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ON THE HÖLDER CONTINUITY OF GRADIENT OF SOLUTIONS TO A CLASS OF ELLIPTIC SYSTEMS

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Dedicated to Professor Bogdan Bojarski
on the occasion of his 75th birthday

Abstract: The interior $C^{1,\gamma}$-regularity for weak solutions with BMO-gradient of nonlinear nonautonomous second order systems satisfying Legendre-Hadamard ellipticity condition is investigated.

Keywords: nonlinear elliptic systems, regularity, Campanato-Morrey spaces

1. Introduction

The question of smoothness of weak solutions to elliptic systems has been extensively studied for more than one century. One of the key steps in this process is higher integrability of gradients of weak solutions. This information guarantees among other Hölder continuity of solutions on plane domains or estimates of Hausdorff dimension of singular sets. The first and fundamental contribution in this direction was the work of Bogdan Bojarski (see [1], [2]). The boom in these investigations proved that the approach is of great importance. Nevertheless, as well known examples show, full regularity in higher dimensional space cannot be achieved without additional structural assumptions.

In this paper we give conditions guaranteeing that if BMO norm of symmetric gradient of a weak solution u to a nonlinear system satisfying Legendre-Hadamard ellipticity condition is sufficiently small then $u \in C^{1,\gamma}_{\mathrm{loc}}(\Omega)$. Here Ω is a domain in $\mathbb{R}^n$ and as we are interested in local estimates we will not assume anything about $\partial\Omega$ or boundary values of the solution. For $u : \Omega \to \mathbb{R}^n$, Du denotes its distributional gradient and $Eu = \frac{1}{2}(Du + Du^T)$ its symmetric part. The studied system has the form

$$-\mathrm{div}(a^{ij}(x, Eu)) = -\mathrm{div}(f^{ij}(x)) \quad \text{on} \;\; \Omega. \tag{1.1}$$

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We suppose that $a^{ij} = a^{ji}$, $f^{ij} = f^{ji}$ for all $i, j = 1, ..., n$. By a weak solution to (1.1) we will understand a function $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ such that for any $\varphi \in \mathcal{D}(\Omega, \mathbb{R}^n)$ it holds [1]

$$\int_\Omega a^{ij}(x, Eu) E_{ij}(\varphi) dx = \int_\Omega f^{ij} E_{ij}(\varphi) dx.$$

We assume that the coefficients $a^{ij}(x,p)$ have linear growth in p. The scope of this paper is to generalize the results of our previous papers [6], [7] to the case of (1.1) where the coefficients depend on x and on symmetric gradient only and satisfy a weaker ellipticity condition, namely

$$\nu|\xi|^2 \leqslant A^{kl}_{ij}(x,p)\xi_{ij}\xi_{kl}$$

for $x \in \Omega$ and symmetric matrices p, ξ where

$$A^{kl}_{ij}(x,p) = \frac{\partial a^{ij}}{\partial p_{kl}}(x,p).$$

Then (1.1) satisfies the Legendre-Hadamard condition instead of usually required strong uniform ellipticity and the results can be applied to models of continuum or fluid mechanics where the requirement of strong uniform ellipticity is unrealistic.

Let us recall that even for systems with real analytic coefficients, right hand sides satisfying strong uniform ellipticity there are solutions with bounded and discontinuous gradient (see [15]) so that the Hölder continuity of the gradient cannot be obtained without some additional conditions. The example of a system of linear elasticity type with bounded measurable coefficients and unbounded gradient can be found in [16]. For a more detailed survey see [13].

In what follows $\Omega \subset \mathbb{R}^n$ is an open set and $\mathbb{S}_n$ the space of all $n \times n$ symmetric matrices. We impose the following structural assumptions on the coefficients of (1.1):

(i) (Smoothness) The functions $a^{ij}(x,p) : \Omega \times \mathbb{S}_n \to \mathbb{R}$ are differentiable in x and p with continuous derivatives for all $i, j = 1, \dots, n$. Without loss of generality we suppose that $a^{ij}(x, 0) = 0$.

(ii) (Growth) For all $(x,p) \in \Omega \times \mathbb{S}_n$ denote

$$A^{kl}_{ij}(x,p) = \frac{\partial a^{ij}}{\partial p_{kl}}(x,p)$$

and suppose

$$|a^{ij}(x,p)|, \quad \left|\frac{\partial a^{ij}}{\partial x_s}(x,p)\right| \leqslant M(1+|p|), \tag{1.2}$$

$$\left|A^{kl}_{ij}(x,p)\right| \leqslant M, \tag{1.3}$$

where $M > 0$.

[1] Throughout the whole text we use the summation convention over repeated indices.

(iii) (Ellipticity) There exists ν such that for every $x \in \Omega$ and every p, $\xi \in \mathbb{S}_n$

$$\nu|\xi|^2 \leqslant A_{ij}^{kl}(x,p)\xi_{ij}\xi_{kl}. \tag{1.4}$$

(iv) (Oscillation of coefficients) There is a real function ω continuous on $[0,\infty)$, which is bounded, nondecreasing, concave, $\omega(0) = 0$ and such that for all $x \in \Omega$ and p, $q \in \mathbb{S}_n$

$$\left|A_{ij}^{kl}(x,p) - A_{ij}^{kl}(x,q)\right| \leqslant \omega\left(|p-q|\right). \tag{1.5}$$

We set $\omega_\infty = \lim_{t\to\infty}\omega(t) \leqslant 2M$.

(v) (Right hand side) f belongs to Sobolev space $W^{1,2}_{\mathrm{loc}}(\Omega)$ and Df to Morrey space $L^{2,\delta-2}_{\mathrm{loc}}(\Omega)$ for $\delta = n + 2\gamma,\ \gamma \in (0,1)$.

In what follows we will understand by pointwise derivative $\frac{d}{dt}\omega$ the right derivative of ω which is finite on $(0,\infty)$.

For $p \in (1,\infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$ denote

$$J_p = \int_0^\infty \frac{\frac{d}{dt}(\omega^{2p'})(t)}{t}\,dt, \tag{1.6}$$

$$S_p = \sup_{t\in(0,\infty)} \frac{d}{dt}(\omega^{2p'})(t) \tag{1.7}$$

and

$$P_p = \min\{J_p, S_p\}. \tag{1.8}$$

Now we formulate the result

Theorem 1.1. *Let u be a weak solution to* (1.1) *such that $Eu \in BMO(\Omega,\mathbb{S}_n)$ and a^{ij}, f satisfy the hypotheses (i)–(v). Then there exists a number $\sigma_0 > 0$, depending only on n, p, M, ν, ω, such that the inequality*

$$\|Eu\|_{BMO(\Omega,\mathbb{S}_n)} < \sigma_0$$

implies that $Du \in C^{0,\gamma-\varepsilon/2}_{\mathrm{loc}}(\Omega,\mathbb{R}^{n^2})$ for any positive ε. Moreover, for any compactly embedded and smoothly bounded subdomain Ω_0 such that $\operatorname{dist}(\Omega_0,\partial\Omega) \geqslant d$ *the inequality*

$$\|Du\|_{C^{0,\gamma-\frac{\varepsilon}{2}}(\overline{\Omega_0},\mathbb{R}^{n^2})} \leqslant C(\varepsilon,f)d^{\gamma-\frac{\varepsilon}{2}}$$

holds.

2. Preliminaries and Notations

Let $n \in \mathbb{N}$, $n \geqslant 3$. If $x \in \mathbb{R}^n$ and $r > 0$, we set $B(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$. Denote by $u_{x,r} = (\kappa_n\, r^n)^{-1}\int_{B(x,r)} u(y)\,dy$ the mean value of a function $u \in L(B(x,r),\mathbb{R}^n)$ over the set $B(x,r)$.

Moreover, we set

$$\phi_D(x,r) = \int_{B(x,r)} |Du(y) - (Du)_{x,r}|^2\, dy,$$

$$\phi_E(x,r) = \int_{B(x,r)} |Eu(y) - (Eu)_{x,r}|^2\, dy.$$

Beside the usually used space $\mathcal{D}(\Omega, \mathbb{R}^n)$, Hölder space $C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^n)$ and Sobolev spaces $W^{k,p}(\Omega, \mathbb{R}^n)$, $W_0^{k,p}(\Omega, \mathbb{R}^n)$ we use Campanato spaces $\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^n)$, Morrey spaces $L^{q,\lambda}(\Omega, \mathbb{R}^n)$ and space of functions with bounded mean oscillations $BMO(\Omega, \mathbb{R}^n)$(see, e.g.[12]). By function space $X_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ we understand the space of all functions which belong to $X(\tilde{\Omega}, \mathbb{R}^n)$ for any bounded subdomain $\tilde{\Omega}$ with smooth boundary which is compactly embedded in Ω.

For definitions and more details see [3], [9], [12] and [14]. In particular, we will use:

Proposition 2.1. *For a bounded domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary we have the following*

(a) *For $q \in (1, \infty)$, $0 < \lambda < \mu < \infty$ it holds*

$$L^{q,\mu}(\Omega, \mathbb{R}^N) \subset L^{q,\lambda}(\Omega, \mathbb{R}^N), \quad \mathcal{L}^{q,\mu}(\Omega, \mathbb{R}^N) \subset \mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N),$$

(b) *$\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)$ is isomorphic to the $C^{0,(\lambda-n)/q}(\overline{\Omega}, \mathbb{R}^N)$, for $n < \lambda \leqslant n + q$,*
(c) *$L^{q,n}(\Omega, \mathbb{R}^N)$ is isomorphic to the $L^\infty(\Omega, \mathbb{R}^N)$, $\mathcal{L}^{q,n}(\Omega, \mathbb{R}^N)$ is isomorphic to $BMO(\Omega, \mathbb{R}^N)$,*
(d) *$L^{q,\lambda}(\Omega, \mathbb{R}^N)$ is isomorphic to the $\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)$, for $0 < \lambda < n$.*

Lemma 2.2. *Let u be a weak solution to* (1.1) *and coefficients a^{ij} satisfy the hypothesis (i)-(iv) with the constants M, ν and a right hand side $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{n^2})$. Then $u \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and for any $x_0 \in \Omega$ and $R \in (0, \frac{1}{2}\operatorname{dist}(x_0, \partial\Omega))$ it holds*

$$\int\limits_{B(x_0,R)} |D^2 u|^2\, dx \leqslant C\left(\frac{M}{\nu}\right)\left(\frac{1}{R^2}\ \phi_D(x_0, 2R) + R^n \right.$$
$$\left. + \int\limits_{B(x_0,2R)} \left(|Du|^2 + |Df|^2\right)\, dx\right). \tag{2.1}$$

Proof. Following the lines of [10], [8] we use Nirenberg's difference quotients method and Caccioppoli's lemma to obtain for any $s = 1, ..., n$

$$\int\limits_{B(x_0,R)} \left|\frac{\partial}{\partial x_s} Eu\right|^2 dx \leqslant C\left(\frac{M}{\nu}\right)\left(\frac{1}{R^2}\int\limits_{B(x_0,2R)} \left|\frac{\partial u}{\partial x_s} - \left(\frac{\partial u}{\partial x_s}\right)_{x_0,2R}\right|^2 dx\right.$$
$$\left. + R^n + \int\limits_{B(x_0,2R)} |Du|^2\, dx + \int\limits_{B(x_0,2R)} |Df|^2\, dx\right).$$

Any component of D^2u can be easily expressed as a linear combination of components of $D(Eu)$ which implies the assertion of the lemma. ■

In what follows we will use an algebraic lemma due to S. Campanato. By a simple induction argument we can prove

Lemma 2.3. *Let β, d, B be positive numbers and let $\alpha \in [0,\beta)$, $\tau \in (0,1)$. Then there exists constant $C = \frac{Bd^\beta}{\tau(1-\tau^{\beta-\alpha})}$ so that for any nonnegative, nondecreasing function ϕ defined on $[0,d]$ and satisfying the inequality*

$$\phi(\tau R) \leqslant \tau^\alpha \phi(R) + BR^\beta \quad \forall\, R : 0 < R \leqslant d, \tag{2.2}$$

it holds

$$\phi(\sigma) \leqslant \left(\frac{\sigma}{\tau d}\right)^\alpha (\phi(d) + C) \quad \forall \sigma : 0 < \sigma \leqslant d. \tag{2.3}$$

As in [8], Lemma 3.0.5, we get the following

Lemma 2.4. *Consider system of the type (1.1) with $a^{ij}(x,p) = A^{kl}_{ij} p_{kl}$, $A^{kl}_{ij} \in \mathbb{R}$ (i.e. linear system with constant coefficients) satisfying (i), (ii) and (iii) and f identically zero. Then there exists a constant $L = L(n, M/\nu) \geqslant 1$ such that for every weak solution $v \in W^{1,2}(\Omega, \mathbb{R}^n)$ and for every $x \in \Omega$ and $0 < \sigma \leqslant R \leqslant \operatorname{dist}(x, \partial\Omega)$ the following estimates*

$$\int\limits_{B(x,\sigma)} |Dv(y) - (Dv)_{x,\sigma}|^2 \, dy \leqslant L \left(\frac{\sigma}{R}\right)^{n+2} \int\limits_{B(x,R)} |Dv(y) - (Dv)_{x,R}|^2 \, dy,$$

$$\int\limits_{B(x,\sigma)} |Ev(y) - (Ev)_{x,\sigma}|^2 \, dy \leqslant L \left(\frac{\sigma}{R}\right)^{n+2} \int\limits_{B(x,R)} |Ev(y) - (Ev)_{x,R}|^2 \, dy$$

hold.

In the next we take use of

Lemma 2.5 ([17], p.37). *Let $\psi : [0,\infty] \to [0,\infty]$ be a non decreasing function which is absolutely continuous on every closed interval of finite length, $\psi(0) = 0$. If $h \geqslant 0$ is measurable, $H(t) = \{y \in \mathbb{R}^n : h(y) > t\}$ and μ is n-dimensional Lebesgue measure then*

$$\int\limits_{\mathbb{R}^n} \psi \circ h \, dy = \int\limits_0^\infty \mu\left(H(t)\right) \psi'(t) \, dt.$$

Proof of Theorem 1.1. Let Ω_0 be a smoothly bounded subdomain of Ω. Denote $d = \operatorname{dist}(\Omega_0, \partial\Omega) > 0$. For $x_0 \in \Omega_0$ we will prove that $Eu \in \mathcal{L}^{2,\delta}$ on a neighborhood of x_0. Let $R \leqslant \frac{1}{2}d$. Since no confusion can occur, we will use the notation $B(R)$, $\phi_E(R)$, $\phi_D(R)$ and $(Eu)_R$ instead of $B(x_0,R)$, $\phi_E(x_0,R)$, $\phi_D(x_0,R)$ and $(Eu)_{x_0,R}$.

Denote

$$A_{ij,0}^{kl} = A_{ij}^{kl}\left(x_0, (Eu)_R\right), \quad \tilde{A}_{ij}^{kl}(x) = \int_0^1 A_{ij}^{kl}\left(x_0, (Eu)_R + t\left(Eu(x) - (Eu)_R\right)\right) dt.$$

Hence

$$a^{ij}(x_0, Eu) - a^{ij}(x_0, (Eu)_R) = \tilde{A}_{ij}^{kl}(x)\left(E_{kl}u(x) - (E_{kl}u)_R\right).$$

Using the foregoing notations we can rewrite the system (1.1) as

$$\begin{aligned} -D_j\left(A_{ij,0}^{kl}E_{kl}u\right) = {} & -D_j\left(\left(A_{ij,0}^{kl} - \tilde{A}_{ij}^{kl}\right)\left(E_{kl}u - (E_{kl}u)_R\right)\right) \\ & - D_j\left(a^{ij}(x_0, Eu) - a^{ij}(x, Eu)\right) - D_j\left(f^{ij}(x) - (f^{ij})_R\right). \end{aligned}$$

Split u as $v + w$ where v is the solution of the Dirichlet problem

$$\begin{aligned} -D_j\left(A_{ij,0}^{kl}E_{kl}v\right) &= 0 \quad \text{in } B(R) \\ v - u &\in W_0^{1,2}\left(B(R), \mathbb{R}^n\right). \end{aligned}$$

and $w \in W_0^{1,2}(B(R), \mathbb{R}^n)$ is the weak solution of the system

$$\begin{aligned} -D_j\left(A_{ij,0}^{kl}E_{kl}w\right) = {} & -D_j\left(\left(A_{ij,0}^{kl} - \tilde{A}_{ij}^{kl}\right)\left(E_{kl}u - (E_{kl}u)_R\right)\right) \\ & - D_j\left(a^{ij}(x_0, Eu) - a^{ij}(x, Eu)\right) \\ & - D_j\left(f^{ij}(x) - (f^{ij})_R\right). \end{aligned} \tag{2.4}$$

For every $0 < \sigma \leqslant R$ from Lemma 2.4 it follows

$$\int_{B(\sigma)} \left|Dv - (Dv)_\sigma\right|^2 dx \leqslant L\left(\frac{\sigma}{R}\right)^{n+2} \int_{B(R)} \left|Dv - (Dv)_R\right|^2 dx$$

hence

$$\begin{aligned} \phi_D(\sigma) &= \int_{B(\sigma)} \left|Du - (Du)_\sigma\right|^2 dx \\ &\leqslant 2L\left(\frac{\sigma}{R}\right)^{n+2} \int_{B(R)} \left|Dv - (Dv)_R\right|^2 dx + 4\int_{B(R)} |Dw|^2\, dx \\ &\leqslant 4L\left(\frac{\sigma}{R}\right)^{n+2} \phi_D(R) + 4\left(1 + 2L\left(\frac{\sigma}{R}\right)^{n+2}\right) \int_{B(R)} |Dw|^2\, dx. \end{aligned} \tag{2.5}$$

Now as $w \in W_0^{1,2}(B_R, \mathbb{R}^N)$ we can choose test function $\varphi = w$ in (2.4) and, using Korn's lemma in the first inequality, we get

$$\begin{aligned}\int\limits_{B(R)} |Dw|^2\,dx \leqslant 2 \int\limits_{B(R)} |Ew|^2\,dx \\ \leqslant \frac{6}{\nu^2}\Bigg(\int\limits_{B(R)} \omega^2\left(|Eu-(Eu)_R|\right)|Eu-(Eu)_R|^2\,dx \\ + \int\limits_{B(R)} \sum_{i,j=1}^{n} \Big(\left|a^{ij}(x_0,Eu)-a^{ij}(x,Eu)\right|^2 \\ + \left|f^{ij}(x)-(f^{ij})_R\right|^2 \Big)\,dx\Bigg).\end{aligned} \tag{2.6}$$

Now, fix σ so small that $\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right) \leqslant 2$. The inequalities (2.5) and (2.6) and Poincaré's inequality imply

$$\begin{aligned}\phi_D(\sigma) \leqslant 4L\left(\frac{\sigma}{R}\right)^{n+2}\phi_D(R) + \frac{48}{\nu^2}\Bigg[\int\limits_{B(R)} \omega^2\left(|Eu-(Eu)_R|\right)|Eu-(Eu)_R|^2\,dx \\ + \int\limits_{B(R)} \sum_{i,j=1}^{n}\left|a^{ij}(x_0,Eu)-a^{ij}(x,Eu)\right|^2 + c(n)R^2\int\limits_{B(R)}|Df|^2\,dx\Bigg] \\ \leqslant 4L\left(\frac{\sigma}{R}\right)^{n+2}\phi_D(R) + \frac{48}{\nu^2}\left[(I_1+I_2)+c(n)R^{\delta}||Df||^2_{L^{2,\delta-2}(\Omega,\mathbb{R}^{n^2})}\right],\end{aligned} \tag{2.7}$$

where 2 is the constant from Korn's inequality and $c(n)$ the constant from Poincaré's inequality.

Then using Hölder's inequality with the exponent p from the assumptions of Theorem 1.1, the embedding theorem and Lemma 2.2 we have

$$\begin{aligned}I_1 \leqslant \Bigg(\int\limits_{B(R)} |Eu-(Eu)_R|^{2p}\,dx\Bigg)^{1/p}\Bigg(\int\limits_{B(R)} \omega^{2p'}\left(|Eu-(Eu)_R|\right)dx\Bigg)^{1/p'} \\ \leqslant C_p^2 R^{2-n/p'}\int\limits_{B(R)}|D^2u|^2\,dx\Bigg(\int\limits_{B(R)} \omega^{2p'}\left(|Eu-(Eu)_R|\right)dx\Bigg)^{1/p'} \\ \leqslant C(n,p,M/\nu)\Bigg(\frac{1}{\kappa_n R^n}\int\limits_{B(R)} \omega^{2p'}\left(|Eu-(Eu)_R|\right)dx\Bigg)^{1/p'} \\ \times\left(\phi_D(2R)+R^{n+2}+R^2||Du||^2_{L^2(B(2R),\mathbb{R}^{n^2})}+R^{\delta}||Df||^2_{L^{2,\delta-2}(\Omega,\mathbb{R}^{n^2})}\right),\end{aligned} \tag{2.8}$$

where C_p stands for the embedding constant from $W^{1,2}(B(1),\mathbb{R}^N)$ into $L^{2p}(B(1),\mathbb{R}^N)$ and $C(n,p,M/\nu)$ is the product of C_p^2 and the constant from Lemma 2.2.

Taking in Lemma 2.5 $\psi(t)=\omega^{2p'}(t)$, $h(y)=|Eu(y)-(Eu)_R|$ on $B(R)$ and $h(y)=0$ otherwise, we have $H_R(t)=\{y\in B(R) : |Eu(y)-(Eu)_R|>t\}$ and for the last integral we get

$$\int_{B(R)} \omega^{2p'}\left(|Eu-(Eu)_R|\right)\, dx = \int_0^\infty \left[\frac{d}{dt}(\omega^{2p'})(t)\right]\mu\left(H_R(t)\right)\, dt.$$

Now we can estimate the integral on the right hand side according to assumptions of the theorem. In the first case we assume that

$$P_p = J_p = \int_0^\infty \frac{\frac{d}{dt}(\omega^{2p'})(t)}{t}\, dt < \infty.$$

As $\mu\left(H_R(t)\right)$ is nonnegative, non-increasing it holds $\mu\left(H_R(t)\right) \leqslant \frac{1}{t}\int_0^t \mu\left(H_R(s)\right)\, ds$ and we have

$$\begin{aligned}
\int_0^\infty \left[\frac{d}{dt}(\omega^{2p'})(t)\right]\mu\left(H_R(t)\right)\, dt &\leqslant \int_0^\infty \frac{d}{dt}(\omega^{2p'})(t)\left(\frac{1}{t}\int_0^t \mu\left(H_R(s)\right)\, ds\right) dt \\
&\leqslant \int_0^\infty \frac{\frac{d}{dt}(\omega^{2p'})(t)}{t}\, dt \int_{B(R)} |Eu-(Eu)_R|\, dx \\
&\leqslant (\kappa_n R^n)^{1/2} J_p \phi_E^{1/2}(R). \qquad (2.9)
\end{aligned}$$

If $P_p = S_p = \sup_{0<t<\infty} \frac{d}{dt}(\omega^{2p'})(t) < \infty$ we have

$$\int_0^\infty \left[\frac{d}{dt}(\omega^{2p'})(t)\right]\mu\left(H_R(t)\right)\, dt \leqslant (\kappa_n R^n)^{1/2} S_p \phi_E^{1/2}(R). \qquad (2.10)$$

Denoting

$$K = \kappa_n^{-\frac{1}{2p'}} C(n,p,M/\nu) P_p^{\frac{1}{p'}} \|Eu\|_{BMO(\Omega,\mathbb{S}_n)}^{\frac{1}{2p'}} \qquad (2.11)$$

and using (2.8), (2.9), (2.10) for the estimate of I_1 we get

$$I_1 \leqslant K\left(\phi_D(2R) + R^{n+2} + R^2 ||Eu||^2_{L^2(B(2R),\mathbb{S}_n)} + R^\delta ||Df||^2_{L^{2,\delta-2}(\Omega,\mathbb{R}^{n^2})}\right). \qquad (2.12)$$

As we suppose that $Eu \in BMO(\Omega,\mathbb{S}_n)$ we have from Proposition 2.1 that $Eu \in L^{2,\lambda}(\Omega,\mathbb{S}_n)$ for any $\lambda < n$ and

$$||Eu||^2_{L^2(B(2R),\mathbb{S}_n)} \leqslant c(\lambda,n) R^\lambda ||Eu||^2_{BMO(\Omega,\mathbb{S}_n)}. \qquad (2.13)$$

Set $\lambda = \delta - 2 = n - 2(1-\gamma)$, $R < 1$. Hence

$$I_1 \leqslant K\phi_D(2R) + Kc(\lambda, n)\left(1 + ||Eu||^2_{BMO(\Omega,\mathbb{S}_n)} + ||Df||^2_{L^{2,\delta-2}(\Omega,\mathbb{R}^{n^2})}\right) R^\delta$$

$$\begin{aligned} I_2 &\leqslant M^2R^2 \int\limits_{B(R)} \left(1+|Eu|^2\right) dx \leqslant M^2 \left(\kappa_n R^{n+2} + R^2 \int\limits_{B(R)} |Eu|^2\, dx\right) \\ &\leqslant M^2\left(\kappa_n + \|Eu\|^2_{BMO(\Omega,\mathbb{S}_n)}\right) R^\delta. \end{aligned} \tag{2.14}$$

We get from (2.7) by means of (2.14)

$$\begin{aligned} \phi_D(\sigma) \leqslant & \left[4L\left(\frac{\sigma}{R}\right)^{n+2} + \frac{48}{\nu^2}K\right]\phi_D(2R) \\ & + \frac{48}{\nu^2}(Kc(\lambda,n)+M^2)\left(\kappa_n + \|Du\|^2_{BMO} + ||f||^2_{\mathcal{L}^{2,\delta}}\right)R^\delta. \end{aligned} \tag{2.15}$$

Choose a positive $\varepsilon < \delta$ and set in Lemma 2.3 $\beta = \delta$, $\alpha = \delta - \varepsilon$, $\tau \in (0,1)$ so small that $4L\tau^{n+2-\alpha} \leqslant \frac{1}{2}$ and $B = \frac{48}{\nu^2}(Kc(\lambda,n)+M^2)\left(\kappa_n + \|Du\|^2_{BMO} + ||f||^2_{\mathcal{L}^{2,\delta}}\right)$. Let σ_0 be so small that

$$\frac{48}{\nu^2}\kappa_n^{-\frac{1}{2p'}} C(n,p,M/\nu) P_p^{\frac{1}{p'}} \sigma_0^{\frac{1}{2p'}} \leqslant \frac{1}{2}.$$

If $\|Du\|^2_{BMO} < \sigma_0$ then also $\frac{48}{\nu^2}K\tau^{-\alpha} \leqslant \frac{1}{2}$ and from (2.15)

$$\phi_D(\tau R) \leqslant \tau^\alpha \phi_D(R) + BR^\delta.$$

Thus Lemma 2.3 implies that

$$\phi_D(\sigma) \leqslant \tilde{C}\sigma^\alpha, \quad \forall\ \sigma \leqslant \frac{d}{2}. \tag{2.16}$$

The thesis follows from Proposition 2.1, part (b). ■

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