leqslant \frac{(p+1)\log \alpha}{\log p} + \frac{\log n}{\log p}.$$

It is enough to show that if p^a is such that $S(F_n) = S(p^a)$, then $a < p$. Assume that this is not so. Then $a \geqslant p$ and, in particular, the above upper bound on a implies that

$$\frac{(p+1)\log \alpha}{\log p} + \frac{\log n}{\log p} \geqslant p. \tag{24}$$

Assume first that $p \geqslant 3$. Then $\log p \geqslant \log 3 > 1$ and $\log \alpha / \log p < 1/2$ and the above inequality (24) implies that

$$\frac{p+1}{2} + \log n > p,$$

yielding $p < 2\log n + 1$. Since also

$$a \leqslant \frac{p+1}{2} + \log n \leqslant 2\log n + 1,$$

we get that $n - 1 \leqslant P(F_n) \leqslant S(F_n) = S(p^a) \leqslant pa \leqslant (2\log n + 1)^2$, giving $n \leqslant 107$.

This was when $p \geqslant 3$. When $p = 2$, then it is known that for $a \geqslant 3$, 2^a divides F_n if and only if $3 \cdot 2^{a-1}$ divides n. Thus, $a \leqslant \log(n/3)/\log 2 + 1$, giving $pa \leqslant 2\log(n/3)/\log 2 + 2$. Thus, $n - 1 \leqslant P(F_n) \leqslant S(F_n) \leqslant pa \leqslant 2\log(n/3)/\log 2 + 2$, which has no solutions for $n \geqslant 4$. In conclusion, $n \in [13, 107]$. However, one checks that in the range $n \in [13, 107]$, $S(F_n) = P(F_n)$. This shows that the inequality $a \geqslant p$ is impossible. Thus, indeed $S(F_n) = S(p^a)$ for some prime power $p^a \| F_n$ with $a < p$.

The proof for the case of the Lucas sequence is similar. In this case, we observe that p is not 2. Indeed, since 8 never divides L_m for any positive integer m, it follows that if $S(L_n) = S(2^a)$ for some a such that $2^a \| L_n$, then $a \leqslant 2$, so

$S(L_n) \leqslant 4$, contradicting the fact that $S(L_n) \geqslant P(L_n) \geqslant n-1 \geqslant 12$ for $n > 12$. Since p is odd, it follows, as in the case of the Fibonacci sequence, that if we put $k(p)$ for the smallest positive integer m such that $p \mid L_{k(p)}$, then

$$\mu_p(L_n) \leqslant \mu_p(L_{k(p)}) + \mu_p(n/k(p)),$$

with equality if and only if $n/k(p)$ is odd. It is also well-known that $k(p)$ exists if and only if $z(p)$ is even in which case $k(p) = z(p)/2$. In particular, $L_{k(p)} \mid F_{z(p)}$ (because $F_{2m} = F_m L_m$), therefore

$$\mu_p(L_n) \leqslant e_p + \frac{\log n}{\log p},$$

which is the same inequality as inequality (23). Now the above argument together with a computer verification of the fact that $S(L_n) = P(L_n)$ holds for all $n \in [13, 107]$ proves that the same result holds for Lucas numbers. This takes care of (i).

For (ii), we first check numerically that $S(L_n) \neq S(F_n)$ for any $n \in [2, 12]$. For $n \geqslant 13$, we use (i) to infer that if $S(F_n) = S(L_n)$, then there exist $p_1^{a_1} \| F_n$ and $p_2^{a_2} \| L_n$ such that $p_1 a_1 = p_2 a_2$ and $a_1 < p_1,\ a_2 < p_2$. Identifying the largest prime factor in the above equation, we get $p_1 = p_2$. Thus, $p = p_1 = p_2$ is a prime dividing both F_n and L_n. Since $L_n^2 - 5F_n^2 = (-1)^n 4$, we get that $p^2 \mid 4$, therefore $p = 2$. But since 8 does not divide L_n, we get that $a_2 \in \{1, 2\}$. Thus, $S(L_n) \leqslant 4$, which is false because $S(L_n) \geqslant P(L_n) \geqslant n-1 \geqslant 12$. This shows that indeed $S(F_n) \neq S(L_n)$ for all $n > 1$. ■

Theorem 8.

(i) *The inequality*

$$S(F_n) \geqslant (c_0 + o(1)) \frac{n\phi(n)}{(\log n)^2} \tag{25}$$

holds as $n \to \infty$, *where* $c_0 = (\log \alpha)/3$. *The same inequality holds when* F_n *is replaced by* L_n.

(ii) *If* $\alpha > 1/2$, *then both series*

$$\sum_{n \geqslant 1} \frac{1}{S(F_n)^\alpha} \qquad \text{and} \qquad \sum_{n \geqslant 1} \frac{1}{S(L_n)^\alpha}$$

are convergent.

Proof. We start with (i). Let n be large. We look at the part of F_n which is build up just of primitive prime factors. Let A_n be this divisor of F_n. The proof of the Primitive Divisor Theorem shows that

$$A_n = \alpha^{(1+o(1))\phi(n)}$$

as $n \to \infty$ (see, for example, [5]). Write $A_n = \prod_{i=1}^k p_i^{a_i}$, where $p_1 < p_2 < \cdots < p_k$

are all primes and $a_1, \ldots, a_k$ are positive integers. Clearly, $p_i \geqslant n-1$ for all $i = 1, \ldots, k$, therefore

$$(n-1)^{a_i} \leqslant p_i^{a_i} \leqslant A_n \leqslant \alpha^{(1+o(1))\phi(n)},$$

giving $a_i \leqslant (c_1 + o(1))\phi(n)/\log(n-1)$ as $n \to \infty$, where $c_1 = \log \alpha$. In particular, $a_i < p_i$ holds for all $i = 1, \ldots, k$, once n is large. Write

$$S = S(A_n) = \max\{S(p_i^{a_i}) : i = 1, \ldots, k\} = \max\{a_i p_i : i = 1, \ldots, k\}.$$

Clearly, $S(F_n) \geqslant S$. It suffices to find a lower bound on S. The inequality $a_i \leqslant S/p_i$ holds for all $i = 1, \ldots, k$. Furthermore, $p_k \leqslant S$ and $p_1, \ldots, p_k$ are primes which are congruent to $\pm 1 \pmod n$. We thus get

$$\begin{aligned} (c_1 + o(1))\phi(n) = \log A_n &= \sum_{i=1}^{k} a_i \log p_i \leqslant S \sum_{i=1}^{k} \frac{\log p_i}{p_i} \\ &\leqslant S \left(\sum_{\substack{p \leqslant S \\ p \equiv \pm 1 \pmod n}} \frac{\log p}{p} \right). \end{aligned} \tag{26}$$

Observe that, by writing the condition $p \equiv 1 \pmod n$ as $p = 1 + n\lambda$ for some positive integer λ, and discarding the fact that the sum is only over primes (but keeping the arithmetic progression modulo n), we get that

$$\begin{aligned} \sum_{\substack{p \leqslant S \\ p \equiv 1 \pmod n}} \frac{\log p}{p} &\leqslant \sum_{\substack{m \leqslant S \\ m \equiv 1 \pmod n}} \frac{\log m}{m} \leqslant \sum_{1 \leqslant \lambda \leqslant S/n} \frac{\log(1+n\lambda)}{1+n\lambda} \\ &< \sum_{1 \leqslant \lambda \leqslant S/n} \frac{\log(n\lambda)}{n\lambda} = \frac{\log n}{n} \sum_{1 \leqslant \lambda \leqslant S/n} \frac{1}{\lambda} + \frac{1}{n} \sum_{1 \leqslant \lambda \leqslant S/n} \frac{\log \lambda}{\lambda} \\ &\leqslant \frac{\log n}{n} \left(1 + \int_1^{S/n} \frac{dt}{t} \right) + \frac{1}{n} \left(c_2 + \int_3^{S/n} \frac{\log t dt}{t} \right) \\ &< \frac{\log n}{n} (1 + \log(S/n)) + \frac{1}{n} \left(c_2 + \frac{1}{2} \log(S/n)^2 \right) \\ &= \frac{1}{n} \left(\log n \log(S/n) + \frac{1}{2} \log(S/n)^2 + O(\log n) \right), \end{aligned}$$

where $c_2 = (\log 2)/2 + (\log 3)/3$. Furthermore, since the inequality $n\lambda - 1 \geqslant n(\lambda - 1) + 1$ holds for all $\lambda \geqslant 2$ and all $n \geqslant 2$, we also have that

$$\begin{aligned} \sum_{\substack{p \leqslant S \\ p \equiv -1 \pmod n}} \frac{\log p}{p} &\leqslant \frac{\log(n-1)}{n-1} + \sum_{\substack{m \leqslant S \\ m \equiv 1 \pmod n}} \frac{\log m}{m} \\ &\leqslant \frac{1}{n} \left(\log n \log(S/n) + \frac{1}{2} \log(S/n)^2 + O(\log n) \right). \end{aligned}$$

Putting everything together, we get that

$$(c_1 + o(1))\phi(n) \leqslant \frac{S}{n}\left(2\log n \log(S/n) + \log(S/n)^2 + O(\log n)\right).$$

The above inequality leads easily to the conclusion that

$$S \geqslant (c_0 + o(1))\frac{\phi(n)n}{(\log n)^2}$$

as $n \to \infty$, where $c_0 = c_1/3$, which is what we wanted to prove. The proof of part (i) for L_n is similar.

For (ii), we use (i) and the fact that $\phi(n) \gg n/\log\log n$ to conclude that for all fixed $\varepsilon > 0$, there exists n_ε such that the inequality $S(F_n) > n^{2-\varepsilon}$ holds for $n > n_\varepsilon$, and the same holds for $S(L_n)$. Thus, given $\alpha > 1/2$, and choosing $\varepsilon > 0$ sufficiently small such such that $\alpha(2-\varepsilon) = 1+\delta$ holds with some $\delta > 0$, it follows that $S(F_n)^\alpha > n^{1+\delta}$ for n sufficiently large, showing that

$$\sum_{n\geqslant 1} \frac{1}{S(F_n)^\alpha} \leqslant O(1) + \sum_{n\geqslant 1} \frac{1}{n^{1+\delta}} = O(1).$$

The same argument applies with F_n replaced by L_n. ■

Remark. Using sieve methods to deal with the sum

$$\sum_{\substack{p\leqslant S \\ p\equiv \pm 1 \pmod n}} \frac{\log p}{p}$$

appearing in the right hand side of inequality (26), like the Brun-Titchmarsh Theorem and the Abel summation formula, we could have saved another logarithmic factor from estimate (25). We do not enter into such details.

All the results from this section apply to other Lucas sequences as well. For example, for the case of $M_n = 2^n - 1$, an argument similar to the one above leads to the conclusion that the inequality

$$S(2^n \pm 1) \geqslant (c_3 + o(1))\frac{\phi(n)n}{(\log n)^2}$$

holds as $n \to \infty$ with $c_3 = 2(\log 2)/3$. Here, the rôle of α is played by the dominant root $\alpha = 2$ of the characteristic equation $x^2 - 3x + 2 = 0$ of the sequence of Mersenne numbers $(M_n)_{n\geqslant 1}$, and the fact that $c_3 = 2(\log\alpha)/3$ is "twice" as large as c_0 (note that c_0 in (25) is only $c_0 = (\log\alpha)/3$) is due to the fact that the primitive prime factors of $2^n - 1$ are all congruent to $1 \pmod n$, as opposed to the primitive prime factors of F_n which can be congruent to either 1 or -1 modulo n. Similarly,

$$\sum_{n\geqslant 1} \frac{1}{S(2^n-1)^\alpha} < \infty$$

holds for all fixed $\alpha > 1/2$.

5. Limits and Densities

Here, we look at some limits and densities with composites of S with the arithmetic functions $\sigma(n)$, $\phi(n)$ and $d(n)$

Proposition 9.

(i)

$$\liminf_{n\to\infty} \frac{\sigma(S(n))}{n} = 0, \qquad \textit{and} \qquad \limsup_{n\to\infty} \frac{\sigma(S(n))}{n} = 1;$$

(ii)

$$\liminf_{n\to\infty} \frac{S(\sigma(n))}{n} = 0, \qquad \textit{and} \qquad \limsup_{n\to\infty} \frac{S(\sigma(n))}{n} \leqslant 2;$$

(iii)

$$\liminf_{n\to\infty} \frac{S(\varphi(n))}{n} = 0, \qquad \textit{and} \qquad \limsup_{n\to\infty} \frac{S(\varphi(n))}{n} \leqslant \frac{1}{2};$$

(iv)

$$\liminf_{n\to\infty} \frac{\sigma(S(n))}{S(n)} = 1, \qquad \textit{and} \qquad \limsup_{n\to\infty} \frac{\sigma(S(n))}{S(n)} = \infty;$$

(v)

$$\liminf_{n\to\infty} \frac{\sigma(nS(n))}{nS(n)} = 1, \qquad \textit{and} \qquad \limsup_{n\to\infty} \frac{\sigma(nS(n))}{nS(n)} = \infty;$$

(vi)

$$\liminf_{n\to\infty} \frac{\varphi(S(n))}{n} = 0, \qquad \textit{and} \qquad \limsup_{n\to\infty} \frac{\varphi(S(n))}{n} = 1;$$

(vii)

$$\liminf_{n\to\infty} \frac{\varphi(S(n))}{S(n)} = 0, \qquad \textit{and} \qquad \limsup_{n\to\infty} \frac{\varphi(S(n))}{S(n)} = 1;$$

(viii)

$$\liminf_{n\to\infty} \frac{S(\varphi(n))}{\varphi(n)} = 0, \qquad \textit{and} \qquad \limsup_{n\to\infty} \frac{S(\varphi(n))}{\varphi(n)} \leqslant 1/2;$$

(ix)

$$\liminf_{n\to\infty} \frac{\varphi(S(n+1))}{\varphi(S(n))} = 0, \qquad \textit{and} \qquad \limsup_{n\to\infty} \frac{\varphi(S(n+1))}{\varphi(S(n))} = \infty;$$

(x)

$$\liminf_{n\to\infty} \frac{S(d(n))}{\omega(n)+\Omega(n)} = 0, \qquad \textit{and} \qquad \limsup_{n\to\infty} \frac{S(d(n))}{\omega(n)+\Omega(n)} = 1.$$

Proof. The lower limit in (i) follows by choosing $n = p^2$, where p is an arbitrarily large prime by observing that $S(n)/n = 2/p$ for such choice of n. The fact that the upper limit in (i) is at most 1 follows by (i) of Theorem 4, and the fact that it is at least 1 follows by letting n go to infinity through primes.

The lower limit in (ii) follows from Theorem 6 by observing that $S(\sigma(n)) = P(\sigma(n)) < n/\log n$ holds for almost all positive integers n, while the upper limit in (ii) follows from (ii) of Theorem 4.

The lower limit in (iii) follows again from Theorem 6 by observing that $S(\phi(n)) = P(\phi(n)) < n/\log n$ holds for almost all positive integers n, while the upper limit in (iii) follows from (v) of Proposition 5.

Both limits in (iv) follow immediately from the fact that the set $S(n)$ contains all positive integers of the form ap, where a is arbitrary and $p > a$. For such numbers, $\sigma(S(n))/S(n) = (\sigma(a)/a)(1+1/p)$ and such ratios are in fact dense in $[1,\infty]$ when $p \to \infty$ and $a < p$ is arbitrary.

Similarly as in (iv), both limits in (v) follow because the set $nS(n)$ contains all numbers of the form ap^{a+1} (which are $nS(n)$ for $n = p^a$) when $a < p$ and p is prime. For such numbers, $\sigma(nS(n))/(nS(n)) = (\sigma(a)/a)(\sigma(p^{a+1})/p^{a+1})$, and these numbers are dense in $[1,\infty]$ once p tends to infinity and $a < p$ is arbitrary.

For the limits in (vi), take $n = p^2$ for the lower limit. The fact that the upper limit is at most 1 is obvious since $\phi(S(n)) \leqslant S(n) \leqslant n$, and the fact that it is at least 1 follows by letting n go to infinity through primes.

The limits in (vii) follow again because $S(n)$ contains all numbers of the form ap with $a < p$ and p prime. For such numbers, we have $\phi(S(n))/S(n) = (\phi(a)/a)$ $(1-1/p)$, and these fractions are dense in $[0,1]$ when p tends to infinity and $a < p$ is arbitrary.

The lower limit in (viii) follows by choosing $n = 2^k$ and the upper limit in (viii) follows from (iii) of Proposition 5 by observing that since $\phi(n)$ is even for $n \geqslant 3$, one has $S(\phi(n)) \leqslant \phi(\phi(n)) \leqslant \phi(n)/2$ for all large n except when $n = q = 2p+1$. In this last case, $S(\phi(n)) = p = \phi(n)/2$.

For (ix), take $n = 2^m - 1$. Then $S(n+1) = S(2^m) \leqslant 2m$, while $S(n) = S(2^m-1) \gg \phi(m)m/(\log m)^2$ by the results from Section 4. Thus, $\phi(S(n+1)) \leqslant S(n+1) \leqslant 2m$, while $\phi(S(n)) \gg S(n)/\log\log S(n) \gg m\phi(m)/(\log m)^3$. Thus,

$$\frac{\phi(S(n+1))}{\phi(S(n))} \gg \frac{\phi(m)}{(\log m)^3} \gg \frac{m}{(\log m)^4},$$

showing the upper limit in (ix). For the lower limit, take $n = 2^m$ and apply the same argument.

Finally, for (x), let p be a large prime and take $n = p^{p^2-1}$ for the lower limit. The fact that the upper limit is at most 1 follows from the remark at the end of Subsection 3.1, and the fact that it is achieved can be seen by taking $n = p^{p-1}$, where p is a prime tending to infinity. ■

As a byproduct of our arguments, we also record the following result whose proof has already appeared above.

Proposition 10.

(i) *The sequence* $(\sigma(S(n))/S(n))$ *is dense in* $[1,+\infty]$*;*
(ii) *The sequence* $(\varphi(S(n))/S(n))$ *is dense in* $[0,1]$*;*
(iii) *The sequence* $(\sigma(S(n))/\varphi(S(n)))$ *is dense in* $[1,\infty]$.

Remark. All results in of the above two Propositions hold when σ is replaced with ψ (Dedekind's function).

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