Functiones et Approximatio
40.2 (2009), 209–250

LARGE TIME EXISTENCE OF SOLUTIONS TO THE NAVIER–STOKES EQUATIONS IN AXIALLY SYMMETRIC DOMAINS WITH INFLOW AND OUTFLOW

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Dedicated to Professor Bogdan Bojarski
on the occasion of his 75th birthday

Abstract: We prove a long time existence of special regular solutions to the Navier–Stokes equations in an axially symmetric domain in $\mathbb{R}^3$, with boundary slip conditions and with inflow and outflow. We assume that an initial angular component of velocity and an angular component of the external force and angular derivatives of the cylindrical components of initial velocity and of the external force are sufficiently small in corresponding norms. We assume also that inflow and outflow is sufficiently close to homogeneous. Then there exists a solution such that velocity belongs to $W^{2,1}_{5/2}(\Omega^T)$ and gradient of pressure to $L_{5/2}(\Omega^T)$, and we do not have restrictions on T.

Keywords: Navier–Stokes equations, axially symmetric domain, global regular solutions, slip boundary conditions.

1. Introduction

In this paper we examine the motion of an incompressible viscous fluid with a fixed flux through a non straight pipe. The case with a straight pipe but in general not axially symmetric was considered in [27]. In that case we were looking for solutions close to two-dimensional (2d) solutions. The existence of global regular 2d solutions was proved in [8] but with non slip boundary conditions. In [27] we have the slip boundary conditions with inflow and outflow. But to prove the existence of global regular solutions in [27] we do not need the existence of 2d solutions. We need only to show an appropriate global estimate. To obtain such estimate a smallness of derivatives of initial velocity and the external force with respect to the variable along the axis of the considered pipe is assumed in some norms. This means that the solution from [27] is close to 2d solutions.

In this paper we are looking for solutions close to the axially symmetric solutions so the pipe must be axially symmetric too. Therefore our solution is such

Mathematics Subject Classification: 35Q35, 35K20, 76D05, 76D03.

Supported partially by MNiSW Polish Grant 1 P03A 021 30 and EC FP6 Marie Curie ToK programme SPADE2.

that the angular derivatives (in the cylindrical coordinates) of velocity and pressure and the angular component of velocity are small in some norms. The existence of global regular axially symmetric solutions with vanishing angular component of velocity was proved in [9, 17]. The existence of such solutions with nonvanishing angular component of velocity is still an open problem. Since we are looking for solutions close to the axially symmetric solutions we have to formulate all problems in this paper in the cylindrical coordinates. Therefore we need solvability to some boundary or initial-boundary value problems for an elliptic system, the heat equation and the Stokes system in weighted Sobolev spaces (see [5, 6, 19, 32–34]).

We examine the following problem

$$\begin{aligned} v_{,t} + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p_0) &= f && \text{in } \Omega^T=\Omega\times(0,T),\\ \operatorname{div} v &= 0 && \text{in } \Omega^T,\\ v\cdot\bar n &= 0 && \text{on } S_1^T=S_1\times(0,T),\\ \bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha+\gamma v\cdot\bar\tau_\alpha &= 0 && \text{on } S_1^T,\quad \alpha=1,2,\\ v\cdot\bar n &= d && \text{on } S_2^T=S_2\times(0,T),\\ \bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha &= 0 && \text{on } S_2^T,\quad \alpha=1,2,\\ v|_{t=0} &= v(0) && \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded axially symmetric domain with the boundary $S = S_1\cup S_2$.

By $v=(v_1,v_2,v_3)\in\mathbb{R}^3$ we denote the velocity of the fluid, $p_0\in\mathbb{R}$ the pressure, $f=(f_1,f_2,f_3)\in\mathbb{R}^3$ the external force, $\bar n$ is the unit outward vector normal to S, $\bar\tau_\alpha$, $\alpha=1,2$, are tangent to S, $\gamma>0$ is the constant slip coefficient. Moreover, the dot denotes the scalar product in $\mathbb{R}^3$.

By $\mathbb{T}(v,p)$ we denote the stress tensor of the form

$$\mathbb{T}(v,p)=\nu\mathbb{D}(v)-pI, \tag{1.2}$$

where ν is the constant viscosity coefficient, $\mathbb{D}(v)$ the dilatation tensor of the form

$$\mathbb{D}(v)=\{v_{i,x_j}+v_{j,x_i}\}_{i,j=1,2,3}, \tag{1.3}$$

and I is the unit matrix.

Let (x_1,x_2,x_3) be a local Cartesian system such that the x_3 axis is the axis of symmetry of Ω. Let (r,φ,z) be the cylindrical coordinates such that $x_1=r\cos\varphi$, $x_2=r\sin\varphi$, $x_3=z$.

Let $\bar e_r=(\cos\varphi,\sin\varphi,0)$, $\bar e_\varphi=(-\sin\varphi,\cos\varphi,0)$, $\bar e_z=(0,0,1)$ be vectors connected with cylindrical coordinates r,φ,z, respectively. Let u be any vector. Then cylindrical coordinates of u are denoted by $u_r=u\cdot\bar e_r$, $u_\varphi=u\cdot\bar e_\varphi$, $u_z=u\cdot\bar e_z$.

Let $R>0$, $a>0$ be given numbers. We assume that Ω is axially symmetric and is located in the rectangle $r\in[0,R]$, $z\in[-a,a]$, S_1 is described by the relation $\psi(r,z)=0$ and is located in a positive distance from the x_3 axis. S_2 is perpendicular to x_3 and $S_2(-a)$, $S_2(a)$ meet the x_3 axis at points $-a$ and a, respectively.

To describe an inflow and an outflow we define

$$d_1 = -v \cdot \bar{n}|_{S_2(-a)}, \qquad d_2 = v \cdot \bar{n}|_{S_2(a)}, \tag{1.4}$$

so $d_i \geqslant 0$, $i = 1, 2$, and by $(1.1)_{2,3}$ and (1.4) we have the compatibility condition

$$\Phi = \int\limits_{S_2(-a)} d_1 dS_2 = \int\limits_{S_2(a)} d_2 dS_2, \tag{1.5}$$

where Φ is flux.

The aim of this paper is to prove the long time existence of regular solutions to problem (1.1) without restrictions on magnitudes of the external force f, initial data $v(0)$, inflow d_1 and outflow d_2. We show the existence of solutions by regularizing weak solutions.

In general we follow the ideas from [27]. However, the considered geometry of Ω is different from that in [18]. Let $L(b) = S_2(b) \cap S_1$, where $b \in \{-a, a\}$. Then we assume that the angle between $S_2(b)$ and S_1 at $x \in L(b)$ is equal to $\frac{\pi}{2}$. The assumption simplifies considerations because otherwise we have to show the existence of considered elliptic problems and the Stokes system in weighted Sobolev spaces appropriate for problems in domains with edges. Omitting this assumption makes the proof of the main result of this paper much less complicated however the main physical features of problem (1.1) remain the same.

We assume that radius of $L(b)$ is $R(b)$, $b \in \{-a, a\}$.

To make boundary condition $(1.1)_4$ homogeneous we introduce new functions (δ, σ) such that

$$\begin{aligned} \delta_t - \nu \operatorname{div} \mathbb{D}(\delta) + \nabla \sigma = 0 \quad &\text{in} \quad \Omega^T, \\ \operatorname{div} \delta = 0 \quad &\text{in} \quad \Omega^T, \\ \delta_n|_{S_2^T(-a)} = -d_1, \ \ \delta_n|_{S_2^T(a)} = d_2, \ \ \delta_n|_{S_1^T} = 0 & \\ \nu \bar{n} \cdot \mathbb{D}(\delta) \cdot \bar{\tau}_\alpha + \gamma \delta \cdot \bar{\tau}_\alpha = 0 \quad &\text{on} \quad S_1^T, \quad \alpha = 1, 2, \\ \nu \bar{n} \cdot \mathbb{D}(\delta) \cdot \bar{\tau}_\alpha = 0 \quad &\text{on} \quad S_2^T, \quad \alpha = 1, 2, \\ \delta|_{t=0} = \delta(0), & \end{aligned} \tag{1.6}$$

where $\delta_n = \delta \cdot \bar{n}$, $\delta_{\tau_\alpha} = \delta \cdot \bar{\tau}_\alpha$, $\alpha = 1, 2$.

The compatibility condition for $(1.6)_{2,3}$ is satisfied in view of (1.5). Moreover, we have to assume that

$$\bar{n} \cdot \delta(0)|_{S_2(-a)} = -d_1|_{t=0}, \quad \bar{n} \cdot \delta(0)|_{S_2(a)} = d_2|_{t=0}. \tag{1.7}$$

Therefore, we introduce the new functions

$$\omega = v - \delta, \qquad p = p_0 - \delta, \tag{1.8}$$

which are solutions to the problem

$$\begin{aligned}
&\omega_{,t}+\omega\cdot\nabla\omega+\omega\cdot\nabla\delta+\delta\cdot\nabla\omega-\operatorname{div}\mathbb{T}(\omega,p)\\
&\qquad=f-\delta_{,t}-\delta\cdot\nabla\delta+\nu\operatorname{div}\mathbb{D}(\delta)\equiv\mathcal{F}(\delta,f)\quad\text{in}\quad\Omega^T,\\
&\operatorname{div}\omega=0\quad\text{in}\quad\Omega^T,\\
&\omega\cdot\bar n=0\quad\text{on}\quad S^T,\\
&\nu\bar n\cdot\mathbb{D}(\omega)\cdot\bar\tau_\alpha+\gamma\omega\cdot\bar\tau_\alpha=0\quad\text{on}\quad S_1^T,\quad\alpha=1,2,\\
&\nu\bar n\cdot\mathbb{D}(\omega)\cdot\bar\tau_\alpha=0\quad\text{on}\quad S_2^T,\quad\alpha=1,2,\\
&\omega|_{t=0}=v(0)-\delta(0)\equiv\omega(0)\quad\text{in}\quad\Omega,
\end{aligned}\tag{1.9}$$

where $\bar\tau_1|_{S_1}=\bar e_\varphi$, $\bar\tau_2|_{S_1}=a_2\bar e_r-a_1\bar e_z$, $\bar n|_{S_1}=a_1\bar e_r+a_2\bar e_z$, $a_1=\frac{\psi_{,r}}{\sqrt{\psi_{,r}^2+\psi_{,z}^2}}$, $a_2=\frac{\psi_{,z}}{\sqrt{\psi_{,r}^2+\psi_{,z}^2}}$, $\bar\tau_1|_{S_2}=\bar e_r$, $\bar\tau_2|_{S_2}=\bar e_\varphi$, $\bar n|_{S_2}=\bar e_z$ and S_1 is generated by rotation around the x_3-axis of the curve described by $\psi(r,z)=0$.

Now we formulate the main result of this paper. Let

$$\begin{aligned}
g&=\mathcal{F}_{r,\varphi}\bar e_r+\mathcal{F}_{\varphi,\varphi}\bar e_\varphi+\mathcal{F}_{z,\varphi}\bar e_z,\\
h&=\omega_{r,\varphi}\bar e_r+\omega_{\varphi,\varphi}\bar e_\varphi+\omega_{z,\varphi}\bar e_z.
\end{aligned}$$

Let

$$\begin{aligned}
X_1(T)&=\|g\|_{L_2(0,T;L_{6/5}(\Omega))}+\|g\|_{L_{2,-(1+\varepsilon_*)}(\Omega^T)}+\|\mathcal{F}_\varphi\|_{L_{2,1-\mu}(\Omega^T)}\\
&\quad+\|h(0)\|_{H^1_{-(1+\varepsilon_*)}(\Omega)}+\|\omega_\varphi(0)\|_{H^1_{1-\mu}(\Omega)},
\end{aligned}$$

$\mu\in(0,1)$, $\varepsilon_*\in(0,1)$ and ε_* can be chosen arbitrary small. The above and below introduced spaces are defined in Section 2.

Let $F=\operatorname{rot}\mathcal{F}$, $F'=F_r\bar e_r+F_z\bar e_z$, $\alpha=\operatorname{rot}\omega$, $\alpha'=\alpha_r\bar e_r+\alpha_z\bar e_z$,

$$\begin{aligned}
X_2(T)&=\|F'\|_{L_2(0,T;L_{6/5}(\Omega))}+\|F_r\|_{L_2(0,T;L_{6/5,-\mu}(\Omega))}+\|F'\|_{L_{2,1-\mu}(\Omega^T)}\\
&\quad+\|\alpha'(0)\|_{L_2(\Omega)}+\|\alpha_r(0)\|_{L_{2,-\mu}(\Omega)}+\|\alpha'(0)\|_{W^1_{2,1-\mu}(\Omega)},\\
Y_1(T)&=\|F_\varphi\|_{L_2(0,T;L_{6/5,-1}(\Omega))}+\|\alpha_\varphi(0)\|_{L_{2,-1}(\Omega)},\\
d_2(T)&=(1+T)(\|f\|_{L_1(0,T;L_2(\Omega))}+\|v(0)\|_{L_2(\Omega)})\equiv(1+T)d_1,\\
Y_2(T)&=\|f\|_{L_{5/2}(\Omega^T)}+\|v(0)\|_{W^{6/5}_{5/2}(\Omega)},\\
K&=\left\|k-\frac{\gamma}{2\nu}\right\|_{W^1_\infty(S_1)},
\end{aligned}$$

where k is the curvature of the curve $S_1\cap P$, where P is the plane passing through the axis of symmetry of Ω.

Theorem 1.1 (existence). *Let $X_i(T)<\infty$, $i=1,2$, $Y_1(T)<\infty$, $d_2(T)<\infty$, $K<\infty$. Let*

$$A=2\sigma[\varphi_2^2(X_1)Y_1^2+c_1K(K+1)(d_2^2+X_1^2)]+cY_2,$$

where $\sigma > 2$, φ_2 is an increasing positive function and c_1 is the constant from (3.66). Let X_3 be so small that

$$2\varphi_1^2(T, X_1, A)X_3^2 \leqslant \left(1 - \frac{1}{\sigma}\right)A, \qquad X_3 = X_1 + X_2,$$

where φ_1 is an increasing positive function. Then there exists a solution to problem (1.1) such that $v \in W_{5/2}^{2,1}(\Omega^T)$, $\nabla p_0 \in L_{5/2}(\Omega^T)$ and

$$\|v\|_{W_{5/2}^{2,1}(\Omega^T)} + |\nabla p_0|_{5/2,\Omega^T} \leqslant A,$$

Theorem 1.1 states a long time existence of solutions to problem (1.1) where the time T is reciprocically proportional to quantity X_3 which measures a distance between the solution and the axially symmetric solutions.

Moreover, to prove Theorem 1.1 we used the existence and estimates for the weak solution. The restriction on X_3 is connected with the main open problem for the Navier–Stokes equations: the regularity of weak solutions.

The existence of global weak solutions was proved long time ago: Leray [10] proved the existence for the Cauchy problem but Hopf [3] for initial boundary value problem with non-slip conditions. Since the regularity of weak solutions has not been proved yet many mathematicians looked for special regular solutions to the Navier–Stokes equations. In [7, 8, 9, 11, 17] the existence of two-dimensional, axially symmetric and helically symetric solutions was proved, respectively. In [14, 15] the existence of global regular solutions in thin domains was shown. In the next step the long time existence of regular of solutions to problem (1.1) close to the axially symmetric solution was proved in [20, 22, 24, 25, 28, 18, 31]. In this case weighted Sobolev spaces must be used. Moreover, the existence of axially symmetric solutions is possible in axially symmetric domains so this implies a restriction on the considered domain. The existence of long time solutions which remain close to the two-dimensional solutions is examined in [4, 16, 23, 26, 27, 29]. In this case Ω is a cylindrical domain but in general not axially-symmetric.

In [13] global existence was proved by prolongation the long time solutions from [16] step by step in time.

Finally, the existence of solutions to problem (1.1) which are close to a linear combination of axially symmetric and two-dimensional solutons was proved in [30].

2. Notation and auxiliary results

By c we denote the generic constant. By $c(\sigma)$, $\varphi(\sigma)$ we denote the generic functions which are always positive and nondecreasing.

To simplify considerations we introduce

$$
\begin{aligned}
|u|_{p,Q} &= \|u\|_{L_p(Q)}, \qquad Q\in\{\Omega,S,\Omega^T,S^T\},\ \ p\in[1,\infty],\\
\|u\|_{s,Q} &= \|u\|_{H^s(Q)}, \qquad Q\in\{\Omega,S\},\ \ s\in\mathbb{R}_+\cup\{0\},\\
\|u\|_{s,Q} &= \|u\|_{H^{s,s/2}(Q)}, \ \ Q\in\{\Omega^T,S^T\},\ \ s\in\mathbb{R}_+\cup\{0\},\\
\text{and}\quad \|u\|_{0,Q} &= |u|_{2,Q},
\end{aligned}
$$

$$
|u|_{p,q,\Omega^T} = \bigg[\int_0^T dt\bigg(\int_\Omega |u(x,t)|^p dx\bigg)^{q/p}\bigg]^{1/q}, \quad p,q\in[1,\infty],
$$

$$
|u|_{p,q,\mu,\Omega^T} = \bigg(\int_0^T dt\|u(t)\|^q_{L_{p,\mu}(\Omega)}\bigg)^{1/q}, \quad p,q\in[1,\infty], \quad \mu\in\mathbb{R}.
$$

Let us introduce the energy norm

$$
\|v\|_{V_2^s(\Omega^T)} = \operatorname*{ess\,sup}_{t\leqslant T}\|v(t)\|_{H^s(\Omega)} + \bigg(\int_0^t \|\nabla v(t)\|^2_{H^s(\Omega)}dt\bigg)^{1/2}, \quad 0\leqslant s\in\mathbb{N}\cup\{0\}.
$$

Now we introduce weighted spaces

$$
\|u\|_{L_{p,\mu}(Q)} = \bigg(\int_Q |u|^p r^{p\mu} dQ\bigg)^{1/p}, \quad p\in[1,\infty], \quad \mu\in\mathbb{R}, \quad Q\in\{\Omega,S,\Omega^T,S^T\},
$$

where dQ is the measure connected with the set Q, with the notation

$$
|u|_{p,\mu,Q} = \|u\|_{L_{p,\mu}(Q)}.
$$

Let us define $H^s_\mu(Q)$ for $Q\in\{\Omega,S\}$, $s\in\mathbb{Z}_+\cup\{0\}$, $\mu\in\mathbb{R}$ by

$$
\|u\|_{H^s_\mu(Q)} = \bigg(\sum_{|\alpha|\leqslant s}\int_Q |D^\alpha_x u|^2 r^{2(\mu-s+|\alpha|)}dQ\bigg)^{1/2} < \infty,
$$

where $L_{2,\mu}(\Omega) = H^0_\mu(\Omega)$ and $H^{s,s/2}_\mu(Q)$ for $Q\in\{\Omega^T,S^T\}$, $s\in\mathbb{Z}_+\cup\{0\}$, $\mu\in\mathbb{R}$ by

$$
\|u\|_{H^{s,s/2}_\mu(Q)} = \bigg(\sum_{|\alpha|+2i\leqslant s}\int_Q |D^\alpha_x\partial^i_t u|^2 r^{2(\mu-s+|\alpha|+2i)}dQ\bigg)^{1/2} < \infty.
$$

To simplify notation we introduce

$$
\|u\|_{s,\mu,Q} = \|u\|_{H^s_\mu(Q)} \quad \text{for } Q\in\{\Omega,S\}
$$

and

$$
\|u\|_{s,\mu,Q} = \|u\|_{H^{s,s/2}_\mu(Q)} \quad \text{for } Q\in\{\Omega^T,S^T\}.
$$

Similarly we introduce spaces $V^s_{p,\mu}(Q)$ by

$$\|u\|_{V^s_{p,\mu}(Q)} = \bigg(\sum_{|\alpha|\leqslant s}\int_Q |D^\alpha_x u|^p r^{p(\mu-s+|\alpha|)}dQ\bigg)^{1/p} \quad \text{for } Q\in\{\Omega,S\}$$

and

$$\|u\|_{V^{s,s/2}_{p,\mu}(Q)} = \bigg(\sum_{|\alpha|+2i\leqslant s}\int_Q |D^\alpha_x\partial^i_t u|^p r^{p(\mu-s+|\alpha|+2i)}dQ\bigg)^{1/p}$$

for $Q\in\{\Omega^T,S^T\}$, $p\in[1,\infty]$, $s\in\mathbb{Z}_+\cup\{0\}$, $\mu\in\mathbb{R}$, $\|u\|_{s,p,\mu,Q} = \|u\|_{V^s_{p,\mu}(Q)}$ for $Q\in\{\Omega,S\}$, $\|u\|_{s,p,\mu,Q} = \|u\|_{V^{s,s/2}_{p,\mu}(Q)}$ for $Q\in\{\Omega^T,S^T\}$.

Finally, we define

$$W^{2,1}_{2,\mu}(\Omega^T) = \{u:\ \|u\|_{W^{2,1}_{2,\mu}(\Omega^T)} = \bigg(\int_{\Omega^T}(u^2_{,xx}+u^2_{,t}+u^2)r^{2\mu}dxdt\bigg)^{1/2} < \infty\},$$

and use the notation

$$\|u\|_{W^{2,1}_{2,\mu}(\Omega^T)} = |||u|||_{2,2,\mu,\Omega^T}.$$

Moreover,

$$W^2_{2,\mu}(\Omega) = \{u:\ \|u\|_{W^2_{2,\mu}(\Omega)} = \bigg(\int_\Omega (u^2_{,xx}+u^2)r^{2\mu}dx\bigg)^{1/2} < \infty\}$$

and

$$\|u\|_{W^2_{2,\mu}(\Omega)} = |||u|||_{2,2,\mu,\Omega}.$$

Now we recall inequalities and imbedding theorems used in this paper.

From [12] we have the imbedding

$$\|u\|_{V^s_{q,\beta+s-l+\frac{n}{p}-\frac{n}{q}}(\Omega)} \leqslant c\|u\|_{V^l_{p,\beta}(\Omega)}, \quad \Omega\subset\mathbb{R}^n, \tag{2.1}$$

and $s-l+\frac{n}{p}-\frac{n}{q}\leqslant 0$.

Let us consider the problem

$$u_{,t}-\Delta u = f \quad \text{in } \Omega^T, \quad u=g \quad \text{on } S^T, \quad u|_{t=0}=u_0. \tag{2.2}$$

From [21] we have:

Lemma 2.1. *Let*

$$f\in L_{2,\mu}(\Omega^T), \quad g\in W^{3/2,3/4}_{2,\mu}(S^T), \quad u_0\in W^1_{2,\mu}(\Omega), \quad \mu\in(0,1). \tag{2.3}$$

Then there exists a solution to problem (2.2) such that $u\in W^{2,1}_{2,\mu}(\Omega^T)$ *and*

$$|||u|||_{2,2,\mu,\Omega^T} \leqslant c(|f|_{2,\mu,\Omega^T} + |||g|||_{3/2,2,\mu,S^T} + |||u_0|||_{1,2,\mu,\Omega}). \tag{2.4}$$

Similarly, let us consider the problem

$$u_{,t} - \Delta u = f \quad \text{in } \Omega^T, \quad \frac{\partial u}{\partial n} = g \quad \text{on } S^T, \quad u|_{t=0} = u_0. \tag{2.5}$$

From [21] we have:

Lemma 2.2. *Let the assumptions (2.3) for f and u_0 hold, let $g \in W_{2,\mu}^{1/2,1/4}(S^T)$, $\mu \in (0,1)$. Then there exists a solution to problem (2.5) such that $u \in W_{2,\mu}^{2,1}(\Omega^T)$ and*

$$|||u|||_{2,2,\mu,\Omega^T} \leqslant c(|f|_{2,\mu,\Omega^T} + |||g|||_{1/2,2,\mu,S^T} + |||u_0|||_{1,2,\mu,\Omega}). \tag{2.6}$$

From Sect. 2 in [2] we have the Hardy inequality

$$|x^{-\beta} u|_{p,\mathbb{R}_+} \leqslant \frac{1}{\left|\beta - \frac{1}{p}\right|} |x^{-\beta+1} u_{,x}|_{p,\mathbb{R}_+} \tag{2.7}$$

which in our case takes the form

$$|u|_{2,\mu-1,\Omega} \leqslant \frac{1}{|\mu|} |u_{,x}|_{2,\mu,\Omega}, \qquad \mu \neq 0. \tag{2.8}$$

For solutions of (1.9) we have

Lemma 2.3. *Let $\eta = \bar{e}_0 \times \bar{x}$, $\bar{e}_0 = (0,0,1)$, $\bar{x} = (x_1, x_2, x_3)$, $\eta = \bar{x}_{,\varphi}$, Ω have the axis of symmetry $\bar{e}_0$. Let ω be a solution to (1.9) and let*

$$\left| \int_\Omega \omega(0) \cdot \eta dx \right| < \infty, \qquad \left| \int_{\Omega^t} \mathcal{F} \cdot \eta dx dt' \right| < \infty. \tag{2.9}$$

Then

$$\int_\Omega \omega \cdot \eta dx + \gamma \sum_{\alpha=1}^2 \int_{S_1^t} \omega \cdot \bar{\tau}_\alpha \eta \cdot \bar{\tau}_\alpha dS_1 = \int_{\Omega^t} \mathcal{F}(\delta, f) \cdot \eta dx dt' + \int_\Omega \omega(0) \cdot \eta dx. \tag{2.10}$$

Lemma 2.4. *Let*

$$E_\Omega(\omega) = \int_\Omega (\omega_{i,x_j} + \omega_{j,x_i})^2 dx \tag{2.11}$$

and let $\operatorname{div} \omega = 0$, $\omega \cdot \bar{n}|_S = 0$, $\left|\int_\Omega \omega \cdot \eta dx\right| < \infty$, $E_\Omega(\omega) < \infty$.
Then

$$\|\omega\|_{1,\Omega}^2 \leqslant c\left(E_\Omega(\omega) + \sum_{\alpha=1}^2 |\omega \cdot \bar{\tau}_\alpha|_{2,S_1}^2\right). \tag{2.12}$$

Proof. Since $E_\Omega(\omega) = 2(|\nabla \omega|_{2,\Omega}^2 - \int_S \omega_i \omega_j n_{i,x_j} dS)$ we have

$$|\nabla \omega|_{2,\Omega}^2 \leqslant c(E_\Omega(\omega) + \sum_{\alpha=1}^2 |\omega \cdot \bar{\tau}_\alpha|_{2,S_1}^2) \equiv I.$$

By the Poincare inequality

$$|\omega|_{2,\Omega} \leqslant c(|\nabla\omega|_{2,\Omega} + \sum_{\alpha=1}^{2} |\omega \cdot \bar{\tau}|_{2,S_1}) \leqslant cI^{1/2}, \tag{2.13}$$

because it is assumed that tangent coordinates of ω to S_1 are given on S_1 and the normal component of ω to S_1 vanishes on S_1. The above inequalities imply (2.12). This concludes the proof. ■

Repeating the proof of Lemma 2.2 [27] implies

Lemma 2.5. *Assume that $d_1 \in L_6(0,T;L_3(S_2))$, $\nabla\delta \in L_2(0,T;L_3(\Omega))$, $\mathcal{F} \in L_2(0,T;L_{6/5}(\Omega))$, $\omega(0) \in L_2(\Omega)$. Then solutions to (1.9) satisfy*

$$\|\omega\|_{V_2^0(\Omega^t)} \leqslant c\exp(\| d_1 \|^6_{3,6,S_2^t} + \| \nabla\delta \|^2_{3,2,\Omega^t})[\| \mathcal{F} \|_{6/5,2,\Omega^t} + |\omega(0)|_{2,\Omega}] \equiv d_1(t), \quad t \leqslant T. \tag{2.14}$$

Proof. Multiplying $(1.9)_1$ by ω, integrating over Ω and using the boundary conditions we obtain

$$\frac{1}{2}\frac{d}{dt}|\omega|^2_{2,\Omega} + \int_\Omega \omega \cdot \nabla\delta \cdot \omega dx + \nu \int_\Omega |\mathbb{D}(\omega)|^2 dx + \gamma \sum_{\alpha=1}^{2} \int_{S_1} |\omega \cdot \bar{\tau}_\alpha|^2 dS_1 + \frac{1}{2}\int_{S_2} \delta \cdot \bar{n}\omega^2 dS_2 = \int_\Omega \mathcal{F} \cdot \omega dx.$$

From (2.12) and $(1.6)_3$ we have

$$\frac{d}{dt}|\omega|^2_{2,\Omega} + \nu\|\omega\|^2_{1,\Omega} \leqslant c(|d_1|^6_{3,S_2} + |\nabla\delta|^2_{3,\Omega})|\omega|^2_{2,\Omega} + c|\mathcal{F}|^2_{6/5,\Omega},$$

where we used that

$$\begin{aligned} -\frac{1}{2}\int_{S_2} \delta \cdot \bar{n}\omega^2 dS_2 &\leqslant \frac{1}{2}\int_{S_2} d_1\omega^2 dS_2 \leqslant \frac{1}{2}|d_1|_{3,S_2}|w|^2_{2,S_2} \\ &\leqslant \varepsilon^{1/3}|\nabla w|^2_{2,\Omega} + c\varepsilon^{-5/3}|d_1|^6_{3,S_2}|w|^2_{2,\Omega}, \quad \varepsilon \in (0,1). \end{aligned}$$

Integrating the above inequality with respect to time yields (2.14). This concludes the proof. ■

To show the existence of a global regular solution to problem (1.1) we need the following quantities (see [20])

$$\begin{aligned} &h = \omega_{r,\varphi}\bar{e}_r + \omega_{\varphi,\varphi}\bar{e}_\varphi + \omega_{z,\varphi}\bar{e}_z, \quad q = p_{,\varphi}, \\ &\alpha = \operatorname{rot}\omega, \quad \chi = \alpha_\varphi = \omega_{r,z} - \omega_{z,r}, \quad w = \omega_\varphi, \quad F = \operatorname{rot}\mathcal{F}, \\ &g = \mathcal{F}_{r,\varphi}\bar{e}_r + \mathcal{F}_{\varphi,\varphi}\bar{e}_\varphi + \mathcal{F}_{z,\varphi}\bar{e}_z. \end{aligned} \tag{2.15}$$

From Section 1.1 [20] we obtain the following problems for h, q, α, w.

Let ω be given then (h,q) is a solution of the problem

$$\begin{aligned} h_{,t} - \operatorname{div}\mathbb{T}(h,q) &= -\omega\cdot\nabla h - h\cdot\nabla\omega + g \equiv G \quad \text{in } \Omega^T,\\ \operatorname{div} h &= 0 \quad \text{in } \Omega^T,\\ h\cdot\bar n &= 0,\\ \nu\bar n\cdot\mathbb{D}(h)\cdot\bar\tau_\alpha + \gamma\delta_{k1}h\cdot\bar\tau_\alpha &=, \quad \text{on } S_k^T,\ k=1,2,\ \alpha=1,2,\\ h|_{t=0} &= h(0) \quad \text{in } \Omega, \end{aligned} \tag{2.16}$$

where δ_{k1} is the Kronecker delta.

For given ω, h, q we have

$$\begin{aligned} w_{,t} + \omega\cdot\nabla w + \frac{\omega_r}{r}w - \nu\Delta w + \nu\frac{w}{r^2} &= \frac{1}{r}q + \frac{2\nu}{r^2}h_r + \mathcal{F}_\varphi \quad \text{in } \Omega^T,\\ \nu\bar n\cdot\nabla w &= -\gamma w + \nu\frac{a_1}{r}w \quad \text{on } S_1^T,\\ \partial_{x_3}w &= 0 \quad \text{on } S_2^T,\\ w|_{t=0} &= w(0) \quad \text{in } \Omega, \end{aligned} \tag{2.17}$$

where the boundary S_1 is described by the equation $\psi(r,z)=0$ and $a_1 = \frac{\psi_{,r}}{\sqrt{\psi_{,r}^2+\psi_{,z}^2}}$, $a_2 = \frac{\psi_{,z}}{\sqrt{\psi_{,r}^2+\psi_{,z}^2}}$. The cylindrical components of vorticity assume the form

$$\begin{aligned} \alpha_r &= \frac{1}{r}(\omega_{z,\varphi} - r\omega_{\varphi,z}),\\ \alpha_\varphi &= \omega_{r,z} - \omega_{z,r} \equiv \chi,\\ \alpha_z &= \frac{1}{r}[(r\omega_\varphi)_{,r} - \omega_{r,\varphi}]. \end{aligned} \tag{2.18}$$

Now we shall obtain boundary conditions for α.

Applying the proofs of Lemmas 3.1, 3.2 from [20] to (1.9) yields

Lemma 2.6. *Assume that S_1 is described by the relation $\psi(r,z)=0$. Let $a_1 = \frac{\psi_{,r}}{\sqrt{\psi_{,r}^2+\psi_{,z}^2}}$, $a_2 = \frac{\psi_{,z}}{\sqrt{\psi_{,r}^2+\psi_{,z}^2}}$. Then the following boundary conditions for ω and α*

take place

$$
\begin{aligned}
a_1\omega_{\varphi,r}+a_2\omega_{\varphi,z}+\frac{\gamma}{\nu}\omega_\varphi&=\frac{a_1}{r}\omega_\varphi\quad \textit{on}\ S_1^T,\\
2a_1a_2(\omega_{r,r}-\omega_{z,z})+(a_2^2-a_1^2)(\omega_{r,z}+\omega_{z,r})&\\
+\frac{\gamma}{\nu}(a_2\omega_r-a_1\omega_z)&=0\quad \textit{on}\ S_1^T,\\
\omega_z=0,\quad \omega_{r,z}=0,\quad \omega_{\varphi,z}&=0\quad \textit{on}\ S_2^T,\\
\alpha_\varphi&=2\bigg(k-\frac{\gamma}{2\nu}\bigg)\omega\cdot\bar\tau_2\quad \textit{on}\ S_1^T,\\
\bar\tau_2\cdot\alpha'&=-\frac{2a_1}{r}\omega_\varphi+\frac{\gamma}{\nu}\omega_\varphi\quad \textit{on}\ S_1^T,\\
(\bar n\cdot\alpha')_{,n}&=\beta_1h_\varphi+\beta_2h_r+\beta_3h_z+\beta_4w_{,r}\\
&\quad+\beta_5w_{,z}+\beta_6w\quad \textit{on}\ S_1^T,\\
\alpha_r=0,\ \alpha_\varphi=0,\ \alpha_{z,z}&=0\quad \textit{on}\ S_2^T,
\end{aligned}
\tag{2.19}
$$

where $\bar\tau_1|_{S_1}=\bar e_\varphi$, $\bar\tau_2|_{S_1}=a_2\bar e_r-a_1\bar e_z$, $\bar n|_{S_1}=a_1\bar e_r+a_2\bar e_z$, $\bar\tau_1|_{S_2}=\bar e_r$, $\bar\tau_2|_{S_2}=\bar e_\varphi$, $\bar n|_{S_2}=\bar e_z$, $\bar\tau_2\cdot\alpha'=a_2\alpha_r-a_1\alpha_z$, $\bar n\cdot\alpha'=a_1\alpha_r+a_2\alpha_z$, $\alpha'=(\alpha_r,\alpha_z)$, $\partial_n=\bar n\cdot\nabla$, β_i, $i=1,\dots,6$, *depend on* a_1,a_2.

In view of Lemma 2.6 we see that $\alpha'=(\alpha_r,\alpha_z)$ is a solution to the problem

$$
\begin{aligned}
&\alpha_{r,t}+\omega\cdot\nabla\alpha_r-(\alpha_r\omega_{r,r}+\alpha_z\omega_{r,z})-\frac{\chi}{r}h_r\\
&\qquad+\frac{2\nu}{r^2}(h_{r,z}-h_{z,r})+\frac{\nu\alpha_r}{r^2}-\nu\Delta\alpha_r=F_r\quad\text{in}\ \Omega^T,\\
&\alpha_{z,t}+\omega\cdot\nabla\alpha_z-(\alpha_r\omega_{z,r}+\alpha_z\omega_{z,z})\\
&\qquad-\frac{\chi}{r}h_z-\nu\Delta\alpha_z=F_z\quad\text{in}\ \Omega^T,\\
&\bar\tau_2\cdot\alpha'=-\frac{2a_1}{r}w+\frac{\gamma}{\nu}w\quad\text{on}\ S_1^T,\\
&(\bar n\cdot\alpha')_{,n}=\beta_1h_\varphi+\beta_2h_r+\beta_3h_z+\beta_4w_{,r}\\
&\qquad+\beta_5w_{,z}+\beta_6w\quad\text{on}\ S_1^T,\\
&\alpha_r=0,\ \alpha_{z,z}=0\quad\text{on}\ S_2^T,\\
&\alpha'|_{t=0}=\alpha'(0)\quad\text{in}\ \Omega,
\end{aligned}
\tag{2.20}
$$

where $\bar\tau_2=a_2\bar e_r-a_1\bar e_z$, $\bar n=a_1\bar e_r+a_2\bar e_z$, $\bar\tau_2\cdot\alpha'=a_2\alpha_r-a_1\alpha_z$, $\bar n\cdot\alpha'=a_1\alpha_r+a_2\alpha_z$, $\partial_n=\bar n\cdot\nabla$, β_i, $i=1,\dots,6$, depend on a_1,a_2 (see Lemma 3.2 from [20]).

Next, χ is a solution to the problem

$$
\begin{aligned}
&\chi_{,t} + \omega\cdot\nabla\chi + (\omega_{r,r}+\omega_{z,z})\chi - \nu\Big[\Big(r\Big(\frac{\chi}{r}\Big)_{,r}\Big)_{,r}\\
&\quad + \frac{1}{r^2}\chi_{,\varphi\varphi} + \chi_{,zz} + 2\Big(\frac{\chi}{r}\Big)_{,r}\Big] = \frac{2\nu}{r}\Big(-h_{\varphi,z} + \frac{1}{r}h_{z,\varphi}\Big)\\
&\qquad - \frac{1}{r}\Big(w_{,z}h_r - w_{,r}h_z + \frac{w}{r}h_z\Big)\\
&\qquad + \frac{2}{r}ww_{,z} + F_\varphi \quad \text{in } \Omega^T,\\
&\chi = 2\Big(k - \frac{\gamma}{2\nu}\Big)\omega\cdot\bar\tau_2 \quad \text{on } S_1^T,\\
&\chi = 0 \quad \text{on } S_2^T,\\
&\chi|_{t=0} = \chi(0) \quad \text{in } \Omega,
\end{aligned}
\tag{2.21}
$$

where k is the curvature of the curve $S' = \{r,z:\ \psi(r,z)=0\}$ which generates S by rotating it around the x_3 axis.

Finally, ω and p are calculated from the elliptic problems

$$
\begin{aligned}
\operatorname{rot}\omega &= \alpha \quad \text{in } \Omega,\\
\operatorname{div}\omega &= 0 \quad \text{in } \Omega,\\
\omega\cdot\bar n &= 0 \quad \text{on } S
\end{aligned}
\tag{2.22}
$$

and

$$
\begin{aligned}
&\Delta p = -\nabla\omega\cdot\nabla\omega + \operatorname{div}\mathcal{F} \quad \text{in } \Omega,\\
&\frac{\partial p}{\partial n} = \mathcal{F}\cdot\bar n + \nu\bar n\cdot\Delta\omega - \bar n\cdot\omega\cdot\nabla\omega \quad \text{on } S.
\end{aligned}
\tag{2.23}
$$

To obtain an estimate for χ we need

Lemma 2.7. *Assume that*

$$
A_i = \alpha_{i1}\omega_{r,r} + \alpha_{i2}\omega_{r,z} + \alpha_{i3}\omega_{z,r} + \alpha_{i4}\omega_{z,z},
\tag{2.24}
$$

where α_{ij} $i,j=1,\dots,4$, *depend on* a_1,a_2. *Assume that*

$$
\det\{-\alpha_{i1}a_2 - \alpha_{i3}a_1 + \alpha_{i4}a_2, -\alpha_{i1}a_1 + \alpha_{i2}a_2 + \alpha_{i4}a_1, -\alpha_{i2}a_1, \alpha_{i3}a_2\}_{i=1,\dots,4} \neq 0.
$$

Assume that the function B *depends on* a_1,a_2 *and their derivatives and depends linearly on components of* $\omega_{,x}$, $\nabla'\frac{h_\varphi}{r}$, $\nabla'\frac{\omega_r}{r}$, *where* $\nabla' = (\partial_r,\partial_z)$. *Then*

$$
\bar n\cdot\nabla\chi|_{S_1} = \sum_{i=1}^4 b_i\partial_s A_i + B, \quad \partial_s = \bar\tau_2\cdot\nabla,
\tag{2.25}
$$

where b_i, $i=1,\dots,4$, *depend on* a_1 *and* a_2.

Proof. From the form of χ we have

$$\bar n\cdot\nabla\chi|_S = a_1\chi_{,r} + a_2\chi_{,z} = a_1(\omega_{r,rz} - \omega_{z,rr}) + a_2(\omega_{r,zz} - \omega_{z,rz}). \tag{2.26}$$

We want to express $\bar n\cdot\nabla\chi|_S$ in terms of $\partial_s A_i$, $i=1,\dots,4$. Performing calculations in $\partial_s A_i$, $i=1,\dots,4$, we obtain the identities

$$\begin{aligned}&\partial_s\alpha_{i1}\omega_{r,r} + \partial_s\alpha_{i2}\omega_{r,z} + \partial_s\alpha_{i3}\omega_{z,r} + \partial_s\alpha_{i4}\omega_{z,z}\\ &+\alpha_{i1}(a_2\omega_{r,rr} - a_1\omega_{r,rz}) + \alpha_{i2}(a_2\omega_{r,rz} - a_1\omega_{r,zz})\\ &+\alpha_{i3}(a_2\omega_{z,rr} - a_1\omega_{z,rz}) + \alpha_{i4}(a_2\omega_{z,rz} - a_1\omega_{z,zz}) = \partial_s A_i, \quad i=1,\dots,4.\end{aligned} \tag{2.27}$$

Since $\bar n\cdot\nabla\chi|_S$ depends on four different second derivatives we have to eliminate the remaining two derivatives. For this purpose we use the continuity equation

$$\omega_{r,r} + \omega_{z,z} = -\frac{1}{r}(h_\varphi + \omega_r) \tag{2.28}$$

Differentiating (2.28) with respect to r and z we get

$$\begin{aligned}\omega_{r,rr} + \omega_{z,rz} &= -\Big(\frac{h_\varphi}{r} + \frac{\omega_r}{r}\Big)_{,r} \equiv d_1,\\ \omega_{r,rz} + \omega_{z,zz} &= -\frac{1}{r}(h_{\varphi,z} + \omega_{r,z}) \equiv d_2.\end{aligned} \tag{2.29}$$

Calculating $\omega_{r,rr}$ and $\omega_{z,zz}$ from (2.29) and inserting them to (2.27) we obtain

$$\begin{aligned}(-\alpha_{i1}a_2 - \alpha_{i3}a_1 + \alpha_{i4}a_2)\omega_{z,rz} &+ (-\alpha_{i1}a_1 + \alpha_{i2}a_2 + \alpha_{i4}a_1)\omega_{r,rz}\\ -\alpha_{i2}a_1\omega_{r,zz} + \alpha_{i3}a_2\omega_{z,rr} &= \partial_s A_i - (\partial_s\alpha_{i1}\omega_{r,r} + \partial_s\alpha_{i2}\omega_{r,z}\\ &\quad + \partial_s\alpha_{i3}\omega_{z,r} + \partial_s\alpha_{i4}\omega_{z,z})\\ &\quad - \alpha_{i1}a_2 d_1 + \alpha_{i4}a_1 d_2, \quad i=1,\dots,4.\end{aligned} \tag{2.30}$$

In view of the assumptions of the lemma we can calculate the second derivatives of ω from (2.30) and insert them to (2.26). In this way we obtain (2.25). This concludes the proof. ■

From [22] we recall

Lemma 2.8. *Let $g\in L_2(0,T;L_{6/5}(\Omega))$, $h(0)\in L_2(\Omega)$, $\omega\in L_2(0,T;W_3^1(\Omega))$. Then solutions to (2.16) satisfy*

$$\|h\|_{V_2^0(\Omega^t)} \leqslant c\exp(c\|\omega_{,x}\|_{3,2,\Omega^t}^2)[\| g\|_{6/5,2,\Omega^t} + |h(0)|_{2,\Omega}], \tag{2.31}$$

for $t\leqslant T$.

Let $\delta\in(0,1)$ and let

$$\mathfrak{N}_\delta(\Omega^T) = L_\infty(0,T;L_{4,-\delta}(\Omega)) \cap L_\infty(0,T;W_{2,-\delta}^1(\Omega)) \cap L_2(0,T;W_3^1(\Omega)). \tag{2.32}$$

Lemma 2.9 (see [22]). *Let* $\delta \in (0,1)$, $\omega \in \mathfrak{N}_\delta(\Omega^T)$, $g \in L_2(0,T;L_{6/5}(\Omega)) \cap L_{2,-(1+\varepsilon_*)}(\Omega^T)$, $h(0) \in H^1(\Omega) \cap H^1_{-(1+\varepsilon_*)}(\Omega)$, $\varepsilon_* \in (0,\delta)$. *Then solutions of (2.16) satisfy*

$$\|h\|_{2,-(1+\varepsilon_*),\Omega^t} + \|q\|_{L_2(0,t;H^1_{-(1+\varepsilon_*)}(\Omega))} \leqslant \varphi(\|\omega\|_{\mathfrak{N}_\delta(\Omega^t)})(|g|_{6/5,2,\Omega^t} + |h(0)|_{2,\Omega}) + c(|g|_{2,-(1+\varepsilon_*),\Omega^t} + \|h(0)\|_{1,-(1+\varepsilon_*),\Omega}), \quad t \leqslant T, \tag{2.33}$$

where φ *is an increasing positive function.*

3. Estimates

In this section we show a long time a priori estimate for solutions to problem (1.1). First for given w and h we obtain an estimate for χ.

Lemma 3.1. *Assume that* $h \in L_2(0,T;H^2_{-1}(\Omega))$, $w \in L_\infty(0,T;H^1_0(\Omega))$, $\omega_{\varphi,z} \in L_2(0,T;L_{4,-3/4-\varepsilon}(\Omega))$, $\varepsilon > 0$ *is a small number which will be chosen later,* $F_\varphi \in L_2(0,T;L_{6/5,-1}(\Omega))$, $S_1 \in C^2$. *Let*

$$K = \left\|k - \frac{\gamma}{2\nu}\right\|_{W^1_\infty(S_1)}.$$

Let $\{\varphi_j(x,t)\}$ *be a partition of unity near* S_1^t, $t \leqslant T$. *Then*

$$\begin{aligned}|\chi(t)|^2_{2,-1,\Omega} + \nu\int_0^t \left\|\nabla\frac{\chi(t')}{r}\right\|^2_{0,\Omega} dt' \leqslant c\exp(c|h|^2_{3,2,-1,\Omega^t})\Big[K(K+1) \\ \times \sum_j \int_{S_1^t \cap \operatorname{supp}\varphi_j} \left(|\omega_{,x}|^2 + |\omega|^2 + \left|\nabla\frac{h}{r}\right|^2\right) dx dt' \\ + (1 + \sup_t \|w\|^2_{1,0,\Omega}) \int_0^t \|h(t')\|^2_{2,-1,\Omega} dt' \\ + \frac{R^{2\varepsilon}}{\varepsilon^2} \sup_t \|w\|^2_{1,0,\Omega} \int_0^t |\omega_{\varphi,z}(t')|^2_{4,-3/4-\varepsilon,\Omega} dt' \\ + |F_\varphi|^2_{6/5,2,-1,\Omega^t} + |\chi(0)|^2_{2,-1,\Omega}\Big], \quad t \leqslant T,\end{aligned} \tag{3.1}$$

where the constants c *do not depend on* t.

Proof. First we introduce the set $\Omega_* = \{x \in \Omega : \ 0 < \varepsilon_* < r\}$ and add the artificial boundary condition

$$\chi|_{r=\varepsilon_*} = 0. \tag{3.2}$$

Multiplying $(2.21)_1$ by $\frac{\chi}{r^2}$ and integrating over Ω_* we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\chi|^2_{2,-1,\Omega_*}+\int\limits_{\Omega_*}[\omega\cdot\nabla\chi+(\omega_{r,r}+\omega_{z,z})\chi]\frac{\chi}{r^2}dx\\
&-\nu\int\limits_{\Omega_*}\Big[\Big(r\Big(\frac{\chi}{r}\Big)_{,r}\Big)_{,r}+\frac{1}{r^2}\chi_{,\varphi\varphi}+\chi_{,zz}+2\Big(\frac{\chi}{r}\Big)_{,r}\Big]\frac{\chi}{r^2}dx\\
&\qquad=-2\nu\int\limits_{\Omega_*}\frac{1}{r^2}\Big(-h_{\varphi,z}+\frac{1}{r}h_{z,\varphi}\Big)\frac{\chi}{r^2}dx\\
&\qquad\quad-\int\limits_{\Omega_*}\frac{1}{r}\Big(w_{,z}h_r-w_{,r}h_z+\frac{w}{r}h_z\Big)\frac{\chi}{r^2}dx\\
&\qquad\quad+2\int\limits_{\Omega_*}\frac{1}{r}ww_{\varphi,z}\frac{\chi}{r^2}dx+\int\limits_{\Omega_*}F_\varphi\frac{\chi}{r^2}dx.
\end{aligned}\tag{3.3}$$

The second term on the l.h.s. of (3.3) equals

$$\frac{1}{2}\int\limits_{\partial\Omega_*}\omega\cdot\bar n\frac{\chi^2}{r^2}d\partial\Omega_*+\int\limits_{\Omega_*}\Big(\omega_{r,r}+\omega_{z,z}+\frac{\omega_r}{r}\Big)\frac{\chi^2}{r^2}dx\equiv I_1,$$

where the first term in I_1 equals

$$\frac{1}{2}\int\limits_{S_*}\omega\cdot\bar n\frac{\chi^2}{r^2}dS+\frac{1}{2}\int\limits_{-a}^{a}\omega\cdot\bar n\frac{\chi^2}{r^2}\Big|_{r=\varepsilon_*}dz=0,$$

where $S_*=\{x\in S:\ 0<\varepsilon_*<r\}$ and we used $(1.1)_3$ and (3.2).

The second term in I_1 assumes the form

$$-\int\limits_{\Omega_*}\frac{h_\varphi}{r}\frac{\chi^2}{r^2}dx$$

which can be estimated by

$$\varepsilon_1\Big|\frac{\chi}{r}\Big|^2_{6,\Omega_*}+c(1/\varepsilon_1)|h_\varphi|^2_{3,-1,\Omega_*}\Big|\frac{\chi}{r}\Big|^2_{2,\Omega_*}.$$

The last term on the l.h.s. of (3.3) takes the form

$$\begin{aligned}
&-\nu\int\limits_{\Omega_*}\Big[\Big(r\Big(\frac{\chi}{r}\Big)_{,r}\Big)_{,r}\frac{\chi}{r}+\Big(r\Big(\frac{\chi}{r}\Big)_{,z}\Big)_{,z}\frac{\chi}{r}\Big]drdzd\varphi\\
&\qquad+\nu\int\limits_{\Omega_*}\Big|\frac{1}{r^2}\chi_{,\varphi}\Big|^2dx-\nu\int\limits_{\Omega_*}2\Big(\frac{\chi}{r}\Big)_{,r}\frac{\chi}{r}drdzd\varphi\equiv I_2.
\end{aligned}$$

Integrating by parts the first integral in I_2 equals

$$\nu\int\limits_{\Omega_*}\Big|\nabla'\frac{\chi}{r}\Big|^2dx-\nu\int\limits_{S_1}\bar n\cdot\nabla\frac{\chi}{r}\chi dS_1+\nu\int\limits_{-a}^{a}\int\limits_0^{2\pi}d\varphi dz\Big(\frac{\chi}{r}\Big)_{,r}\chi\Big|_{r=\varepsilon_*},$$

where $\nabla'=(\partial_r,\partial_z)$.

Finally, the last integral in I_2 reads

$$-\nu\int\limits_{\Omega_*}\partial_r\Big(\frac{\chi^2}{r^2}\Big)drdzd\varphi=-\nu\int\limits_{S_1}\frac{\chi^2}{r^2}dS_1+\nu\int\limits_{-a}^{a}\int\limits_0^{2\pi}\frac{\chi^2}{r^2}\Big|_{r=\varepsilon_*}dzd\varphi.$$

The first term on the r.h.s. of (3.3) is estimated by

$$\left|2\nu\int\limits_{\Omega_*}\left(\frac{h_\varphi}{r^3}\left(\frac{\chi'}{r}\right)_{,z}-\frac{h_z}{r^3}\frac{1}{r}\left(\frac{\chi'}{r}\right)_{,\varphi}\right)dx\right|\leqslant\varepsilon_2\Big|\nabla\frac{\chi}{r}\Big|^2_{2,\Omega_*}+c((1/\varepsilon_2)|h|^2_{2,-3,\Omega_*},$$

where the last norm is estimated by $\|h\|^2_{2,-1,\Omega_*}$.

By the Hölder and Young inequalities we estimate the second term on the r.h.s. of (3.3) by

$$\varepsilon_3\Big|\frac{\chi}{r}\Big|^2_{6,\Omega_*}+c(1/\varepsilon_3)\|w\|^2_{1,0,\Omega_*}|h|^2_{3,-2,\Omega_*},$$

where the imbedding

$$|h|_{3,-2,\Omega_*}\leqslant c\|h\|_{2,-1,\Omega_*}$$

will be used.

We estimate the third term on the r.h.s. of (3.3) by

$$\varepsilon_4\Big|\frac{\chi}{r^{2-\varepsilon'}}\Big|^2_{2,\Omega_*}+\frac{c}{\varepsilon_4}\|w\|^2_{1,0,\Omega_*}|\omega_{\varphi,z}|^2_{4,-3/4-\varepsilon',\Omega_*},$$

where $\varepsilon'>0$ is a small number which will be chosen later.

The last three cases are more explicitly described in the proof of Lemma 4.1 from [20].

Finally, the last term on the r.h.s. of (3.3) is estimated by

$$\varepsilon_5\Big|\frac{\chi}{r}\Big|^2_{6,\Omega_*}+c(1/\varepsilon_5)\Big|\frac{F_\varphi}{r}\Big|^2_{6/5,\Omega_*}.$$

Emploing the above estimates in (3.3) and using that $\varepsilon_i\leqslant\varepsilon/8$, $i=1,3,5$, $\varepsilon_2,\varepsilon_4$

are sufficiently small and (3.2) we obtain

$$\begin{aligned}\frac{d}{dt}|\chi|^2_{2,-1,\Omega_*}+\nu\left|\nabla\frac{\chi}{r}\right|^2_{2,\Omega_*}\leqslant\varepsilon\left|\frac{\chi}{r}\right|^2_{6,\Omega_*}&+c\int\limits_{S_1}\frac{\chi^2}{r^2}dS_1+c\bigg|\int\limits_{\Omega_*}\frac{h_\varphi}{r}\frac{\chi^2}{r^2}dx\bigg|\\&+2\nu\int\limits_{S_1}\bar n\cdot\nabla\frac{\chi}{r}\chi dS_1\\&+c(1/\varepsilon)\Big[(1+\|w\|^2_{1,0,\Omega_*})\|h\|^2_{2,-1,\Omega_*}\\&+\frac{R^{2\varepsilon'}}{\varepsilon'^2}\|w\|^2_{1,0,\Omega_*}|\omega_{\varphi,z}|^2_{4,-3/4-\varepsilon',\Omega_*}\\&+|F_\varphi|^2_{6/5,-1,\Omega_*}\Big],\end{aligned}\tag{3.4}$$

where we used the Hardy inequality

$$\left|\frac{\chi}{r}\right|^2_{2,-1+\varepsilon',\Omega}\leqslant\frac{1}{|\varepsilon'|^2}\int\limits_\Omega\left|\nabla\frac{\chi}{r}\right|^2r^{2\varepsilon'}dx\leqslant\frac{R^{2\varepsilon'}}{|\varepsilon'|^2}\left|\nabla\frac{\chi}{r}\right|^2_{2,\Omega}.$$

In view of the Poincare inequality

$$\left|\frac{\chi}{r}\right|_{2,\Omega_*}\leqslant c\left|\frac{\chi}{r}\right|_{2,S_1}+c\left|\nabla\frac{\chi}{r}\right|_{2,\Omega_*}$$

and for sufficiently small ε we obtain from (3.4) the inequality

$$\begin{aligned}\frac{d}{dt}|\chi|^2_{2-1,\Omega_*}+\nu\left\|\frac{\chi}{r}\right\|^2_{1,\Omega_*}\leqslant c|\chi|^2_{2,-1,S_1}&+c\bigg|\int\limits_{\Omega_*}\frac{h_\varphi}{r}\frac{\chi^2}{r^2}dx\bigg|+c\nu\int\limits_{S_1}\bar n\cdot\nabla\frac{\chi}{r}\chi dS_1\\&+c\Big[(1+\|w\|^2_{1,0,\Omega_*})\|h\|^2_{2,-1,\Omega_*}\\&+\frac{R^{2\varepsilon'}}{|\varepsilon'|^2}\|w\|^2_{1,0,\Omega_*}|\omega_{\varphi,z}|^2_{4,-3/4-\varepsilon',\Omega_*}\\&+|F_\varphi|^2_{6/5,-1,\Omega_*}\Big].\end{aligned}\tag{3.5}$$

From the boundary condition $(2.21)_2$ the first term on the r.h.s. of (3.5) is estimated by

$$c\left|\left(k-\frac{\gamma}{2\nu}\right)\omega\cdot\bar\tau_2\right|^2_{2,S_1}\leqslant c\sup_{S_1}\left(\left|k-\frac{\gamma}{2\nu}\right|^2a_1^2\right)|\omega|^2_{2,S_1}\leqslant cK^2|\omega|^2_{2,S_1}.$$

Applying the Hölder and Young inequalities we estimate the second term on the r.h.s. of (3.5) by

$$\varepsilon_6\left|\frac{\chi}{r}\right|^2_{6,\Omega_*}+c(1/\varepsilon_6)|h_\varphi|^2_{3,-1,\Omega_*}|\chi|^2_{2,-1,\Omega_*}.$$

Using the above estimates in (3.5) and assuming that ε_6 is sufficiently small yields

$$\frac{d}{dt}|\chi|^2_{2,-1,\Omega_*}+\nu\Big\|\frac{\chi}{r}\Big\|^2_{1,\Omega_*}\leqslant cK^2|\omega|^2_{2,S_1}+c|h_\varphi|^2_{3,-1,\Omega_*}|\chi|^2_{2,-1,\Omega_*}+c\int\limits_{S_1}\bar n\cdot\nabla\frac{\chi}{r}\chi dS_1$$
$$+c\bigg[(1+\|w\|^2_{1,0,\Omega_*})\|h\|^2_{2,-1,\Omega_*}$$
$$+\frac{R^{2\varepsilon'}}{|\varepsilon'|^2}\|w\|^2_{1,0,\Omega_*}|\omega_{\varphi,z}|^2_{4,-3/4-\varepsilon',\Omega_*}+|F_\varphi|^2_{6/5,-1,\Omega_*}\bigg].$$

Integrating the above inequality with respect to time, replacing ε' by ε and passing with ε_* to 0 we obtain

$$|\chi(t)|^2_{2,-1,\Omega}+\nu\int\limits_0^t\Big\|\frac{\chi(t')}{r}\Big\|^2_{1,\Omega}dt'\leqslant c\exp(c|h|^2_{3,2,-1,\Omega^t})$$
$$\times\bigg[\bigg|\int\limits_{S_1^t}\bar n\cdot\nabla\frac{\chi}{r}\chi dS_1dt'\bigg|+K^2|\omega|^2_{2,S_1^t}$$
$$+(1+\sup_t\|w\|^2_{1,0,\Omega})\int\limits_0^t\|h(t')\|^2_{2,-1,\Omega}dt'$$
$$+\frac{R^{2\varepsilon}}{\varepsilon^2}\sup_t\|w\|^2_{1,0,\Omega}\int\limits_0^t|\omega_{\varphi,z}(t')|^2_{4,-3/4-\varepsilon,\Omega}dt'$$
$$+|F_\varphi|^2_{6/5,2,-1,\Omega^t}+|\chi(0)|^2_{2,-1,\Omega}\bigg],\quad t\leqslant T. \tag{3.6}$$

To examine the first term on the r.h.s. of (3.6) we introduce a partition of unity $\varphi_j(x,t)$ in a neighborhood of S_1^T. Since $\sum_j\varphi_j(x,t)=1$ we have

$$I_1\equiv\sum_j\int\limits_{S_1^t}\varphi_j(x,t')\bar n\cdot\nabla\frac{\chi}{r}\chi dS_1dt'$$
$$=\sum_j\int\limits_{S_1^t}\varphi_j(x,t')\frac{1}{r}\bar n\cdot\nabla\chi\chi dS_1dt'-\sum_j\int\limits_{S_1^t}\varphi_j(x,t')a_1\frac{\chi^2}{r^2}dS_1dt'\equiv I_2+I_3.$$

In view of Lemma 2.4,

$$I_2=\sum_j\bigg[\int\limits_{S_1^t}\varphi_j(x,t')\frac{1}{r}\sum_{i=1}^4(b_i\partial_sA_i+B)\Big(k-\frac{\gamma}{2\nu}\Big)\omega\cdot\bar\tau_2dS_1dt'.$$

Integrating by parts yields

$$I_2 = -\sum_j \int\limits_{S_1^t} \sum_{i=1}^4 A_i \partial_s \Big[\varphi_j \frac{1}{r} b_i \Big(k - \frac{\gamma}{2\nu}\Big) \omega \cdot \bar\tau_2 \Big] dS_1 dt' + \sum_j \int\limits_{S_1^t} \varphi_j \frac{1}{r} B \Big(k - \frac{\gamma}{2\nu}\Big) \omega \cdot \bar\tau_2 dS_1 dt' \equiv I_4 + I_5,$$

where

$$\begin{aligned}|I_4| &\leqslant \sum_j \int\limits_{S_1^t \cap \operatorname{supp}\varphi_j} \varphi_j \frac{1}{r} \Big| k - \frac{\gamma}{2\nu} \Big| \omega_{,x}^2 dS_1 dt' \\ &\quad + \sum_j \int\limits_{S_1^t \cap \operatorname{supp}\varphi_j} \Big| \sum_{i=1}^4 \partial_s \Big(\varphi_j \frac{b_i}{r} \Big(k - \frac{\gamma}{2\nu}\Big)\Big) \Big| |\omega_{,x}|\, |\omega| dS_1 dt' \\ &\leqslant cK \sum_j \int\limits_{S_1^t \cap \operatorname{supp}\varphi_j} \omega_{,x}^2 dS_1 dt' + cK \sum_j \int\limits_{S_1^t \cap \operatorname{supp}\varphi_j} |\omega_{,x}| |\omega| dS_1 dt',\end{aligned}$$

and

$$|I_5| \leqslant cK \sum_j \int\limits_{S_1^t \cap \operatorname{supp}\varphi_j} |\omega| \Big(|\omega_{,x}| + |\omega| + \Big| \nabla \frac{h}{r} \Big| \Big) dS_1 dt'.$$

Finally,

$$|I_3| \leqslant cK^2 \sum_j \int\limits_{S_1^t \cap \operatorname{supp}\varphi_j} |\omega|^2 dS_1 dt'.$$

Summarizing, we obtain

$$|I_1| \leqslant cK(K+1) \sum_j \int\limits_{S_1^t \cap \operatorname{supp}\varphi_j} \Big(\omega_{,x}^2 + \omega^2 + \Big| \nabla \frac{h}{r} \Big|^2 \Big) dS_1 dt'.$$

Using this estimate in (3.6) we obtain (3.1). This concludes the proof. ■

To obtain a long time estimate necessary for the proof of global existence we have to estimate all norms from the r.h.s. of (3.1). First we shall examine the second factor from the third term on the r.h.s. of (3.1). For this purpose we use $(2.18)_1$ in the form

$$\omega_{\varphi,z} = -\alpha_r + \frac{1}{r} h_z. \tag{3.7}$$

By (2.1) we have

$$\begin{aligned}|\omega_{\varphi,z}|_{4,-3/4-\varepsilon,\Omega} &\leqslant |\alpha_r|_{4,-3/4-\varepsilon,\Omega} + |h|_{4,-7/4-\varepsilon,\Omega} \\ &\leqslant c(\|\alpha_r\|_{1,-1/2-\varepsilon,\Omega} + \|h\|_{1,-3/2-\varepsilon,\Omega}).\end{aligned} \tag{3.8}$$

To estimate the first norm on the r.h.s. of (3.8) we need energy type estimates for $\alpha' = (\alpha_r, \alpha_z)$.

Lemma 3.2. *Assume that* $\omega \in L_4(\Omega^T) \cap L_3(0,T;W_3^1(\Omega)) \cap L_\infty(0,T;H^1(\Omega))$, $h \in H^{2,1}_{1-\mu}(\Omega^T)$, $w \in H^{2,1}_{1-\mu}(\Omega^T)$, $F' \in L_{2,1-\mu}(\Omega^T)$, $\alpha'(0) \in W^1_{2,1-\mu}(\Omega)$, $\alpha' \in L_\infty(0,T;L_{2,1-\mu}(\Omega))$, $\alpha_r \in L_{2,-(1+\mu)}(\Omega^T)$, $a_1, a_2 \in C^2$, $\beta_i \in C^1$, $i = 1,\dots,6$. *Then solutions of problem (2.20) satisfy*

$$
\begin{aligned}
|||\alpha'|||_{2,2,1-\mu,\Omega^t} \leqslant{}& \varphi(|\omega|_{4,\Omega^t}, |\nabla\omega|_{3,\Omega^t})|\alpha'|_{2,1-\mu,\Omega^t} + c\|\nabla v\|_{2,\infty,\Omega^t}\|h\|_{2,-\mu,\Omega^t}\\
&+ c(\|h\|_{2,-\mu,\Omega^t} + \|w\|_{2,1-\mu,\Omega^t}) + c|\alpha_r|_{2,-(1+\mu),\Omega^t}\\
&+ c|F'|_{2,1-\mu,\Omega^t} + c|||\alpha'(0)|||_{1,2,1-\mu,\Omega}, \qquad t \leqslant T,
\end{aligned}
\tag{3.9}
$$

where φ *is an increasing positive function.*

Proof. Applying [21] to problem (2.20) yields (see also Lemmas 2.1, 2.2)

$$
\begin{aligned}
|||\alpha'|||_{2,2,1-\mu,\Omega^t} \leqslant c\Big(&|\omega\cdot\nabla\alpha'|_{2,1-\mu,\Omega^t} + |\alpha_r\omega_{r,r} + \alpha_z\omega_{r,z}|_{2,1-\mu,\Omega^t}\\
&+ |\alpha_r\omega_{z,r} + \alpha_z\omega_{z,z}|_{2,1-\mu,\Omega^t} + \Big|\frac{\chi}{r}h'\Big|_{2,1-\mu,\Omega^t}\\
&+ \Big|\frac{1}{r^2}(h_{r,z} - h_{z,r})\Big|_{2,1-\mu,\Omega^t} + \Big|\frac{\alpha_r}{r^2}\Big|_{2,1-\mu,\Omega^t}\\
&+ |F'|_{2,1-\mu,\Omega^t} + \Big|\Big|\Big|\frac{2a_1}{r}w - \frac{\gamma}{\nu}w\Big|\Big|\Big|_{3/2,2,1-\mu,S_1^t}\\
&+ |||\beta_1 h_\varphi + \beta_2 h_r + \beta_3 h_z + \beta_4 w_{\varphi,r} + \beta_5 w_{\varphi,z}\\
&+ \beta_6 w_\varphi|||_{1/2,2,1-\mu,S_1^t} + |||\alpha'(0)|||_{1,2,1-\mu,\Omega}\Big),
\end{aligned}
\tag{3.10}
$$

where $h' = (h_r, h_z)$.

Now we estimate the terms from the r.h.s. of (3.10). The first term is estimated by

$$
|\nabla\alpha'|_{4,1-\mu,\Omega^t}|\omega|_{4,\Omega^t} \leqslant \varepsilon_1|||\alpha'|||_{2,2,1-\mu,\Omega^t} + \varphi(1/\varepsilon_1, |\omega|_{4,\Omega^t})|\alpha'|_{2,1-\mu,\Omega^t},
$$

where φ is an increasing positive function.

The second and the third by

$$
\begin{aligned}
c|\alpha'\cdot\nabla\omega|_{2,1-\mu,\Omega^t} &\leqslant c|\alpha'|_{6,1-\mu,\Omega^t}|\nabla\omega|_{3,\Omega^t}\\
&\leqslant \varepsilon_2|||\alpha'|||_{2,2,1-\mu,\Omega^t} + \varphi(1/\varepsilon_2, |\nabla\omega|_{3,\Omega^t})|\alpha'|_{2,1-\mu,\Omega^t},
\end{aligned}
$$

where φ as above.

The fourth term by

$$
\begin{aligned}
\bigg(\int_0^t |\chi|^2_{2,\Omega}|h|^2_{\infty,-\mu,\Omega}dt'\bigg)^{1/2} &\leqslant c\bigg(\int_0^t |\chi|^2_{2,\Omega}\|h\|^2_{2,-\mu,\Omega}dt'\bigg)^{1/2}\\
&\leqslant c\|\nabla\omega\|_{2,\infty,\Omega^t}\|h\|_{L_2(0,t;H^2_{-\mu}(\Omega))}.
\end{aligned}
$$

The fifth term by

$$
c\|h\|_{L_2(0,t;H^2_{-\mu}(\Omega))}.
$$

The eighth by

$$c\|w\|_{2,1-\mu,\Omega^t}.$$

Finally, the ninth by

$$c(\|h\|_{2,-\mu,\Omega^t}+\|w\|_{2,1-\mu,\Omega^t}).$$

To show the last estimate see the proof of Lemma 4.3 from [20].

From the above estimates we obtain (3.9). This ends the proof. ■

Next we have

Lemma 3.3. *Assume that* $h\in H^{2,1}_{-1}(\Omega^T)$, $w\in H^{2,1}_{1-\mu}(\Omega^T)$, $\omega\in L_2(0,T;W^1_3(\Omega))\cap L_\infty(0,T;H^1(\Omega))\cap L_2(0,T;L_\infty(\Omega))$, $F'\in L_2(0,T;L_{6/5}(\Omega))$, $\alpha'(0)\in L_2(\Omega)$. *Then solutions to problem (2.20) satisfy the estimate*

$$\begin{aligned}|\alpha'(t)|^2_{2,\Omega}&+\nu\int_0^t\|\alpha'(t')\|^2_{1,\Omega}dt'+\nu|\alpha_r|^2_{2,-1,\Omega^t}\\&\leqslant c\exp(c|\omega_{,x}|^2_{3,2,\Omega^t})\\&\quad\times\Big[\varphi(|\omega_{,x}|_{2,\infty,\Omega^t},|\omega|_{\infty,2,\Omega^t},|\omega_{,x}|_{3,2,\Omega^t})\\&\quad\times(\|h\|^2_{2,-1,\Omega^t}+\|w\|^2_{2,1-\mu,\Omega^t})\\&\quad+|F'|^2_{6/5,2,\Omega^t}+|\alpha(0)|^2_{2,\Omega}\Big],\qquad t\leqslant T,\end{aligned}\tag{3.11}$$

where φ *is an increasing positive function.*

Proof. To show (3.11) we introduce functions $\tilde\alpha'=(\tilde\alpha_r,\tilde\alpha_z)$ as solutions of the problem

$$\begin{aligned}\tilde\alpha_{r,t}-\nu\Delta\tilde\alpha_r&=0&&\text{in }\Omega^T,\\\tilde\alpha_{z,t}-\nu\Delta\tilde\alpha_z&=0&&\text{in }\Omega^T,\\a_2\tilde\alpha_r-a_1\tilde\alpha_z&=-\frac{2a_1}{r}w+\frac{\gamma}{\nu}w\equiv g_1&&\text{on }S^T_1,\\(a_1\tilde\alpha_r+a_2\tilde\alpha_z)_{,n}&=\beta_1h_\varphi+\beta_2h_r+\beta_3h_z\\&\quad+\beta_4w_{,r}+\beta_5w_{,z}+\beta_6w\equiv g_2&&\text{on }S^T_1,\\\alpha_r=0,&\quad\alpha_{z,z}=0&&\text{on }S^T_2,\\\tilde\alpha_r|_{t=0}=0,&\quad\tilde\alpha_z|_{t=0}=0&&\text{in }\Omega.\end{aligned}\tag{3.12}$$

Defining the functions

$$\bar\alpha_r=\alpha_r-\tilde\alpha_r,\quad\bar\alpha_z=\alpha_z-\tilde\alpha_z,\quad\bar\alpha'=(\bar\alpha_r,\bar\alpha_z),\quad\tilde\alpha'=(\tilde\alpha_r,\tilde\alpha_z),\tag{3.13}$$

we see that they are solutions to the problem

$$
\begin{aligned}
&\bar\alpha_{r,t}+\omega\cdot\nabla\bar\alpha_r-(\bar\alpha_r\omega_{r,r}+\bar\alpha_z\omega_{r,z})-\frac{\chi}{r}h_r+\frac{2\nu}{r^2}(h_{r,z}-h_{z,r})\\
&\qquad+\nu\frac{\bar\alpha_r}{r^2}-\nu\Delta\bar\alpha_r=F_r-\omega\cdot\nabla\tilde\alpha_r+(\tilde\alpha_r\omega_{r,r}+\tilde\alpha_z\omega_{r,z})-\nu\frac{\tilde\alpha_r}{r^2}\quad\text{in }\Omega^T,\\
&\bar\alpha_{z,t}+\omega\cdot\nabla\bar\alpha_z-(\bar\alpha_r\omega_{z,r}+\bar\alpha_z\omega_{z,z})\\
&\qquad-\frac{\chi}{r}h_z-\nu\Delta\bar\alpha_z=F_z-\omega\cdot\nabla\tilde\alpha_z+(\tilde\alpha_r\omega_{z,r}+\tilde\alpha_z\omega_{z,z})\quad\text{in }\Omega^T,\\
&\qquad a_2\bar\alpha_r-a_1\bar\alpha_z=0\qquad\text{on }S_1^T,\\
&\qquad(a_1\bar\alpha_r+a_2\bar\alpha_z)_{,n}=0\qquad\text{on }S_1^T,\\
&\qquad\bar\alpha_r=0,\ \ \bar\alpha_{z,z}=0\qquad\text{on }S_2^T,\\
&\qquad\bar\alpha'|_{t=0}=\bar\alpha'(0)\qquad\text{in }\Omega.
\end{aligned}
\tag{3.14}
$$

For solutions of (3.12) we have (for more details see the proof of Lemma 4.4 [20])

$$
|||\tilde\alpha'|||_{2,2,1-\mu,\Omega^t}\leqslant c(\|h\|_{2,-1,\Omega^t}+\|w\|_{2,1-\mu,\Omega^t}).\tag{3.15}
$$

Now we obtain an energy estimate for solutions to (3.14). Multiplying $(3.14)_1$ by $\bar\alpha_r$, $(3.14)_2$ by $\bar\alpha_z$, integrating the results over Ω and adding yields

$$
\begin{aligned}
&\frac12\frac{d}{dt}|\bar\alpha'|^2_{2,\Omega}-\int\limits_\Omega\Big[\bar\alpha_r^2\omega_{r,r}+\bar\alpha_r\bar\alpha_z(\omega_{r,z}+\omega_{z,r})+\bar\alpha_z^2\omega_{z,z}\Big]dx\\
&\quad-\int\limits_\Omega\frac{\chi}{r}(h_r\bar\alpha_r+h_z\bar\alpha_z)dx+2\nu\int\limits_\Omega\frac{1}{r^2}(h_{r,z}-h_{z,r})\bar\alpha_rdx\\
&\quad+|\bar\alpha_r|^2_{2,-1,\Omega}-\nu\int\limits_\Omega(\Delta\bar\alpha_r\bar\alpha_r+\Delta\bar\alpha_z\bar\alpha_z)dx\\
&=\int\limits_\Omega(F_r\bar\alpha_r+F_z\bar\alpha_z)dx-\int\limits_\Omega(\omega\cdot\nabla\tilde\alpha_r\bar\alpha_r+\omega\cdot\nabla\tilde\alpha_z\bar\alpha_z)dx\\
&\quad-\nu\int\limits_\Omega\frac{\tilde\alpha_r}{r^2}\bar\alpha_rdx+\int\limits_\Omega\Big[(\tilde\alpha_r\omega_{r,r}+\tilde\alpha_z\omega_{r,z})\bar\alpha_r\\
&\quad+(\tilde\alpha_r\omega_{z,r}+\tilde\alpha_z\omega_{z,z})\bar\alpha_z\Big]dx.
\end{aligned}
\tag{3.16}
$$

The term with laplacians equals (see the proof of Lemma 4.4 [20])

$$
\nu|\nabla\bar\alpha'|^2_{2,\Omega}
$$

The second term on the l.h.s. of (3.16) is estimated by

$$
\varepsilon|\bar\alpha'|^2_{6,\Omega}+c(1/\varepsilon)|\omega_{,x}|^2_{3,\Omega}|\bar\alpha'|^2_{2,\Omega}.
$$

The third by

$$\varepsilon(|\bar\alpha_r|^2_{2,-1,\Omega}+|\bar\alpha_z|^2_{2,\Omega})+c(1/\varepsilon)\int\limits_\Omega\frac{\chi^2}{r^2}(h_r^2+h_z^2)dx.$$

The fourth by

$$\varepsilon|\bar\alpha_r|^2_{2,-1,\Omega}+c(1/\varepsilon)\|h\|^2_{2,-1,\Omega}.$$

The first term on the r.h.s. of (3.16) is bounded by

$$\varepsilon|\bar\alpha'|^2_{6,\Omega}+c(1/\varepsilon)|F'|^2_{6/5,\Omega},$$

the second by

$$\varepsilon|\nabla\bar\alpha'|^2_{2,\Omega}+c(1/\varepsilon)|\omega|^2_{\infty,\Omega}|\tilde\alpha'|^2_{2,\Omega},$$

the third by

$$\varepsilon|\bar\alpha_r|^2_{2,-1,\Omega}+c(1/\varepsilon)|\tilde\alpha_r|^2_{2,-1,\Omega},$$

and finally the last by

$$\varepsilon|\bar\alpha'|_{6,\Omega}+c(1/\varepsilon)|\omega_{,x}|^2_{3,\Omega}|\tilde\alpha'|^2_{2,\Omega}.$$

Summarizing the above results we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\bar\alpha'|^2_{2,\Omega}+\nu|\nabla\bar\alpha'|^2_{2,\Omega}+\nu|\bar\alpha_r|^2_{2,-1,\Omega}&\\ \leqslant\varepsilon(|\bar\alpha_z|^2_{6,\Omega}+|\bar\alpha_z|^2_{2,\Omega})&+c(1/\varepsilon)|\omega_{,x}|^2_{3,\Omega}|\bar\alpha'|^2_{2,\Omega}\\ &+c(1/\varepsilon)|\chi|^2_{2,\Omega}\|h\|^2_{2,-1,\Omega}+c\|h\|^2_{2,-1,\Omega}\\ &+c|F'|^2_{6/5,\Omega}+c|\omega|^2_{\infty,\Omega}|\tilde\alpha'|^2_{2,\Omega}+c|\tilde\alpha_r|^2_{2,-1,\Omega}\\ &+c(1/\varepsilon)|\omega_{,x}|^2_{3,\Omega}|\tilde\alpha'|^2_{2,\Omega}.\end{aligned}\tag{3.17}$$

In view of (4.25) from [20] we have

$$|\bar\alpha_z|_{2,\Omega}\leqslant c(|\nabla\bar\alpha_z|_{2,\Omega}+\|\bar\alpha_r\|_{1,\Omega}).\tag{3.18}$$

In virtue of (3.18) inequality (3.17) takes the form

$$\begin{aligned}\frac{d}{dt}|\bar\alpha'|^2_{2,\Omega}+\nu\|\bar\alpha'\|^2_{1,\Omega}+\nu|\bar\alpha_r|^2_{2,-1,\Omega}\leqslant c|\omega_{,x}|^2_{3,\Omega}|\bar\alpha'|^2_{2,\Omega}&\\ +c|\omega_{,x}|^2_{2,\Omega}\|h\|^2_{2,-1,\Omega}+c\|h\|^2_{2,-1,\Omega}&\\ +c|F'|^2_{6/5,\Omega}+c|\omega|^2_{\infty,\Omega}|\tilde\alpha'|^2_{2,\Omega}&\\ +c|\tilde\alpha_r|^2_{2,-1,\Omega}+c|\omega_{,x}|^2_{3,\Omega}|\tilde\alpha'|^2_{2,\Omega}.&\end{aligned}\tag{3.19}$$

By the energy method we obtain for solutions to problem (3.12) the inequality (see (4.32) [20])

$$|\tilde\alpha'|^2_{2,\Omega}+|\nabla\tilde\alpha'|^2_{2,\Omega^t}+|||\tilde\alpha'|||^2_{2,2,1-\mu,\Omega^t}\leqslant c(\|h\|^2_{2,-1,\Omega^t}+\|w\|^2_{2,1-\mu,\Omega^t}).\tag{3.20}$$

By (4.35) [20] (see also (4.32) and (4.34) in [20]) we have

$$\|\tilde{\alpha}_r\|_{2,1-\mu,\Omega^t} \leqslant c(\|w\|_{2,1-\mu,\Omega^t} + \|h\|_{2,-1,\Omega^t}) \equiv cI, \tag{3.21}$$

so

$$\|\tilde{\alpha}_r\|_{1,-\mu,\Omega^t} \leqslant cI. \tag{3.22}$$

Integrating (3.19) with respect to time and using (3.20) and (3.22) yields

$$\begin{aligned} |\bar{\alpha}'(t)|^2_{2,\Omega} + \nu\int_0^t \|\bar{\alpha}'(t')\|^2_{1,\Omega}dt' + \nu|\bar{\alpha}_r|^2_{2,-1,\Omega^t} & \\ \leqslant c\exp(c|\omega_{,x}|^2_{3,2,\Omega^t})\Big[&|\omega_{,x}|^2_{2,\infty,\Omega^t}\|h\|^2_{2,-1,\Omega^t} \\ &+ (1 + |\omega|^2_{\infty,2,\Omega^t} + |\omega_{,x}|^2_{3,2,\Omega^t}) \\ &\times (\|h\|^2_{2,-1,\Omega^t} + \|w\|^2_{2,1-\mu,\Omega^t}) \\ &+ |F'|^2_{6/5,2,\Omega^t} + |\alpha'(0)|^2_{2,\Omega}\Big]. \end{aligned} \tag{3.23}$$

Using (3.15) and (3.22) in (3.23) we obtain (3.11). This concludes the proof. ■

Finally, we have

Lemma 3.4. *Assume that*

$$\begin{aligned} \omega \in L_\infty(0,T;W^1_2(\Omega)) \cap L_2(0,T;W^1_\infty(\Omega)) \cap L_\infty(0,T;W^1_{p'}(\Omega)) \\ \cap L_2(0,T;W^1_3(\Omega)) \cap L_4(\Omega^T) \cap L_3(0,T;W^1_3(\Omega)) \equiv \mathfrak{N}_1(\Omega^T), \qquad p' < 3. \end{aligned}$$

Assume that $h \in H^{2,1}_{-1}(\Omega^T)$, $w \in H^{2,1}_{1-\mu}(\Omega^T)$, $F' \in L_2(0,T;L_{6/5}(\Omega))$, $\alpha'(0) \in L_2(\Omega)$, $\alpha_r(0) \in L_{2,-\mu}(\Omega)$, $F_r \in L_2(0,T;L_{6/5,-\mu}(\Omega))$, $F' \in L_{2,1-\mu}(\Omega^T)$, $\alpha'(0) \in W^1_{2,1-\mu}(\Omega)$, $\mu \in (0,1)$. *Then*

$$\begin{aligned} \nu\int_0^t \|\alpha_r(t')\|^2_{1,-\mu,\Omega} \leqslant \varphi(\|\omega\|_{\mathfrak{N}_1(\Omega^T)})\Big[&\|h\|^2_{2,-1,\Omega^t} + \|w\|^2_{2,1-\mu,\Omega^t} \\ &+ |F'|^2_{6/5,2,\Omega^t} + |F_r|^2_{6/5,2,-\mu,\Omega^t} \\ &+ |\alpha'(0)|^2_{2,\Omega} + |\alpha_r(0)|^2_{2,-\mu,\Omega} \\ &+ |F'|^2_{2,1-\mu,\Omega^t} + |||\alpha'(0)|||^2_{1,2,1-\mu,\Omega}\Big], \qquad t \leqslant T, \end{aligned} \tag{3.24}$$

where φ *is an increasing positive function.*

Proof. In view of (3.20), (3.22) we have to find an estimate for $\|\bar{\alpha}_r\|_{1,-\mu,\Omega^t}$.

Multiplying $(3.14)_1$ by $\bar\alpha_r r^{-2\mu}$ and integrating over Ω implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\bar\alpha_r|^2_{2,-\mu,\Omega}&+\int\limits_\Omega \omega\cdot\nabla\bar\alpha_r\bar\alpha_r r^{-2\mu}dx-\int\limits_\Omega(\bar\alpha_r\omega_{r,r}+\bar\alpha_z\omega_{r,z})\bar\alpha_r r^{-2\mu}dx\\&-\int\limits_\Omega\frac{\chi}{r}h_r\bar\alpha_r r^{-2\mu}dx+2\nu\int\limits_\Omega\frac{1}{r^2}(h_{r,z}-h_{z,r})\bar\alpha_r r^{-2\mu}dx\\&+\nu|\bar\alpha_r|^2_{2,-(1+\mu),\Omega}-\nu\int\limits_\Omega\Delta\bar\alpha_r\bar\alpha_r r^{-2\mu}dx\\&=\int\limits_\Omega F_r\bar\alpha_r r^{-2\mu}dx-\int\limits_\Omega\omega\cdot\nabla\tilde\alpha_r\bar\alpha_r r^{-2\mu}dx\\&+\int\limits_\Omega(\tilde\alpha_r\omega_{r,r}+\tilde\alpha_z\omega_{r,z})\bar\alpha_r r^{-2\mu}dx-\nu\int\limits_\Omega\frac{\tilde\alpha_r}{r^2}\bar\alpha_r r^{-2\mu}dx.\end{aligned}\tag{3.25}$$

Now we examine the particular terms in (3.25). The second term on the l.h.s. equals

$$\frac{1}{2}\int\limits_\Omega\omega\cdot\nabla\bar\alpha_r^2 r^{-2\mu}dx=\frac{1}{2}\int\limits_\Omega\omega\cdot\nabla(\bar\alpha_r^2 r^{-2\mu})dx+\mu\int\limits_\Omega\bar\alpha_r^2 r^{-2\mu-1}\omega\cdot\nabla r dx,$$

where the first term vanishes and the second is estimated by

$$\varepsilon|\bar\alpha_r|^2_{2,-(1+\mu),\Omega}+c(1/\varepsilon)|\omega|^2_{\infty,\Omega}|\bar\alpha_r|^2_{2,-\mu,\Omega}.$$

The third term on the l.h.s. of (3.25) is bounded by

$$\varepsilon|\bar\alpha_r|^2_{6,-\mu,\Omega}+c(1/\varepsilon)|\omega_{,x}|^2_{p',\Omega}|\bar\alpha'|^2_{p,-\mu,\Omega},$$

where $\frac{1}{p}+\frac{1}{p'}=\frac{5}{6}$, $p'<3$, $p>2$, the fourth by

$$\varepsilon|\bar\alpha_r|^2_{2,-(1+\mu),\Omega}+c(1/\varepsilon)|\chi|^2_{2,\Omega}|h|^2_{\infty,-\mu,\Omega},$$

the fifth by

$$\varepsilon|\bar\alpha_r|^2_{2,-(1+\mu),\Omega}+c(1/\varepsilon)|h_{,x}|^2_{2,-(1+\mu),\Omega}.$$

Integrating by parts the term with laplacian takes the form

$$\begin{aligned}\nu\int\limits_\Omega\bar\alpha^2_{r,x}r^{-2\mu}dx&-\nu\int\limits_\Omega\operatorname{div}(\nabla\bar\alpha_r\bar\alpha_r r^{-2\mu})dx\\&-2\mu\nu\int\limits_\Omega\nabla\bar\alpha_r\bar\alpha_r r^{-2\mu-1}\nabla r dx\equiv\nu|\nabla\bar\alpha_r|^2_{2,-\mu,\Omega}+I_1+I_2,\end{aligned}$$

where (see the proof of Lemma 4.5 [20])

$$|I_1|\leqslant\varepsilon_1|||\bar\alpha'|||_{2,2,1-\mu,\Omega}+c(1/\varepsilon_1)\|\bar\alpha'\|_{1,\Omega}$$

and

$$|I_2| \leqslant \nu\Big[\frac{\varepsilon}{2}|\bar\alpha_{r,x}|^2_{2,-\mu,\Omega} + \frac{2\mu^2}{\varepsilon}|\bar\alpha_r|^2_{2,-(1+\mu),\Omega}\Big].$$

The first term on the r.h.s. of (3.25) is estimated by

$$\varepsilon|\bar\alpha_r|^2_{6,-\mu,\Omega} + c(1/\varepsilon)|F_r|^2_{6/5,-\mu,\Omega},$$

the second by

$$\varepsilon|\nabla\tilde\alpha_r|^2_{2,-\mu,\Omega} + c(1/\varepsilon)|\omega|^2_{\infty,\Omega}|\bar\alpha_r|^2_{2,-\mu,\Omega},$$

the third by

$$\varepsilon|\bar\alpha_r|^2_{6,-\mu,\Omega} + c(1/\varepsilon)|\omega_{,x}|^2_{p',\Omega}|\tilde\alpha'|^2_{p,-\mu,\Omega}, \quad \frac{1}{p}+\frac{1}{p'}=\frac{5}{6}, \quad p'<3, \quad p>2,$$

and finally the last by

$$\varepsilon|\bar\alpha_r|^2_{2,-(1+\mu),\Omega} + c(1/\varepsilon)|\tilde\alpha_r|^2_{2,-(1+\mu),\Omega}.$$

In view of the above estimates and for sufficiently small ε we obtain from (3.25) the inequality

$$\begin{aligned}\frac{d}{dt}|\bar\alpha_r|^2_{2,-\mu,\Omega} + \nu\|\bar\alpha_r\|^2_{1,-\mu,\Omega} &\leqslant c|\omega|^2_{\infty,\Omega}|\bar\alpha_r|^2_{2,-\mu,\Omega} + c|\omega_{,x}|^2_{p',\Omega}|\bar\alpha'|^2_{p,-\mu,\Omega}\\ &\quad + c|\omega_{,x}|^2_{2,\Omega}\|h\|^2_{2,-\mu,\Omega} + c\|h\|^2_{2,-\mu,\Omega}\\ &\quad + \varepsilon_1|||\bar\alpha'|||^2_{2,2,1-\mu,\Omega} + c(1/\varepsilon_1)\|\bar\alpha'\|^2_{1,\Omega}\\ &\quad + c|F_r|^2_{6/5,-\mu,\Omega} + c|\nabla\tilde\alpha_r|^2_{2,-\mu,\Omega}\\ &\quad + c|\omega_{,x}|^2_{p',\Omega}|\tilde\alpha'|^2_{p,-\mu,\Omega} + c|\tilde\alpha_r|^2_{2,-(1+\mu),\Omega},\end{aligned} \tag{3.26}$$

where $\frac{1}{p}+\frac{1}{p'}=\frac{5}{6}$, $p'<3$, $p>2$.

Integrating (3.26) with respect to time yields

$$\begin{aligned}|\bar\alpha_r|^2_{2,-\mu,\Omega} + \nu\int_0^t\|\bar\alpha_r\|^2_{1,-\mu,\Omega}dt' &\leqslant c\exp(c|\omega|^2_{\infty,2,\Omega^t})\Big[|\omega_{,x}|^2_{p',\infty,\Omega^t}\\ &\quad\times\big(|\bar\alpha'|^2_{p,2,-\mu,\Omega^t} + |\tilde\alpha'|^2_{p,2,-\mu,\Omega^t}\big)\\ &\quad+ |\omega_{,x}|^2_{2,\infty,\Omega^t}\|h\|^2_{2,-\mu,\Omega^t} + \|h\|^2_{2,-\mu,\Omega^t}\\ &\quad+\varepsilon_1\int_0^t|||\bar\alpha'(t')|||^2_{2,2,1-\mu,\Omega}dt' + c(1/\varepsilon_1)\\ &\quad\times\int_0^t\|\bar\alpha'(t')\|^2_{1,\Omega}dt' + |F_r|^2_{6/5,2,-\mu,\Omega^t}\\ &\quad+|\tilde\alpha_r|^2_{2,-(1+\mu),\Omega^t} + |\alpha_r(0)|^2_{2,-\mu,\Omega}\Big].\end{aligned} \tag{3.27}$$

Using (3.20)–(3.22) in (3.27) implies

$$
\begin{aligned}
|\bar\alpha_r|^2_{2,-\mu,\Omega} + \nu\int_0^t \|\alpha_r\|^2_{1,-\mu,\Omega}dt' \leqslant \varphi(|\omega|_{\infty,2,\Omega^t}, |\omega_{,x}|_{p',\infty,\Omega^t}, |\omega_{,x}|_{2,\infty,\Omega^t}) \\
\times\Big[|\bar\alpha'|^2_{p,2,-\mu,\Omega^t} + \|h\|^2_{2,-1,\Omega^t} + \|w\|^2_{2,1-\mu,\Omega^t} \\
+\varepsilon_1\int_0^t |||\bar\alpha'(t')|||^2_{2,2,1-\mu,\Omega}dt' \\
+c(1/\varepsilon_1)\int_0^t \|\bar\alpha'(t')\|^2_{1,\Omega}dt' + |F_r|^2_{6/5,2,-\mu,\Omega^t} \\
+|\alpha_r(0)|^2_{2,-\mu,\Omega}\Big],
\end{aligned}
\tag{3.28}
$$

where φ is an increasing positive function.

Employing (3.23) in (3.28) gives

$$
\begin{aligned}
\nu\int_0^t \|\alpha_r(t')\|^2_{1,-\mu,\Omega}dt' \leqslant \varphi(|\omega|_{\infty,2,\Omega^t}, |\omega_{,x}|_{p',\infty,\Omega^t}, |\omega_{,x}|_{2,\infty,\Omega^t}, |\omega_{,x}|_{3,2,\Omega^t}) \\
\times\Big[\|h\|^2_{2,-1,\Omega^t} + \|w\|^2_{2,1-\mu,\Omega^t} + |F'|^2_{6/5,2,\Omega^t} \\
+|F_r|^2_{6/5,2,-\mu,\Omega^t} + |\alpha'(0)|^2_{2,\Omega} + |\alpha_r(0)|^2_{2,-\mu,\Omega} \\
+\varepsilon\int_0^t |||\bar\alpha'(t')|||^2_{2,2,1-\mu,\Omega}dt'\Big].
\end{aligned}
\tag{3.29}
$$

Exploiting (3.9) with (3.11) in (3.29) and assuming that ε is sufficiently small we get (3.24). This concludes the proof. ∎

In virtue of (3.24) we obtain from (3.8) the inequality

$$
\begin{aligned}
|\omega_{\varphi,z}|^2_{4,2,-3/4-\varepsilon,\Omega^t} \leqslant c\Big(\int_0^t \|\alpha_r(t')\|^2_{1,-1/2-\varepsilon,\Omega}dt' + \|h\|^2_{2,-1,\Omega^t}\Big) \\
\leqslant \varphi(\|\omega\|_{\mathfrak{N}'_1(\Omega^t)})\Big[\|h\|^2_{2,-1,\Omega^t} + \|w\|^2_{2,1-\mu,\Omega^t} \\
+|F'|^2_{6/5,2,\Omega^t} + |F_r|^2_{6/5,2,-\mu,\Omega^t} + |\alpha'(0)|^2_{2,\Omega} \\
+|\alpha_r(0)|^2_{2,-\mu,\Omega} + |F'|^2_{2,1-\mu,\Omega^t} + |||\alpha'(0)|||^2_{1,2,1-\mu,\Omega}\Big], \\
\frac{1}{2}+\varepsilon<\mu, \quad \mu\in(0,1),
\end{aligned}
\tag{3.30}
$$

where

$$\|\omega\|_{\mathfrak{N}_1'(\Omega^T)}=|\!|\omega|\!|_{\infty,2,\Omega^T}+|\!|\omega_{,x}|\!|_{p',\infty,\Omega^T}+|\!|\omega_{,x}|\!|_{2,\infty,\Omega^T}+|\!|\omega_{,x}|\!|_{3,2,\Omega^T},\quad p'<3. \tag{3.31}$$

We need an estimate for $\|w\|_{2,1-\mu,\Omega^t}$. However, it must be shown in a different way than in Lemma 5.1 from [20].

Lemma 3.5. *Assume that* $h\in L_{2,-(1+\mu)}(\Omega^T)$, $w\in L_{2,-(1+\mu)}(\Omega^T)$, $q\in L_{2,-\mu}(\Omega^T)$, $\omega\in L_\infty(0,T;L_4(\Omega))$, $\mathcal{F}_\varphi\in L_{2,1-\mu}(\Omega^T)$, $w(0)\in H^1_{1-\mu}(\Omega)$, $\mu\in(0,1)$, $a_1\in C^1$. *Then*

$$\begin{aligned}\|w\|_{2,1-\mu,\Omega^t}&\leqslant\varphi(|\!|\omega|\!|_{4,\infty,\Omega^t})|w|_{2,1-\mu,\Omega^t}+c(|w|_{2,-(1+\mu),\Omega^t}\\&\quad+|h|_{2,-(1+\mu),\Omega^t}+|q|_{2,-\mu,\Omega^t}+|\mathcal{F}_\varphi|_{2,1-\mu,\Omega^t}\\&\quad+\|w(0)\|_{1,1-\mu,\Omega}),\qquad t\leqslant T,\end{aligned}\tag{3.32}$$

where φ is an increasing positive function.

Proof. Applying [21] (see also Lemmas 2.1, 2.2) to problem (2.18) yields

$$\begin{aligned}\|w\|_{2,1-\mu,\Omega^t}&\leqslant c\Big(|\omega\cdot\nabla w|_{2,1-\mu,\Omega^t}+\Big|\frac{\omega_rw}{r}\Big|_{2,1-\mu,\Omega^t}+|w|_{2,-(1+\mu),\Omega^t}\\&\quad+|q|_{2,-\mu,\Omega^t}+|h|_{2,-(1+\mu),\Omega^t}+|\mathcal{F}_\varphi|_{2,1-\mu,\Omega^t}\\&\quad+\|w\|_{1/2,1-\mu,S_1^t}+\Big\|\frac{a_1}{r}w\Big\|_{1/2,1-\mu,S_1^t}+\|w(0)\|_{1,1-\mu,\Omega}\Big).\end{aligned}\tag{3.33}$$

Let us examine the particular terms from the r.h.s. of (3.33). The first term is estimated by

$$\sup_t|\omega|_{2p,\Omega}\|\nabla w\|_{L_2(0,t;L_{2p',1-\mu}(\Omega))}\equiv I_1,$$

where $1/p+1/p'=1$.

Using the interpolation inequality

$$\|\nabla w\|_{L_2(0,T;L_{2p',1-\mu}(\Omega))}\leqslant\varepsilon\|w\|_{2,1-\mu,\Omega^T}+c(1/\varepsilon)|w|_{2,1-\mu,\Omega^T}$$

which holds for $p'<3$, so we can choose $p=p'=2$. Hence we obtain

$$I_1\leqslant\varepsilon_1\|w\|_{2,1-\mu,\Omega^t}+\varphi(1/\varepsilon_1,|\!|\omega|\!|_{4,\infty,\Omega^t})|w|_{2,1-\mu,\Omega^t},$$

where φ is an increasing positive function. The second term on the r.h.s. of (3.33) is treated in the way

$$\Big|\omega_r\frac{w}{r}\Big|_{2,1-\mu,\Omega^t}\leqslant\sup_t|\omega_r|_{2p,\Omega}\Big\|\frac{w}{r}\Big\|_{L_2(0,t;L_{2p',1-\mu}(\Omega))}\equiv I_2,$$

where $1/p+1/p'=1$.

Using the interpolation inequality

$$\begin{aligned}\Big\|\frac{w}{r}\Big\|_{L_2(0,T;L_{2p',1-\mu}(\Omega))}&=\|w\|_{L_2(0,T;L_{2p',-\mu}(\Omega))}\\&\leqslant\varepsilon\|w\|_{2,1-\mu,\Omega^T}+c(1/\varepsilon)|w|_{2,1-\mu,\Omega^T},\end{aligned}$$

which holds for $p' < 3$, so we can choose $p = p' = 2$. Hence

$$I_2 \leqslant \varepsilon_2 \|w\|_{2,1-\mu,\Omega^t} + \varphi(1/\varepsilon_2, |\omega|_{4,\infty,\Omega^t})|w|_{2,1-\mu,\Omega^t}.$$

The first boundary term on the r.h.s. of (3.33) is estimated by

$$\|w\|_{1/2,1-\mu,S_1^t} \leqslant \varepsilon_3 \|w\|_{2,1-\mu,\Omega^t} + c(1/\varepsilon_3)|w|_{2,1-\mu,\Omega^t}. \tag{3.34}$$

To estimate the second boundary term we introduce the set

$$\Omega(S_1, \delta_0) = \{x \in \Omega : \ \mathrm{dist}(x, S_1) < \delta_0\}, \tag{3.35}$$

where δ_0 is so small that

$$\Omega(S_1, \delta_0) \cap \{x_3 - \mathrm{axis}\} = \phi. \tag{3.36}$$

Then

$$\begin{aligned}
\left\|\frac{a_1}{r} w\right\|_{1/2,1-\mu,S_1^t} &\leqslant c\left\|\frac{a_1}{r} w\right\|_{1,1-\mu,\Omega^t(S_1,\delta_0)} \\
&= c\bigg(\int_0^t \left\|\frac{a_1}{r} w\right\|^2_{1,1-\mu,\Omega(S_1,\delta_0)} dt'\bigg)^{1/2} \\
&\quad + c\bigg(\int_{\Omega(S_1,\delta_0)} dx \int_0^t \int_0^t \frac{\left|\frac{a_1}{r}\omega(x,t') - \frac{a_1}{r}\omega(x,t'')\right|^2}{|t'-t''|^2} r^{2(1-\mu)} dt' dt'\bigg)^{\frac{1}{2}} \\
&\leqslant c\bigg(\int_0^t \|w(t')\|^2_{1,1-\mu,\Omega} dt'\bigg)^{1/2} \\
&\quad + c\bigg(\int_\Omega dx \int_0^t \int_0^t \frac{|w(x,t') - w(x,t'')|^2}{|t'-t''|^2} r^{2(1-\mu)} dt' dt''\bigg)^{1/2} \\
&\leqslant c\|w\|_{1,1-\mu,\Omega^t},
\end{aligned} \tag{3.37}$$

where we used that $a_1 \in C^1$.

Applying an interpolation inequality yields

$$\|w\|_{1,1-\mu,\Omega^t} \leqslant \varepsilon_4 \|w\|_{2,1-\mu,\Omega^t} + c(1/\varepsilon_4)|w|_{2,1-\mu,\Omega^t}. \tag{3.38}$$

Employing the above estimates in (3.33) implies (3.32) for sufficiently small $\varepsilon_1 - \varepsilon_4$. This concludes the proof. ∎

We need to estimate $|w|_{2,-(1+\mu),\Omega^t}$ which appears in (3.32).

Lemma 3.6. *Assume that* $\omega \in L_2(0,T;L_\infty(\Omega))$, $w(0) \in L_{2,-\mu}(\Omega)$, $q \in L_{2,-\mu}(\Omega^T)$, $\mathcal{F}_\varphi \in L_1(0,T;L_2(\Omega))$, $h \in L_{2,-(1+\mu)}(\Omega^T)$, $\mu \in (0,1)$, $|a_1| < c$. *Then*

$$\begin{aligned} &|w(t)|^2_{2,-\mu,\Omega} + \nu|\nabla w|^2_{2,-\mu,\Omega^t} + \nu|w|^2_{2,-(1+\mu),\Omega^t} + \gamma|w|^2_{2,-\mu,S^t} \\ &\qquad \leqslant c\exp(\|\omega\|^2_{\infty,2,\Omega^t} + t)\Big[|q|^2_{2,-\mu,\Omega^t} \\ &\qquad\quad + |h|^2_{2,-(1+\mu),\Omega^t} + |\mathcal{F}_\varphi|^2_{2,1,\Omega^t} + |w(0)|^2_{2,-\mu,\Omega}\Big]. \end{aligned} \tag{3.39}$$

Proof. Multiplying $(2.17)_1$ by $wr^{-2\mu}$, integrating over Ω_* (see the proof of Lemma 3.1) and assuming the artificial boundary condition

$$w|_{r=\varepsilon_*} = 0 \tag{3.40}$$

we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|w|^2_{2,-\mu,\Omega_*} - \nu\int\limits_{\Omega_*}\Delta w w r^{-2\mu}dx + \nu|w|^2_{2,-(1+\mu),\Omega_*} \\ &\qquad = -\int\limits_{\Omega_*}\Big(\omega\cdot\nabla w + \frac{\omega_r}{r}w\Big)wr^{-2\mu}dx + \int\limits_{\Omega_*}\frac{1}{r}qwr^{-2\mu}dx \\ &\qquad + 2\nu\int\limits_{\Omega_*}\frac{1}{r^2}h_r wr^{-2\mu}dx + \int\limits_{\Omega_*}\mathcal{F}_\varphi wr^{-2\mu}dx. \end{aligned} \tag{3.41}$$

The second term on the l.h.s. of (3.41) equals

$$\begin{aligned} &-\nu\int\limits_{S_1}\frac{a_1}{r}w^2r^{-2\mu}dS_1 + \nu\int\limits_{-a}^{a}dz\int\limits_0^{2\pi}d\varphi\frac{\partial w}{\partial r}wr^{-2\mu+1}\Big|_{r=r_*} - \int\limits_{S_2}\bar n\cdot\nabla w wr^{-2\mu}dS_2 \\ &\qquad +\nu\int\limits_{\Omega_*}|\nabla w|^2r^{-2\mu}dx + \gamma\int\limits_{S_1}w^2r^{-2\mu}dS_1 - 2\mu\nu\int\limits_{\Omega}\nabla w wr^{-2\mu-1}\nabla r dx \equiv I_1. \end{aligned}$$

In view of (3.40) the second term in I_1 vanishes. By $(2.17)_3$ the integral over S_2 in I_1 disappears. Using that $|a_1| \leqslant c$ and the fact that S_1 is located in a positive distance from x_3-axis the first term in I_1 is estimated by

$$\varepsilon_1|w_{,x}|^2_{2,-\mu,\Omega} + c(1/\varepsilon_1)|w|^2_{2,-\mu,\Omega}.$$

Applying the Hölder and Young inequalities we estimate the last term in I_1 by

$$\nu\frac{\varepsilon_0}{2}|\nabla w|^2_{2,-\mu,\Omega} + \nu\frac{2\mu^2}{\varepsilon_0}|w|^2_{2,-(1+\mu),\Omega}.$$

The first term on the r.h.s. of (3.41) takes the form

$$
\begin{aligned}
-\int\limits_{\Omega_*}\Big(\omega\cdot\nabla ww+\frac{\omega_r}{r}w^2\Big)r^{-2\mu}dx&=-\int\limits_{\Omega_*}\Big(\frac{1}{2}\omega\cdot\nabla w^2+\frac{\omega_r}{r}w^2\Big)r^{-2\mu}dx\\
&=-\int\limits_{\Omega_*}\Big[\frac{1}{2}\omega\cdot\nabla(w^2r^{-2\mu})+(1+\mu)\frac{\omega_r}{r}w^2r^{-2\mu}\Big]dx\\
&=-(1+\mu)\int\limits_{\Omega_*}\frac{\omega_r}{r}w^2r^{-2\mu}dx\equiv I_2,
\end{aligned}
\tag{3.42}
$$

where $(1.9)_2$, the boundary conditions $(1.9)_3$ and (3.40) were exploited.

Hence

$$
|I_2|\leqslant\varepsilon_2|w|^2_{2,-(1+\mu),\Omega_*}+c(1/\varepsilon_2)|\omega|^2_{\infty,\Omega_*}|w|^2_{2,-\mu,\Omega_*}.
$$

The last three terms on the r.h.s. of (3.41) are estimated by

$$
\varepsilon_3|w|^2_{2,-(1+\mu),\Omega_*}+c(1/\varepsilon_3)(|q|^2_{2,-\mu,\Omega_*}+|h|^2_{2,-(1+\mu),\Omega_*}+|\mathcal{F}_\varphi|^2_{2,1,\Omega_*}).
$$

Using the above estimates in (3.41) and assuming that $\varepsilon_1-\varepsilon_3$ are sufficiently small and $\varepsilon_0\in(2\mu^2,2)$, where $\mu\in(0,1)$, we obtain

$$
\begin{aligned}
\frac{d}{dt}|w|^2_{2,-\mu,\Omega_*}+\nu|\nabla w|^2_{2,-\mu,\Omega_*}+\nu|w|^2_{2,-(1+\mu),\Omega_*}+\gamma|w|^2_{2,-\mu,S_*}&\\
\leqslant c(|\omega|^2_{\infty,\Omega_*}+1)|w|^2_{2,-\mu,\Omega_*}&\\
+c(|q|^2_{2,-\mu,\Omega_*}+|h|^2_{2,-(1+\mu),\Omega_*}+|\mathcal{F}_\varphi|^2_{2,1,\Omega_*}).&
\end{aligned}
\tag{3.43}
$$

Integrating (3.43) with respect to time and passing with ε_* to 0 we obtain (3.39). This concludes the proof. ■

From (3.32) and (3.39) we obtain

$$
\begin{aligned}
\|w\|_{2,1-\mu,\Omega^t}\leqslant\varphi(t,|\omega|_{4,\infty,\Omega^t},|\omega|_{\infty,2,\Omega^t})\Big[&|q|_{2,-\mu,\Omega^t}\\
&+|h|_{2,-(1+\mu),\Omega^t}+|\mathcal{F}_\varphi|_{2,1-\mu,\Omega^t}+\|w(0)\|_{1,1-\mu,\Omega}\Big].
\end{aligned}
\tag{3.44}
$$

Let

$$
\|\omega\|_{\mathfrak{N}_2(\Omega^T)}=\|\omega\|_{\mathfrak{N}_\delta(\Omega^T)}+|\omega|_{4,\infty,\Omega^T}+|\omega|_{\infty,2,\Omega^T},\tag{3.45}
$$

where the first norm on the r.h.s. is defined by (2.32) and let

$$
\begin{aligned}
X_1(T)=|g|_{6/5,2,\Omega^T}+|g|_{2,-(1+\varepsilon_*),\Omega^T}+|\mathcal{F}_\varphi|_{2,1-\mu,\Omega^T}&\\
+\|h(0)\|_{1,-(1+\varepsilon_*),\Omega}+\|w(0)\|_{1,1-\mu,\Omega}.&
\end{aligned}
\tag{3.46}
$$

Then in view of (2.33) we obtain from (3.44) the inequality

$$
\|w\|_{2,1-\mu,\Omega^t}\leqslant\varphi(t,\|\omega\|_{\mathfrak{N}_2(\Omega^t)})X_1(t),\quad t\leqslant T.\tag{3.47}
$$

Let

$$X_2(T) = \|F'\|_{6/5,2,\Omega^T} + \|F_r\|_{6/5,2,-\mu,\Omega^T} + |F'|_{2,1-\mu,\Omega^T} + |\alpha'(0)|_{2,\Omega} + |\alpha_r(0)|_{2,-\mu,\Omega} + |||\alpha'(0)|||_{1,2,1-\mu,\Omega} \tag{3.48}$$

and

$$\mathfrak{N}(\Omega^T) = \mathfrak{N}_1(\Omega^T) \cap \mathfrak{N}_2(\Omega^T). \tag{3.49}$$

Using (3.47) in (3.30) yields

$$\|\omega_{\varphi,z}\|_{4,2,-3/4-\varepsilon,\Omega^t} \leqslant \varphi(t, \|\omega\|_{\mathfrak{N}(\Omega^t)})X_3(t), \tag{3.50}$$

where

$$X_3(t) = X_1(t) + X_2(t). \tag{3.51}$$

In view of (3.50), (2.33) we obtain from (3.1) the inequality

$$\Big\|\frac{\chi}{r}\Big\|_{V_2^0(\Omega^t)} \leqslant \varphi(X_1)\sqrt{K(K+1)}\Big(\sum_j \|\omega\|_{L_2(0,t;H^1(S_j))} + X_1\Big) + \varphi(t, \|\omega\|_{\mathfrak{N}(\Omega^t)})(1 + \sup_t \|w\|_{1,0,\Omega})X_3 + \varphi(X_1)Y_1, \tag{3.52}$$

where $S_j = S \cap \operatorname{supp}\varphi_j$, $\{\varphi_j\}$ is the partition of unity and

$$Y_1(t) = \|F_\varphi\|_{6/5,2,-1,\Omega^t} + |\chi(0)|_{2,-1,\Omega}, \tag{3.53}$$

and we used that $\mathfrak{N}_1(\Omega^T) \subset \mathfrak{N}_1'(\Omega^T)$, where $\mathfrak{N}_1(\Omega^T)$ is defined in assumptions of Lemma 3.4 and $\mathfrak{N}_1'(\Omega^T)$ by (3.31)

Now we have to find an estimate for $\sup_t \|w(t)\|_{1,0,\Omega}$. In view of 5.2.22 from [23] and (5.15) from [20] we obtain

$$\|w(t)\|_{1,0,\Omega}^2 \leqslant c\exp(c\|\omega\|_{\infty,2,\Omega^t}^2)\Big[(1 + \|\omega\|_{\infty,4,\Omega^t}^4)\sup_t |w(t)|_{2,1,\Omega}^2 + |w|_{2,\Omega^t}^2 + |q|_{2,\Omega^t}^2 + |h|_{2,-1,\Omega^t}^2 + |\mathcal{F}_\varphi|_{2,1,\Omega^t}^2 + e^{-t}\|w(0)\|_{1,0,\Omega}^2\Big], \tag{3.54}$$

where

$$|w(t)|_{2,1,\Omega} \leqslant |w(0)|_{2,1,\Omega} + c\int_0^t (|q(t')|_{2,\Omega} + |h(t')|_{2,-1,\Omega} + |\mathcal{F}_\varphi(t')|_{2,1,\Omega})dt'. \tag{3.55}$$

Hence

$$\|w(t)\|_{1,0,\Omega} \leqslant \varphi(t, \|\omega\|_{\mathfrak{N}_2(\Omega^t)})(X_1 + |w|_{2,\Omega^t}). \tag{3.56}$$

Using (3.39) gives

$$|w|_{2,\Omega^t} \leqslant \varphi(t, \|\omega\|_{\mathfrak{N}_2(\Omega^t)})X_1. \tag{3.57}$$

Hence

$$\|w(t)\|_{1,0,\Omega} \leqslant \varphi(t, \|\omega\|_{\mathfrak{N}_2(\Omega^t)})X_1. \tag{3.58}$$

Using (3.58) in (3.52) yields

$$\begin{aligned}\Big\|\frac{\chi}{r}\Big\|_{V_2^0(\Omega^t)} &\leqslant \varphi(X_1)\sqrt{K(K+1)}\Big(\sum_j\|\omega\|_{L_2(0,t;H^1(S_j))}+X_1\Big)\\&\quad+\varphi(t,\|\omega\|_{\mathfrak{N}(\Omega^t)},X_1)X_3+\varphi(X_1)Y_1.\end{aligned}\tag{3.59}$$

From (3.11) we get

$$\|\alpha'\|_{V_2^0(\Omega^t)}\leqslant\varphi(t,X_1,\|\omega\|_{\mathfrak{N}(\Omega^t)})X_2.\tag{3.60}$$

Hence, from problem (2.22) we have

$$\begin{aligned}|\omega|_{10,\Omega^t}+|\nabla\omega|_{10/3,\Omega^t}&\leqslant\varphi_1(t,X_1,\|\omega\|_{\mathfrak{N}(\Omega^t)})X_3+\varphi_2(X_1)Y_1\\&\quad+\sqrt{K(K+1)}\Big(\sum_j\|\omega\|_{L_2(0,t;H^1(S_j))}+X_1\Big).\end{aligned}\tag{3.61}$$

Finally, we obtain an a priori estimate for solutions to problem (1.1). First we recall that

$$\begin{aligned}\mathfrak{N}(\Omega^T)&=\mathfrak{N}_1(\Omega^T)\cap\mathfrak{N}_2(\Omega^T)\\&=L_\infty(0,T;W^1_{2,-\delta}(\Omega))\cap L_2(0,T;W^1_\infty(\Omega))\cap L_\infty(0,T;W^1_{p'}(\Omega))\\&\quad\cap L_3(0,T;W^1_3(\Omega)),\end{aligned}$$

where $p'\in(2,3)$ and $\delta\in(0,1)$.

Lemma 3.7. *Let $T<\infty$ be given, let $X_1(T)<\infty$ (see (3.46)), $X_2(T)<\infty$ (see 3.48)), $Y_1(T)<\infty$ (see (3.53)). Let*

$$|\mathcal{F}|_{5/2,\Omega^T}+\|\omega(0)\|_{6/5,5/2,\Omega}<\infty.$$

Let solutions of (1.9) satisfy estimates for the weak solutions (see (2.14)). Then there exists the constant

$$A=2\sigma[\varphi_2^2(X_1)Y_1^2+c_1(K+1)(d_2^2+X_1^2)+c(|\mathcal{F}|_{5/2,\Omega^T}+\|\omega(0)\|_{6/5,5/2,\Omega})],$$

where $\sigma>2$, function φ_2 appears in (3.61) and c_1 is the constant introduced in (3.66), such that for X_3 so small that (3.72) holds we have the estimate

$$\|\omega\|_{2,5/2,\Omega^T}+|\nabla p|_{5/2,\Omega^T}\leqslant A.\tag{3.62}$$

Proof. Let us consider problem (1.9). In view of (3.61) we have

$$\begin{aligned}\|\omega\|_{2,5/2,\Omega^T}+|\nabla p|_{5/2,\Omega^T}&\leqslant\varphi_1^2(T,X_1,\|\omega\|_{2,5/2,\Omega^T})X_3^2+\varphi_2^2(X_1)Y_1^2\\&\quad+K(K+1)\Big(\sum_j\|\omega\|^2_{L_2(0,T;H^1(S_j))}+X_1^2\Big)\\&\quad+c(|\mathcal{F}|_{5/2,\Omega^T}+\|\omega(0)\|_{6/5,5/2,\Omega}),\end{aligned}\tag{3.63}$$

where we used that

$$\|\omega\|_{\mathfrak{N}(\Omega^T)} \leqslant c\|\omega\|_{2,5/2,\Omega^T}. \tag{3.64}$$

To examine the third term on the r.h.s. of (3.63) we apply the interpolation inequality

$$|u|_{2,S} \leqslant c|\nabla u|_{2,\Omega}^{1/2}|u|_{2,\Omega}^{1/2} + c|u|_{2,\Omega},$$

which after integration with respect to time takes the form

$$|u|_{2,S^T} \leqslant c|\nabla u|_{2,\Omega^T}^{1/2}|u|_{2,\Omega^T}^{1/2} + c|u|_{2,\Omega^T}. \tag{3.65}$$

Using (3.65) we obtain

$$\begin{aligned} \|\omega\|_{L_2(0,T;H^1(S_j))} &= \|\omega\|_{L_2(0,T;L_2(S_j))} + \|\nabla\omega\|_{L_2(0,T;L_2(S_j))} \\ &\leqslant c|\nabla^2\omega|_{2,\Omega_j^T}^{1/2}|\nabla\omega|_{2,\Omega_j^T}^{1/2} + c|\nabla\omega|_{2,\Omega_j^T} \\ &\quad + c|\nabla\omega|_{2,\Omega_j^T}^{1/2}|\omega|_{2,\Omega_j^T}^{1/2} + c|\omega|_{2,\Omega_j^T} \\ &\leqslant c_1 d_2^{1/2}|\nabla^2\omega|_{2,\Omega_j^T}^{1/2} + c_1 d_2, \end{aligned} \tag{3.66}$$

where $\Omega_j^T = \Omega^T \cap \operatorname{supp}\varphi_j$ and (2.15) was employed.

By the Hölder inequality we have

$$|\nabla^2\omega|_{2,\Omega_j^T} \leqslant |\Omega_j^T|^{1/10}|\nabla^2\omega|_{5/2,\Omega_j^T}. \tag{3.67}$$

Utilizing (3.66) and (3.67) in (3.63) yields

$$\begin{aligned} \|\omega\|_{2,5/2,\Omega^T} + |\nabla p|_{5/2,\Omega^T} &\leqslant \varphi_1^2(T, X_1, \|\omega\|_{2,5/2,\Omega^T})X_3^2 + \varphi_2^2(X_1)Y_1^2 \\ &\quad + c_1K(K+1)\Big[\sup_j|\Omega_j^T|^{1/10}|\nabla^2\omega|_{5/2,\Omega^T}d_2 \\ &\quad + d_2^2 + X_1^2\Big] + c(|\mathcal{F}|_{5/2,\Omega^T} + \|\omega(0)\|_{6/5,5/2,\Omega}). \end{aligned} \tag{3.68}$$

Assuming that

$$c_1\sup_j|\Omega_j^T|^{1/10}K(K+1)d_2 \leqslant \frac{1}{2} \tag{3.69}$$

we obtain from (3.68) the inequality

$$\begin{aligned} \|\omega\|_{2,5/2,\Omega^T} + |\nabla p|_{5/2,\Omega^T} &\leqslant 2\varphi_1^2(T, X_1, \|\omega\|_{2,5/2,\Omega^T})X_3^2 + 2\varphi_2^2(X_1)Y_1^2 \\ &\quad + 2c_1K(K+1)(d_2^2 + X_1^2) \\ &\quad + 2c(|\mathcal{F}|_{5/2,\Omega^T} + \|\omega(0)\|_{6/5,5/2,\Omega}). \end{aligned} \tag{3.70}$$

Let us introduce the quantity

$$A = 2\sigma[\varphi_2^2(X_1)Y_1^2 + c_1K(K+1)(d_2^2 + X_1^2) + c(|\mathcal{F}|_{5/2,\Omega^T} + \|\omega(0)\|_{6/5,5/2,\Omega})]$$

and let

$$\|\omega\|_{2,5/2,\Omega^T} + |\nabla p|_{5/2,\Omega^T} \leqslant A. \tag{3.71}$$

Then inequality (3.70) implies

$$2\varphi_1^2(T, X_1, A)X_3^2 \leqslant \left(1 - \frac{1}{\sigma}\right)A, \tag{3.72}$$

which can be satisfied for sufficiently small X_3. In view of (3.72) estimate (3.62) holds. This concludes the proof. ■

To obtain (3.62) we need that $X_2(T)$ is sufficiently small. Hence we have to show that $\|F'\|_{L_2(0,T;H^1(\Omega))}$ is sufficiently small. For this purpose we need

Lemma 3.8. *Let*

$$l_1 = \sum_{i=1}^2 (|d_{i,r}|_{2,S_2^T} + |d_{i,\varphi}|_{2,S_2^T} + \gamma|d_i|_{2,S_2^T}) < \infty,$$

$$l_2 = \sum_{i=1}^2 \left(\|d_{i,r}\|_{W_2^{5/2,5/4}(S_2^T)} + \|d_{i,\varphi}\|_{W_2^{5/2,5/4}(S_2^T)} + \gamma\|d_i\|_{W_2^{5/2,5/4}(S_2^T)}\right) < \infty,$$

$$l_3 = \sum_{i=1}^2 \Big(\|d_{i,t}\|_{L_2(0,T;W_2^{3/2}(S_2))} + \|d_{i,tt}\|_{L_2(S_2^T)} + \|d_i\|_{L_2(0,T;W_2^{7/2}(S_2))}$$
$$+ \|d_{i,\tau}\|_{L_2(0,T;W_2^{5/2}(S_2))} + \|d_{i,\tau}\|_{L_2(S_2;W_2^{5/4}(0,T))} + \|d_i\|_{L_2(S_2;W_2^{5/4}(0,T))}\Big) < \infty,$$

where ∂_τ denotes the tangent derivative to S_2. There exists a constant $a > 0$ such that

$$\|F\|_{L_2(0,T;H^1(\Omega))} \leqslant c(\|\operatorname{rot} f\|_{W_2^{1,1/2}(\Omega^T)} + l_1^a c(l_1, l_2, l_3)), \tag{3.73}$$

where $c(l_1, l_2, l_3)$ is an increasing positive function of its arguments.

Proof. Since $F = \operatorname{rot}\mathcal{F}$ and $\mathcal{F} = f - \delta_{,t} - \delta\cdot\nabla\delta + \nu\Delta\delta$, where δ is a solution to problem (1.6) we have that

$$F = \operatorname{rot} f - (\operatorname{rot}\delta)_{,t} - \delta\cdot\nabla\operatorname{rot}\delta - \operatorname{rot}\delta\cdot\nabla\delta + \nu\Delta\operatorname{rot}\delta.$$

We are looking for solutions to problem (1.6) in the form

$$\delta = \nabla\psi + \eta, \tag{3.74}$$

where ψ is a solution to the problem

$$\begin{gathered}\Delta\psi=0,\quad \int\limits_\Omega \psi dx=0\\ \frac{\partial\psi}{\partial n}\Big|_{S_2(-a)}=-d_1,\quad \frac{\partial\psi}{\partial n}\Big|_{S_2(a)}=d_2,\quad \frac{\partial\psi}{\partial n}\Big|_{S_1}=0\end{gathered}\tag{3.75}$$

and η satisfies

$$\begin{aligned}\eta_t-\nu\Delta\eta+\nabla\sigma&=-\nabla\psi_t+\nu\Delta\nabla\psi\quad\text{in}\ \ \Omega^T,\\ \operatorname{div}\eta&=0\quad\text{in}\ \ \Omega^T,\\ \eta\cdot\bar n|_S&=0\quad\text{on}\ \ S^T,\\ \nu\bar n\cdot\mathbb{D}(\eta)\cdot\bar\tau_\alpha+\gamma\eta\cdot\bar\tau_\alpha&=-\nu\bar n\cdot\mathbb{D}(\nabla\psi)\cdot\bar\tau_\alpha-\gamma\nabla\psi\cdot\bar\tau_\alpha\\ &\equiv B_{1\alpha},\quad\alpha=1,2,\quad\text{on}\ \ S_1^T,\\ \nu\bar n\cdot\mathbb{D}(\eta)\cdot\bar\tau_\alpha&=-\nu\bar n\cdot\mathbb{D}(\nabla\psi)\cdot\bar\tau_\alpha\\ &\equiv B_{2\alpha},\quad\alpha=1,2,\quad\text{on}\ \ S_2^T,\\ \eta|_{t=0}&=\delta(0)-\nabla\psi(0)=0\quad\text{in}\ \ \Omega,\end{aligned}\tag{3.76}$$

where $(3.76)_6$ can be assumed in view of (1.7).

For solutions to problem (3.75) compatibility condition (1.5) holds. Then there exists the Green function to problem (3.75) such that

$$\psi(x,t)=\int\limits_{S_2(-a)}G(x-y)d_1(y,t)dy+\int\limits_{S_2(a)}G(x-y)d_2(y,t)dy\tag{3.77}$$

Now we calculate the r.h.s. of $(3.76)_{4,5}$. First

$$\begin{aligned}B_{11}(r,\varphi,z)&=-2\nu n_i\partial_{x_i}\partial_{x_j}\psi\tau_{1j}-\gamma\psi_{,x_i}\tau_{1i}\\ &=-2\nu\int\limits_{S_2(-a)}G(r-r',\varphi-\varphi',z)\partial_{r'}\partial_{\varphi'}d_1(r',\varphi',t)dS_2\\ &\quad-2\nu\int\limits_{S_2(a)}G(r-r',\varphi-\varphi',z)\partial_{r'}\partial_{\varphi'}d_2(r',\varphi',t)dS_2\\ &\quad-\gamma\bigg(\int\limits_{S_2(-a)}G(r-r',\varphi-\varphi',z)\partial_{\varphi'}d_1(r',\varphi',t)dS_2\\ &\quad+\int\limits_{S_2(a)}G(r-r',\varphi-\varphi',z)\partial_{\varphi'}d_2(r',\varphi',t)dS_2\bigg)\quad\text{on}\ \ S_1,\end{aligned}$$

where we used that S_1 and S_2 meet under angle $\frac{\pi}{2}$. Next

$$\begin{aligned} B_{12}(r,\varphi,z) &= -2\nu n_i\partial_{x_i}\partial_{x_j}\psi e_{3j} - \gamma\psi_{,x_3} \\ &= -2\nu \int\limits_{S_2(-a)} \partial_{x_3}G(r-r',\varphi-\varphi',x_3)\partial_{r'}d_1(r,\varphi',t)dS_2 \\ &\quad - 2\nu \int\limits_{S_2(a)} \partial_{x_3}G(r-r',\varphi-\varphi',x_3)\partial_{r'}d_2(r',\varphi',t)dS_2 \\ &\quad - \gamma \int\limits_{S_2(-a)} \partial_{x_3}Gd_1dS_2 - \gamma \int\limits_{S_2(a)} \partial_{x_3}Gd_2dS_2 \quad \text{on } \ S_1. \end{aligned}$$

Next

$$\begin{aligned} B_{21} &= -2\nu n_i\partial_{x_i}\partial_{x_j}\psi\tau_{1j} = -2\nu\partial_z\partial_r\psi \\ &= -2\nu\bigg[\int\limits_{S_2(-a)} \partial_zG(r-r',\varphi-\varphi',z)\partial_{r'}d_1(r',\varphi',t)dS_2 \\ &\quad + \int\limits_{S_2(a)} \partial_zG(r-r',\varphi-\varphi',z)\partial_{r'}d_2(r',\varphi',t)dS_2\bigg] \quad \text{on } \ S_2, \end{aligned}$$

and finally

$$\begin{aligned} B_{22} &= -2\nu n_i\partial_{x_i}\partial_{x_j}\psi\tau_{2j} = -2\nu\partial_z\partial_\varphi\psi \\ &= -2\nu\bigg[\int\limits_{S_2(-a)} \partial_zG(r-r',\varphi-\varphi',z)\partial_{\varphi'}d_1(r',\varphi',t)dS_2 \\ &\quad + \int\limits_{S_2(a)} \partial_zG(r-r',\varphi-\varphi',z)\partial_{\varphi'}d_2(r',\varphi',t)dS_2\bigg] \quad \text{on } \ S_2. \end{aligned}$$

Since F depends on $\operatorname{rot}\delta$, we have in view of (3.74) that $\operatorname{rot}\delta = \operatorname{rot}\eta$. Let $\beta = \operatorname{rot}\eta$.

Then β is a solution to the problem

$$
\begin{aligned}
&\beta_t - \Delta\beta = 0 \quad \text{in } \Omega^T,\\
&\beta_r = 2\bigg(k - \frac{\gamma}{2\nu}\bigg)\eta\cdot\bar\tau_2 + B_{12} \quad \text{on } S_1^T,\\
&\beta\cdot\bar\tau_2 = -\frac{2a_1}{r}\eta_\varphi + \frac{\gamma}{\nu}\eta_\varphi - B_{11} \quad \text{on } S_1^T,\\
&\beta_{n,n} = \frac{a_1}{r}\bigg(\frac{\gamma}{\nu} - \frac{2a_1}{r}\bigg)\eta_{\varphi,\varphi}\\
&\quad + \bigg[\bigg(a_{2,r} - a_{1,z} + \frac{a_2}{r}\bigg)\bigg(\frac{a_1}{r} - \frac{\gamma}{\nu}\bigg) + \frac{1}{r}\bigg(a_1a_{2,r} - \frac{a_1a_2}{r} + a_2a_{2,z}\bigg)\\
&\quad + \bar\tau_2\cdot\nabla\bigg(\frac{a_1}{r}\bigg)\bigg]\eta_\varphi + \bigg(a_1\delta_1 - a_2\delta_2 - \frac{\gamma}{\nu} + \frac{a_1}{r}\bigg)\bar\tau_2\cdot\nabla\eta_\varphi\\
&\quad - \frac{k}{r}\eta\cdot\bar\tau_2 - \frac{a_1}{\nu r}B_{11,\varphi} + \frac{1}{\nu}\bigg(a_{2,r} - a_{1,z} + \frac{a_2}{r}\bigg)B_{11}\\
&\quad + \frac{1}{\nu}\bar\tau_2\cdot\nabla B_{11} \quad \text{on } S_1^T,\\
&\beta_r = -\frac{1}{\nu}B_{22},\ \ \beta_\varphi = \frac{1}{\nu}B_{21},\ \ \beta_{2,z} = \frac{1}{\nu}(B_{22,r} + B_{22} - B_{21,\varphi}) \quad \text{on } S_2^T,\\
&\beta|_{t=0} = \beta(0) \quad \text{in } \Omega.
\end{aligned}
\tag{3.78}
$$

We need

$$
\begin{aligned}
\|F\|_{L_2(0,T;H^1(\Omega))} &\leqslant \|F\|_{W_2^{1,1/2}(\Omega^T)}\\
&\leqslant \|\operatorname{rot} f\|_{W_2^{1,1/2}(\Omega^T)} + \|\beta\|_{W_2^{3,3/2}(\Omega^T)}\\
&\quad + \|\delta\nabla\beta\|_{W_2^{1,1/2}(\Omega^T)} + \|\beta\cdot\nabla\delta\|_{W_2^{1,1/2}(\Omega^T)}\\
&\leqslant \|\operatorname{rot} f\|_{W_2^{1,1/2}(\Omega^T)} + \|\beta\|_{W_2^{3,3/2}(\Omega^T)}\\
&\quad + c\|\delta\|_{W_2^{3,3/2}(\Omega^T)}\|\beta\|_{W_2^{3,3/2}(\Omega^T)}.
\end{aligned}
\tag{3.79}
$$

Now we shall estimate the r.h.s. of (3.79). Multiplying $(3.76)_1$ by η, integrating over Ω^T, using the expressions for B_{ij}, $i,j = 1,2$, we obtain

$$
\|\eta\|_{V_2^0(\Omega^T)} \leqslant c\sum_{i=1}^2(|d_{i,r}|_{2,S_2^T} + |d_{i,\varphi}|_{2,S_2^T} + \gamma|d_i|_{2,S_2^T}) \equiv l_1. \tag{3.80}
$$

For solutions to problem (3.78) we have

$$\begin{aligned}\|\beta\|_{W_2^{3,3/2}(\Omega^T)}&\leqslant\varepsilon\|\eta\|_{W_2^{4,4/2}(\Omega^T)}+c(1/\varepsilon)\|\eta\|_{V_2^0(\Omega^T)}\\&\quad+c\sum_{i=1}^2(\|d_{i,r}\|_{W_2^{5/2,5/4}(S_2^T)}+\|d_{i,\varphi}\|_{W_2^{5/2,5/4}(S_2^T)}\\&\quad+\gamma\|d_i\|_{W_2^{5/2,5/4}(S_2^T)})\\&\equiv\varepsilon\|\eta\|_{W_2^{4,4/2}(\Omega^T)}+c(1/\varepsilon)\|\eta\|_{V_2^0(\Omega^T)}+l_2\end{aligned}\tag{3.81}$$

For solutions to problem (3.76) we have

$$\|\eta\|_{W_2^{4,4/2}(\Omega^T)}\leqslant cl_3.\tag{3.82}$$

Finally,

$$\begin{aligned}\|F\|_{L_2(0,T;H^1(\Omega))}&\leqslant c(\|\operatorname{rot} f\|_{W_2^{1,1/2}(\Omega^T)}\\&\quad+[\varepsilon(l_2+l_3)+c(1/\varepsilon)l_1](1+l_2+l_3)).\end{aligned}\tag{3.83}$$

Using that $c(\cdot)$ is an increasing function of its argument we obtain (3.73). This ends the proof. ■

4. Existence

We prove the existence of solutions to problem (1.9) by the Leray-Schauder fixed point theorem. Therefore we consider the problem

$$\begin{aligned}&\omega_{,t}-\operatorname{div}\mathbb{T}(\omega,p)=-\lambda[\tilde\omega(\omega')\cdot\nabla\tilde\omega(\omega')+\tilde\omega(\omega')\cdot\nabla\delta+\delta\cdot\nabla\tilde\omega(\omega')]+\mathcal{F}\quad\text{in }\Omega^T,\\&\operatorname{div}\omega=0\quad\text{in }\Omega^T,\\&\omega\cdot\bar n=0,\ \ \bar n\cdot\mathbb{T}(\omega,p)\cdot\bar\tau_\alpha+\gamma\omega\cdot\bar\tau_\alpha=B_{1\alpha}(\delta),\quad\alpha=1,2,\quad\text{on }S_1^T,\\&\omega\cdot\bar n=0,\ \ \nu\bar n\cdot\mathbb{T}(\omega,p)\cdot\bar\tau_\alpha=B_{2\alpha}(\delta),\ \ \alpha=1,2,\quad\text{on }S_2^T,\\&\omega|_{t=0}=\omega(0)\quad\text{in }\Omega,\end{aligned}\tag{4.1}$$

where $\lambda\in[0,1]$, $\omega'\in\mathfrak{N}(\Omega^T)$ and $\tilde\omega\in\mathfrak{M}_0(\Omega^T)=\{\omega:|\omega|_{10,\Omega^T}+|\nabla\omega|_{10/3,\Omega^T}<\infty\}$.

In view of Section 3 we have the mapping

$$\Phi_1:\mathfrak{N}(\Omega^T)\to\mathfrak{M}_0(\Omega^T)$$

so

$$\mathfrak{N}(\Omega^T)\ni\omega'\to\Phi_1(\omega')=\tilde\omega\in\mathfrak{M}_0(\Omega^T).$$

For any $\lambda\in[0,1]$ and $\tilde\omega\in\mathfrak{M}_0(\Omega^T)$ problem (4.1) generates the mapping

$$\Phi_2:\mathfrak{M}_0(\Omega^T)\ni\tilde\omega\to\Phi_2(\tilde\omega)=\omega\in W_{5/2}^{2,1}(\Omega^T).$$

Therefore we define the mapping

$$\omega = \Phi(\omega', \lambda) = \Phi_2(\Phi_1(\omega'), \lambda)$$

such that for any $\lambda \in [0,1]$ we have

$$\Phi : \mathfrak{N}(\Omega^T) \to W^{2,1}_{5/2}(\Omega^T) \subset \mathfrak{N}(\Omega^T), \tag{4.2}$$

where the last imbedding is compact.

Lemma 4.1. *The mapping Φ is compact, continuous on $\mathfrak{N}(\Omega^T) \times [0,1]$ and index $\Phi|_{\lambda=0} = 1$.*

Proof. Compactness follows from imbedding (4.2). Continuity on $\mathfrak{N}(\Omega^T) \times [0,1]$ can be proved in the same way as in Lemma 5.3 from [31]. For $\lambda = 0$ there exists a unique solution to problem (4.1) (see [1]). This ends the proof. ■

In view of Lemma 4.1 and (3.62), where a fixed point of mapping Φ is estimated we can apply the Leray-Schauder fixed point theorem. Hence Theorem 1.1 from Section 1 is proved.

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Received: 29 May 2008; **revised:** 12 March 2009