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RENORMALIZED ESTIMATES FOR SOLUTIONS TO THE NAVIER-STOKES EQUATION

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Dedicated to Professor Bogdan Bojarski
on the occasion of his 75th birthday

Abstract: For weak solutions to the three-dimensional Navier-Stokes equations the interior regularity problem for the renormalized velocity $u(1+|u|^2)^{-\alpha/2}$ and pressure $p(1+|u|^2)^{-\beta/2}$ is investigated. If a velocity component is locally semibounded and ∇u slightly more regular than suitable weak solutions the regularity estimates for the renormalized velocity are improved. Furthermore, estimates for the negative part of a renormalized pressure are presented.

Keywords: Navier-Stokes equations, interior regularity, Morrey conditions

1. Introduction

In this paper we consider the classical Navier-Stokes equations

$$u_t - \mu\Delta u + (u\cdot\nabla)u = -\nabla p + f, \tag{1.1}$$
$$\operatorname{div} u = 0 \tag{1.2}$$

in a domain $(0,T)\times\Omega\subset\mathbb{R}^1\times\mathbb{R}^3$ together with an initial condition $u|_{t=0}=u_0$. We deal only with interior estimates, so the boundary condition does not play any role, the reader may think of slip, non-slip or Neumann-boundary or mixed conditions, and Ω is bounded for simplicity. For the viscosity constant we assume $0<\mu\in\mathbb{R}$, for the outer force f and the initial value we require

$$\begin{aligned} &f\in L^\infty\big(W^{1,\infty}(\Omega)\big), \quad \operatorname{div} f = 0 \quad \text{a.e.},\\ &\qquad\qquad u_0\in W^{2,q_0}(\Omega) \quad \text{for some} \quad q_0>5,\ \operatorname{div} u_0 = 0 \end{aligned} \tag{1.3}$$

for simplicity. Here for $1\leqslant q\leqslant\infty$ and $m\in\mathbb{N}_0$, the space $W^{m,q}(\Omega)\subset L^q(\Omega)$ denotes the usual Sobolev space (see [1], e.g.), we use also the notation $H^m(\Omega)$ in the Hilbert space case $q=2$. By the classical methods as they are exposed in the books of [9, 11, 15, 14] we know the existence of a *suitable* weak solution (u,p)

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which satisfies the following properties

$$\begin{aligned} &u \in L^\infty(L^2(\Omega)) \cap L^2\big(H^1(\Omega)\big) \cap L^{10/3}\big(L^{10/3}(\Omega)\big), \quad \nabla^2 u \in L^{4/3-\delta}(L^{4/3-\delta}) \\ &p \in L^{5/3}\big(L^{5/3}_{\mathrm{loc}}(\Omega)\big), \quad \nabla p \in L^{5/4}\big(L^{5/4}_{\mathrm{loc}}(\Omega)\big) \end{aligned} \tag{1.4}$$

and the corresponding norms are estimated uniformly with respect to the data. Here, $L^r(V) = L^r\big((0,T)\,;V\big)$ are the usual L^r space of V-valued functions on $[0,T]$, see [12] e.g. For simplicity we mostly write L^r etc instead of $L^r(\Omega)$, if no confusion arises.

If we apply the divergence operator to (1.1) and use (1.2), we obtain the pressure equation

$$-\Delta p = \sum_{i,l=1}^{n} D_i u_l D_l u_i \tag{1.5}$$

Since it is known that the right hand side of (1.5) is (locally) in the Hardy space $\mathbb{H}$ [11] it is possible to apply the theory of Hardy spaces and we obtain

$$\nabla^2 p \in L^1\big(L^1_{\mathrm{loc}}(\Omega)\big) \tag{1.6}$$

together with an estimate of $\|\nabla^2 p\|_{L^1(L^1(\Omega_0))}$ by the data (1.3), here Ω_0 is any relative compact subdomain of Ω. From (1.6) and the Navier Stokes equations (1.1), (1.2) there follow the estimates (see e.g. the arguments in Lemma 3.1)

$$\int_0^T \int_{\Omega_0} \frac{|\nabla D u_\nu|^2}{(1+|Du_\nu|)^{1+\delta}}\,dx\,dt + \sup_{0\leqslant T} \|Du_\nu\|_{L^1(\Omega_0)} \leqslant K_{\Omega_0,T}\,, \qquad \nu = 1,2,3\,, \tag{1.7}$$

where K depends on the data (the simply available ones), for all $\Omega_0 \subset\subset \Omega$, and $\delta > 0$. Here D stands for the partial derivatives $\partial/\partial x_i = D_i$, $i = 1,2,3$.

It is well known that the inclusion $u \in L^5(L^5) = L^5\big(0,T;L^5_{(\mathrm{loc})}(\Omega)\big)$ implies regularity of the solution u - a sufficient regularity of the data is provided (see [14], e.g). Recently Escauriazia, Seregin and Šverák [7] found, among other results, that also the condition

$$u \in L^\infty\big(L^3(\Omega)\big)$$

implies regularity. An example for results on partial regularity is the famous Caffarelli-Kohn-Nirenberg condition: if

$$R^{-1} \iint_{Q_R(x_0)} |\nabla u|^2\,dx\,dt \leqslant \varepsilon_0, \quad R \leqslant R_0$$

for some ε_0, R_0 sufficiently small, then the solution (u,p) is regular in a neighborhood of x_0, t_0 here $Q_R(x_0) = [t_0, t_0 - R^2] \times B_R(x_0)$ (see [5, 10, 16] and [13] for further results on partial regularity).

In this situation it is reasonable to pose the *simpler question* whether the *renormalized quantities*

$$\frac{u}{(1+|u|^2)^{\alpha/2}} \quad \text{and} \quad \frac{p}{(1+|u|^2)^{\beta/2}}$$

are regular.

A more modest question is, whether these *renormalized quantities* enjoy *better L^q-properties* for the derivatives in comparison with the original quantities u and p. Using partial integration the following result can be obtained comparably easy (see Section 2 for the proof).

Lemma 1.1. *Let $u \in L^2(H^1)$ be a weak solution of* (1.1), (1.2). *Then*

$$\int_0^T \int_{\Omega_0} \frac{|Du_\nu|^{5/2-\delta}}{1+|u_\nu|}\, dx\, dt \leqslant K_{\Omega_0}\,, \qquad \nu = 1,2,3 \tag{1.8}$$

for all $\delta \in]0,1[$ and all $\Omega_0 \subset\subset \Omega$, with a constant K_{Ω_0} depending on δ and the data.

In [2, 3] Da Veiga showed that the condition

$$\frac{p}{1+|u|} \in L^r(L^q), \qquad \frac{2}{r} + \frac{n}{q} < 1 \tag{1.9}$$

implies regularity of a weak solution. In fact he needs this condition only on the set $\{x : |u| > k\}$. In this spirit we investigate equations and interior regularity estimates for the normalized pressure and velocity field.

Plan and essential results of this paper are the following: In Section 2 we derive estimates for the renormalized velocity without additional assumptions. In Section 3 and 4 we consider velocity fields with a locally semibounded component u_ν and obtain the following result:
If in addition $\nabla u \in L^{2+2\varepsilon}\big(L^{2+2\varepsilon}_{\text{loc}}(\Omega)\big)$, then for any $\Omega_1 \subset\subset \Omega_0 \subset\subset \Omega$ there hold the inequalities (Theorem 4.1)

$$\int_0^T \int_{\Omega_1} \frac{|Du_\nu|^{3+3\delta}}{(1+|u_\nu|)^2}\, dx\, dt \leqslant K_{\Omega_0}\,, \tag{1.10}$$

$$\int_0^T \int_{\Omega_1} \frac{|\nabla Du_\nu|^{(1+\delta)3/2}}{(1+|u_\nu|)^{1-\varepsilon+\rho}}\, dx\, dt \leqslant K_{\Omega_0}, \tag{1.11}$$

where $|\rho|$ is arbitrarily small and $\delta = \delta(\varepsilon) > 0$.

Section 5 deals with the renormalized pressure. Using (1.5) we derive an equation for the renormalized pressure $z = p/(1+|u|^2)^{q/2}$, however despite looking innocent at a first glance, this equation has rather bad coefficients depending on t allowing only for local $L^{\delta+3/2}$-estimates of $\nabla^2 z(t)$ in in terms of the corresponding local norms of this quantity in $L^1_{\text{loc}}(\Omega)$. Therefore we consider an alternative approach to obtain renormalized pressure estimates. If φ is a local solution to $-\Delta\varphi = p$, this solution is locally semibounded, an under conditions slightly more restrictive than (1.4) the inclusion $p_-^2|\varphi - k_0|^{-1} \in L^1(L^1_{\text{loc}})$ can be proved (Theorem 5.9). In Section 6 we discuss briefly equations for the renormalized velocity. In the semibounded case we suggest to use a scalar equation for the renormalized velocity which contains a degenerate quadratic term having a sign. Under additional assumptions for the renormalized pressure -in the spirit of [2]- here one might expect C^α-regularity for the solutions. With rather simple arguments one can obtain Morrey conditions at the limiting case (i.e. any larger Morrey-exponent would imply the desired C^α-regularity, see Theorem 6.1 and the subsequent remark).

In a forthcoming second part we intend to present further analysis of the scalar renormalized Navier-Stokes equation together with estimates for the renormalized fractional time derivatives of the velocities and various estimates for the renormalized pressure and its spatial gradient.

2. The Starting Point

We start with the proof of Lemma 1.1.

Proof. We fix $\Omega_0 \subset\subset \Omega$ and a nonnegative localization function $\tau \in C_0^\infty(\Omega)$ with $\tau = 1$ on Ω_0. By (1.7) and Hölder's inequality we obtain that for any $s > 0$

$$-\int_0^T \int_{\Omega_0} \frac{D^2 u_\nu}{(1 + |Du_\nu|^2)^{(1+\delta)/4}} |Du_\nu| \cdot \frac{u_\nu}{(1 + |u_\nu|^s)^{1/s}} \tau \, dx \leqslant K_{\Omega_0}.$$

In fact, the first two factors in the integral above are in L^2, while the third is in L^∞. We rewrite the first two factors as

$$\frac{D^2 u_\nu |Du_\nu|}{(1 + |Du_\nu|^2)^{(1+\delta)/4}} = \frac{2}{3-\delta} (\operatorname{sign} Du_\nu) D(1 + |Du_\nu|^2)^{(3-\delta)/4}.$$

Now we perform partial integration and with the identity

$$D\left(\frac{u_\nu}{(1 + |u_\nu|^s)^{1/s}}\right) = \frac{Du_\nu}{(1 + |u_\nu|^s)^{1+1/s}}$$

we arrive at

$$\int_0^T \int_{\Omega_0} \frac{(1 + |Du_\nu|^2)^{(3-\delta)/4} |Du_\nu|}{(1 + |u_\nu|^s)^{1+1/s}} \tau \, dx \, dt + \text{ pollution term } \leqslant K_{\Omega_0} .$$

The pollution term contains derivatives of τ and a function which can be estimated by $K(1 + |Du_\nu|^2)^{3/4}$ and, hence, creates a bounded term. Choosing $s = \delta'$ and replacing $\delta/2$ by δ we arrive at

$$v \int_0^T \int_{\Omega_0} \frac{|Du_\nu|^{5/2-\delta}}{(1 + |u_\nu|)^{1+\delta'}} \, dx \, dt \leqslant K_{\Omega_0} , \qquad \nu = 1, 2, 3.$$

Since δ and δ' are independent we obtain the statement of Lemma 1.1. (Observe that $(\int\int G^{\frac{1}{1+\delta'}}\,dx\,dt)^{1+\delta'} \leqslant K\int\int G\,dx\,dt$.) ■

We can prove an elementary renormalized estimate for the second derivatives of u, too.

Lemma 2.1. *Let (u,p) be a weak solution of* (1.1), (1.2) *with $u \in L^2(0,T,H^1_{\mathrm{loc}}) \cap L^\infty(L^2)$. Then*

$$\int_0^T\int_{\Omega_0}\left[\frac{|\nabla Du_\nu|}{(1+|u_\nu|)^{1/5}}\right]^{10/7-\delta'} dx\,dt \leqslant K_{\Omega_0}$$

for all $\delta' \in]0,1[$, where the constant K_{Ω_0} depends on the data.

Proof. For $l < 2$, Hölder's inequality with exponents $2/l$, $2/(2-l)$ and Young's inequality lead to

$$\begin{aligned}\int_0^T\int_{\Omega_0}\frac{|\nabla Du_\nu|^l}{(1+|u_\nu|)^{l/5}}\,dx\,dt &= \int_0^T\int_{\Omega_0}\frac{|\nabla Du_\nu|^l}{(1+|Du_\nu|)^{(1+\delta)l/2}}\,\frac{(1+|Du_\nu|)^{(1+\delta)l/2}}{(1+|u_\nu|)^{l/5}}\\ &\leqslant \int_0^T\int_{\Omega_0}\frac{|\nabla Du_\nu|^2}{(1+|Du_\nu|)^{1+\delta}}\,dx\,dt \qquad (2.1)\\ &\quad+\int_0^T\int_{\Omega_0}\frac{(1+|Du_\nu|)^{(1+\delta)l/(2-l)}}{(1+|u_\nu|)^{2l/5(2-l)}}\,dx\,dt\,.\end{aligned}$$

The first summand is bounded for every small δ due to (1.7). To estimate the second one is a bit more intricate. If we take $l = 10/7$, then

$$a := 2l/5(2-l) = 1, \quad b := (1+\delta)l/(2-l) = (1+\delta)5/2,$$

thus the second value is too large to apply (1.8). If we choose $l = \frac{10}{7} - \delta'$, then $a = \frac{2}{5}\big(\frac{5}{2}-\delta_1\big) < 1$, and $b = (1+\delta)\big(\frac{5}{2}-\delta_1\big)$, this can be rewritten to

$$b = \left(\frac{5}{2}-\delta_0\right)a + b - \left(\frac{5}{2}-\delta_0\right)a = \left(\frac{5}{2}-\delta_0\right)a + \left(\delta+\frac{2}{5}\delta_0\right)\left(\frac{5}{2}-\delta_1\right).$$

Now we divide

$$\frac{(1+|Du_\nu|)^{(1+\delta)l/(2-l)}}{(1+|u_\nu|)^{2l/5(2-l)}} = \left(\frac{(1+|Du_\nu|)^{(\frac{5}{2}-\delta_0)a}}{(1+|u_\nu|)^a}\right)(1+|Du_\nu|)^{(\delta+\frac{2}{5}\delta_0)(\frac{5}{2}-\delta_1)}$$

apply Hölder's inequality with exponents $1/a$ and $5/(2\delta_1)$, and use (1.8) with δ replaced by δ_0. Here we have to observe that δ_1 is fixed by the choice of l, but δ, δ_0 can be taken arbitrarily small, hence $\int\int(1+|Du_\nu|)^{(\delta+\frac{2}{5}\delta_0)(\frac{5}{2}-\delta_1)\frac{5}{2\delta_1}}$ is bounded due to embedding theorems. ■

Under the additional assumption of *semiboundedness* one can considerably improve these inequalities with the help of Bernis' inequalities [4]. In the Navier-Stokes case, where $\nabla^2 u \in L^{4/3-\delta}_{\mathrm{loc}}(L^{4/3-\delta}_{\mathrm{loc}})$ ($\delta > 0$ arbitrarily small, see [6], e.g.) they imply

$$\int_0^T \int_{\Omega_1} \left(\frac{|Du_\nu|^2}{u_\nu + k_\nu} \right)^{4/3-\delta} \leqslant K \int_0^T \int_{\Omega_0} |D^2 u_\nu|^{4/3-\delta} + K,$$

if $u_\nu + k_\nu \geqslant 1$. This means we gain the power $8/3 - 2\delta$ for Du_ν in the numerator.

3. Estimates for Approximate Derivatives

In the following, we work with a modest additional a priori assumption on a component of the velocity u_ν, namely:

There exists an $\varepsilon > 0$ such that

$$\int_0^T \int_{\Omega_0} |\nabla u|^{2+2\varepsilon} \, dx \, dt \leqslant K_{\Omega_0}. \tag{3.1}$$

With this assumption there holds a refinement of (1.7) , namely

$$\int_0^T \int_{\Omega_0} \frac{|\nabla D u_\nu|^2}{|Du_\nu|^{1-2\varepsilon}} \, dx \, dt \leqslant K_{\Omega_0} \, . \tag{3.2}$$

Note that in the denominator we have $|Du_\nu|^{1-2\varepsilon}$ here instead of $(1 + |Du_\nu|)^{1-2\varepsilon}$, but we will not make use of this fact.

We prove (3.2) by using an approximate version, which is of further use also. To this end we introduce the difference quotients in the i-th spatial direction defined by

$$D_i^h u(t,x) = h^{-1}\big(u(t, x + e_i h) - u(t,x)\big),$$
$$D_i^{-h} u(t,x) = h^{-1}\big(u(t,x) - u(t, x - e_i h)\big),$$

where e_i denotes the i-th unit vector. We observe that

$$D_i^h u = D_i(v), \qquad \text{with } \ v = I_i^h u,$$

where I_i^h defines a special integral operator, namely

$$I_i^h u(t,x) = h^{-1} \int_0^h u(t, x + e_i \xi) \, d\xi.$$

Lemma 3.1. *Let (u,p) be a weak solution of the Navier-Stokes equation* (1.1),(1.2), $u \in L^2(H^1_{\mathrm{loc}}) \cap L^\infty(L^2_{\mathrm{loc}})$, *where the data fulfill* (1.3). *Additionally,* $\nabla u \in L^{2+2\varepsilon}_{\mathrm{loc}}$ *is assumed, where* $0<\varepsilon<2/3$. *Then there holds*

$$\int_0^T \int_{\Omega_0} \frac{|\nabla D_i(I_i^h u_\nu)|^2}{|D_i(I_i^h u_\nu)|^{1-2\varepsilon}}\, dx\, dt \leqslant K_{\Omega_0}, \tag{3.3}$$

the constant K_{Ω_0} *depends on* ε *and the data, but not on the parameter* h.

Proof. We apply the operation $D^h = D_i^h$ to the ν-th equation of the Navier-Stokes System and test with $\tau^2 |D^h u_\nu|^{2\varepsilon} \operatorname{sign}(D^h u_\nu)$. Here τ^2 is a localization function with support $\tau \subset\subset \Omega$ and $\tau = 1$ on Ω_0.[1] Then we arrive at the equation

$$\begin{aligned}
&\int_0^T \int_\Omega (1+2\varepsilon)^{-1} \tau^2 \big(|D^h u_\nu|^{1+2\varepsilon}\big)_t\, dx\, dt + 2\varepsilon \int_0^T \int_\Omega \tau^2 |\nabla D^h u_\nu|^2 |D^h u_\nu|^{2\varepsilon-1}\, dx\, dt \\
&\qquad + (1+2\varepsilon)^{-1} \int_0^T \int_\Omega \tau^2 \sum_{j=1}^3 u_j D_i |D^h u_\nu|^{1+2\varepsilon}\, dx\, dt \\
&\qquad + \int_0^T \int_\Omega \tau^2 \sum_{j=1}^3 D^h u_j E^h D_j u_\nu |D^h u_\nu|^{2\varepsilon} \operatorname{sign}(D^h u_\nu)\, dx\, dt \\
&\qquad + \text{pollution terms} + \int_0^T \int_\Omega \tau^2 D^h \nabla p (D^h u_\nu)^{2\varepsilon} \operatorname{sign}(D^h u_\nu)\, dx\, dt \\
&= \int_0^T \int_\Omega \tau^2 f |D^h u_\nu|^{2\varepsilon} \operatorname{sign}(D^h u_\nu)\, dx\, dt\,.
\end{aligned}$$

Here E^h is the shift operator, and we have to convince us that all integrals exist for small positive ε. Since the $L^2(W^{1,2})_{\mathrm{loc}}$-norms of u and the $L^2(L^{2+2\varepsilon}_{\mathrm{loc}})$-norms of u_ν are bounded uniformly as $h \to 0$, the right hand side of the last equation and the fourth term (containing E^h) on the left hand side are uniformly bounded as $h \to 0$. Furthermore, the third term can be transformed via partial integration using $\operatorname{div} u = 0$. The resulting term is easily estimated since $u_j |D^h u_\nu|^{1+2\varepsilon}$ is bounded uniformly in $L^1(L^1)$ as long as $|Du_\nu|^{1+2\varepsilon} \in L^{10/7}$, the dual of $L^{10/3}(L^{10/3}) \ni u_j$. This is the case for $\varepsilon \leqslant 2/3$. The pollution terms (containing $\nabla\tau$) can be simply handled, too. The first summand is rewritten via partial integration, there arises a pollution term, which is easily estimated, and boundary terms, carrying the correct sign at the time T. The boundary term for $t=0$ is estimated due to hypothesis (1.3). Thus we arrive at the inequality

$$(1+2\varepsilon)^{-1} \int_\Omega \tau^2 |D^h u_\nu|^{1+2\varepsilon}\, dx|_{t=T} + 2\varepsilon \int_0^T \int_\Omega \tau^2 |\nabla D^h u_\nu|^2 |D^h u_\nu|^{2\varepsilon-1}\, dx\, dt \leqslant K_{\Omega_0}\,.$$

Since we may write $D^h u_\nu = D_i I_i^h u_\nu$, the theorem follows. In addition we may pass to the limit $h \to 0$ and obtain (3.2). ■

4. Renormalized Estimates for Semi-Bounded Components of the velocity

In this section we shall assume that the ν-th component of u is *semibounded*, without loss of generality we assume

$$\begin{gathered} \textit{For each } \Omega_0 \subset\subset \Omega \textit{ there exists a constant } c = c(\Omega_0) \textit{ such that} \\ u_\nu \geqslant -c \qquad \textit{a.e. in } \Omega_0. \end{gathered} \tag{4.1}$$

Note that we do *not* necessarily require (4.1) for *all* $\nu = 1, 2, 3$. We will see that (4.1) improves the renormalized estimate of Lemma 1.1 considerably. With the additional assumption

$$\nabla u_\nu \in L^{2+2\varepsilon}\big(L^{2+2\varepsilon}_{\mathrm{loc}}\big) \tag{4.2}$$

we obtain

Theorem 4.1. *Suppose (u, p) is a weak solution of the Navier-Stokes system where $u \in L^\infty(L^2) \cap L^2(H^1)$ and for some $\nu \in \{1, 2, 3\}$ the component u_ν of u satisfies (4.1) and the additional $L^{2+2\varepsilon}$-inclusion (4.2) while (1.3) holds for the data f and u_0. Then, for all small $\delta'' > 0$, there holds*

$$\int_0^T \int_{\Omega_0} \frac{|\nabla u_\nu|^{3+3\delta}}{1 + |u_\nu|^2}\, dx\, dt \leqslant K_{\Omega_0}, \quad \textit{with } \delta = \frac{2}{3}\varepsilon - \delta'' \tag{4.3}$$

$$\int_0^T \int_{\Omega_0} \frac{|\nabla^2 u_\nu|^{(3+3\delta)/2}}{1 + |u_\nu|^{1/2-\varepsilon+\rho}}\, dx\, dt \leqslant K_{\Omega_0}, \quad |\rho| \textit{ arbitrarily small} \tag{4.4}$$

for all $\Omega_0 \subset\subset \Omega$, with some constant K_{Ω_0} depending in the data and the $L^{2+2\varepsilon}\big(L^{2+2\varepsilon}_{\mathrm{loc}}\big)$–norms in condition (4.2).

The proof of Theorem 4.1 is based on the two interpolation inequalities in Proposition 4.2 and a bootstrap argument. The following proposition will be applied later to the function

$$v = I_i^h u\,,$$

where the operator $I_i^h u$ is defined in Section 3. The general idea is: Estimate integrals of higher powers of $D_i v$ and $D_i^2 v$ by lower ones, but *pay* by dividing with powers of v.

Proposition 4.2. *Let $\Omega \subset \mathbb{R}^3$ open, $i \in \{1, 2, 3\}$, $\varepsilon \in]0, \frac{1}{2}[$ fixed. For $\alpha \geqslant 0$, $q \in]1, 2[$, set*

$$p = \frac{4q}{2q + 1 - 2\varepsilon}\,, \quad \beta = \frac{2 - p}{2}\alpha\,, \quad \gamma = p + \beta\,. \tag{4.5}$$

Let $v \geqslant k_0 > 0$ a.e. in $\Omega_0 \subset\subset \Omega$, $k_0 = k_0(\Omega_0)$, and assume $v \in L^p\left(L^p_{\mathrm{loc}}\right)$, $D_i v \in L^{2q}\left(L^{2q}_{\mathrm{loc}}\right)$, $D_i^2 v \in L^2$ and v fulfills the estimate

$$\int_0^T \int_{\Omega_0} |D_i v|^{2\varepsilon-1} |D_i^2 v|^2 \, dx \, dt \leqslant K_{\Omega_0} \,. \tag{4.6}$$

Then for any pair of compact subsets, $\Omega_1 \subset\subset \Omega_0 \subset\subset \Omega$ the following estimates hold:

$$\int_0^T \int_{\Omega_0} \frac{|D_i^2 v|^p}{v^\beta} \, dx \, dt \leqslant \left(\int_0^T \int_{\Omega_0} \frac{|D_i v|^{2q}}{v^\alpha} \, dx \, dt \right)^{2-p} + (K_{\Omega_0})^p, \tag{4.7}$$

$$\int_0^T \int_{\Omega_1} \frac{|D_i v|^{2p}}{v^\gamma} \, dx \, dt \leqslant C(\Omega_1, \Omega_0) \left\{ \left(\int_0^T \int_{\Omega_0} \frac{|D_i^2 v|^p}{v^\beta} \, dx \, dt \right)^{(2-p)/2} + 1 \right\}. \tag{4.8}$$

The constants K_{Ω_0}, $C(\Omega_1, \Omega_0)$ depend on the $L^p(L^p_{\mathrm{loc}})$-norms of v and the constant K_{Ω_0} in (4.6) but not *on the $L^{2p}(L^{2p}_{\mathrm{loc}})$-norms of ∇v.*

Proof. Observe that $(4.5)_1$ together with $\varepsilon \in [0, 1/2[$ implies $p < 2$, hence we may apply Hölder's inequality with exponents $\frac{2}{p}$, $\frac{2}{2-p}$ and obtain

$$\int_0^T \int_{\Omega_0} \frac{|D_i^2 v|^p}{v^\beta} \, dx \, dt \leqslant \left(\int_0^T \int_{\Omega_0} \frac{|D_i^2 v|^2}{|D_i v|^{1-2\varepsilon}} \, dx \, dt \right)^{\frac{p}{2}}$$
$$\times \left(\int_0^T \int_{\Omega_0} \frac{|D_i v|^{(1-2\varepsilon)p/(2-p)}}{v^{2\beta/(2-p)}} \, dx \, dt \right)^{\frac{(2-p)}{2}}$$

By $(4.5)_2$, we have $\alpha = 2\beta/(2-p)$, and, also by $(4.5)_1$, via a simple calculation $\frac{p}{2-p} = \frac{2q}{1-2\varepsilon}$. Hence (4.7) follows by Young's inequality.

To show inequality (4.8), we fix a non-negative cut-off function $\tau \in C_0^\infty(\Omega_0)$ with $\tau = 1$ on Ω_1. Then

$$\int_0^T \int_{\Omega_1} \frac{|D_i v|^{2p}}{v^\gamma} \, dx \, dt \leqslant \int_0^T \int_{\Omega_0} \tau^{2p} \frac{|D_i v|^{2p}}{v^\gamma} \, dx \, dt =: I$$

and it is enough to estimate the integral I on the right hand side. We use the notation $[\psi]^a = |\psi|^a \operatorname{sign} \psi$ for any function ψ and $a \neq 0$. From (4.5) we obtain

$\gamma > 1$ and partial integration leads to

$$I = \int_0^T \int_{\Omega_0} \tau^{2p} \frac{D_i v [D_i v]^{2p-1}}{v^\gamma} \, dx \, dt = (1-2p) \int_0^T \int_{\Omega_0} \tau^{2p} \frac{v}{v^\gamma} D_i^2 v [D_i v]^{2p-2} \, dx \, dt$$
$$+ \gamma \int_0^T \int_{\Omega_0} \tau^{2p} \frac{v}{v^{\gamma+1}} |D_i v|^{2p} \, dx \, dt + \text{ pollution term.}$$

Hence

$$I = \frac{2p-1}{\gamma-1} \int_0^T \int_{\Omega_0} \tau^{2p} v^{1-\gamma} D_i^2 v [D_i v]^{2p-2} \, dx \, dt + \frac{1}{1-\gamma} \text{ pollution term.} \tag{4.9}$$

The pollution term is

$$2p \int_0^T \int_{\Omega_0} \tau^{2p-1} D_i \tau \frac{v}{v^\gamma} [D_i v]^{2p-1} \, dx \, dt \tag{4.10}$$

and we apply Young's inequality pointwise on the right hand side of (4.9) and the integral (4.10). This yields

$$I \leqslant c \int_0^T \int_{\Omega_0} \frac{\tau^{2p}}{v^\gamma} \Big(\varepsilon |D_i v|^{2p} + c(\varepsilon) \big(v^p |D_i^2 v|^p + v^{2p} |D_i \tau|^{2p} \big) \Big) \, dx \, dt \, .$$

Hence

$$\begin{aligned} I &\leqslant c \int_0^T \int_{\Omega_0} \tau^{2p} \frac{1}{v^\gamma} \left\{ v^p |D_i^2 v|^p + v^p |D_i \tau|^p \right\} \, dx \, dt \\ &\leqslant c \int_0^T \int_{\Omega_0} \frac{|D_i^2 v|^p}{v^\beta} \, dx \, dt + \frac{c}{k_0^\gamma} \int_0^T \int_{\Omega_0} v^{2p} \, dx \, dt \end{aligned}$$

and we obtain (4.8). ■

We now use Proposition 4.2 to create an iteration process starting with $\alpha = \alpha_0 = 0$ and $q = q_0 (= 1 + \varepsilon) > 1$. We have seen: if $\int_0^T \int_{\Omega_0} v^{-\alpha} |D_i v|^{2q} \, dx \, dt$ can be estimated, then also $\int_0^T \int_{\Omega_0'} v^{-\gamma} |D_i v|^{2p} \, dx \, dt$, where $\Omega_0' \subset\subset \Omega_1$ and, by (4.5),

$$p = p(q) = \frac{4q}{2q + 1 - 2\varepsilon} \, , \qquad \gamma(q, \alpha) = \frac{1}{2} p(q) (2 - \alpha) + \alpha .$$

Lemma 4.3. *Let the functions $\varphi : [1,2] \to \mathbb{R}$ and $\gamma : [1,2] \times [0,2] \to \mathbb{R}$ be defined by*

$$\varphi(q) = \frac{4q}{2q+1-2\varepsilon}\,, \qquad \gamma(q,\alpha) = \frac{1}{2}\varphi(q)(2-\alpha) + \alpha\,.$$

Then the sequences $(q_n), (\alpha_n)$ defined by $q_{n+1} = \varphi(q_n)$, $\alpha_{n+1} = \gamma(q_n, \alpha_n)$ with starting values $q = q_0 \in [1, 3/2)$, $\alpha_0 = 0$ converge monotonously and

$$\lim_{n\to\infty} q_n = q^* = \frac{3}{2} + \varepsilon\,, \qquad \lim_{n\to\infty} \alpha_n = \alpha^* = 2\,. \tag{4.11}$$

For the sequence (β_n) with $\beta_n := \frac{1}{2}(2-q_n)\alpha_{n-1}$ it follows $\lim_{n\to\infty} \beta_n = \frac{1}{2} - \varepsilon$.

Proof. We consider the closed subset $A = [1,2] \times [0,2] \subset \mathbb{R}^2$ and the map Φ defined by

$$\Phi(q,\alpha) = \Big(\varphi(q), \frac{1}{2}\varphi(q)(2-\alpha) + \alpha\Big)\,.$$

Elementary calculations show: Φ maps A into A, in particular $\frac{4}{3-2\varepsilon} \leqslant \varphi(q) < 2$, hence $\alpha_{n+1} > \alpha_n$, and for the derivative φ' we get

$$0 < \varphi'(q) = \frac{4(1-2\varepsilon)}{(2q+1-2\varepsilon)^2} \leqslant \varphi'(1) = \frac{4(1-2\varepsilon)}{(3-2\varepsilon)^2} < \frac{2}{3}$$

for all $q \geqslant 1$, which implies also $q_{n+1} > q_n$.
Since

$$D\Phi(q,\alpha) = \begin{pmatrix} \varphi'(q) & 0 \\ \frac{1}{2}\varphi'(q)(2-a) & 1 - \frac{1}{2}\varphi(q) \end{pmatrix},$$

it is easy to see that

$$|\Phi(q,\alpha) - \Phi(\bar{q},\bar{\alpha})| \leqslant |L|\,|(q,\alpha) - (\bar{q},\bar{\alpha})|$$

with

$$L = \max\left\{\frac{2}{3}, 1 - \frac{1}{2}\cdot\frac{4}{3-2\varepsilon}\right\} < 1\,.$$

Thus (q_n, α_n) converges to the uniquely determined fix point (q^*, α^*) of the mapping Φ which is given by (4.11). The last assertion is obvious since $\beta_n = \frac{\alpha_{n-1}}{2}(2 - \varphi(q_n))$. ■

Proof of Theorem 4.1. We apply Proposition 4.2 with $v = I_i^h u_\nu$ where I_i^h has been defined in Section 3. Since we know already $u \in L^{10/3}(L^{10/3})$, we have $D_i v = D_i^h u_\nu \in L^{10/3}(L^{10/3})$ for fixed h. Hence the condition $D_i v \in L^{2p}\big(L^{2p}_{\mathrm{loc}}\big)$ is satisfied for $p \leqslant 3/2+\varepsilon$ as long as $\varepsilon < 1/6$. If (4.2) holds for u_ν, then $u_\nu \in L^{10/3+\varepsilon}(L^{10/3+\varepsilon})$ and no restriction for ε is needed. Since $u \in L^2(H^1_{\mathrm{loc}})$, we have also $D_i^2 v \in L^2(L^2)$.

To v defined as above, we apply Proposition 4.2 repeatedly, starting with $\alpha_0 = 0$, $q_0 = 1 + \varepsilon$. By Lemma 4.3, this leads, after a *finite* number of steps to iterated values

$$\begin{aligned} q_n &= \tfrac{3}{2} + \varepsilon - \delta' > \tfrac{3}{2} + \delta(\varepsilon), & 0 < \delta' << \varepsilon, \\ \alpha_n &= 2 - \delta'', & 0 < \delta'' << \varepsilon, \\ p_n &= \tfrac{3}{2} + \varepsilon - \delta''', & 0 < \delta''' << \varepsilon. \end{aligned}$$

Observe that the domains of integration, Ω_1, Ω_0, shrink with each step, but since we iterate only a finite time, this does not matter. (Alternatively, one could have worked with localization functions $\tau^{s(n)}$ instead of τ^2 in the integrals of Proposition 4.2.) Finally we may pass to the limit $h \to 0$ as in the proof of Lemma 3.1.

Thus we conclude from Proposition 4.2 and Lemma 4.3 that

$$\int_0^T \int_{\Omega'} \frac{|D_i u_\nu|^{2q_n}}{(u_\nu + k_0)^{\alpha_n}}\, dx\, dt + \int_0^T \int_{\Omega'} \frac{|D_i^2 u_\nu|^{q_n}}{(u_\nu + k_0)^{\beta_n}}\, dx\, dt \leqslant K_{\Omega'}, \quad \Omega' \subset\subset \Omega \tag{4.12}$$

where

$$\beta_n = \frac{1}{2} - \varepsilon - \rho, \quad 0 < |\rho| << \varepsilon\,.$$

To express the exponents α_n and β_n in terms of $\frac{1}{2} - \varepsilon$ in (4.12) is nasty. However, in (4.12) we may change α_n into 2 and β_n into $\frac{1}{2} - \varepsilon$. This is admissible if we alter the constant in the first inequality. The same procedure works for the second inequality if $\rho > 0$; if $\rho < 0$, we may use Hölder's inequality. One chooses δ_1 such that $(\frac{1}{2} - \varepsilon) = \frac{1}{2} - \varepsilon + |\rho|$ and changes δ' such that $q_n/(1 + \delta_1) = \frac{3}{2} + \varepsilon - \delta'$. This proves the theorem. ■

Remark. Without the assumption (3.1) similar arguments as above would lead to the estimates

$$\int\int \frac{|Du_\nu|^{3-\delta}}{(k_\nu + u_\nu)^{2+\delta'}} dx\, dt + \int\int \frac{|D^2 u_\nu|^{(3-3\delta)/2}}{(k_\nu + u_\nu)^{1/2+\delta'}} \leqslant K$$

where $\delta, \delta' > 0$ are arbitrarily small.

5. Estimates for the Renormalized Pressure

In this section we deal with the renormalized pressure

$$z = p(1 + |u|^2)^{-q/2}. \tag{5.1}$$

Let us assume that u satisfies the renormalized inclusions for some $q \in (0, 2]$ and some $\delta > 0$

$$|\nabla u|^2 (1 + |u|^2)^{-q/2} \in L^{(1+\delta)3/2}\left(L_{\text{loc}}^{(1+\delta)3/2}\right), \tag{5.2}$$

$$|\nabla^2 u|(1 + |u|^2)^{-1/2} \in L^{(1+\delta)3/2}\left(L_{\text{loc}}^{(1+\delta)3/2}\right). \tag{5.3}$$

In fact, if the components of u are semibounded, we have (5.2), (5.3) with $q = \frac{4}{3(1+\delta)}$ by Theorem 4.1. From (1.5) we obtain by elementary calculations that p fulfills, a.e.,

$$-\Delta\left(\frac{p}{(1+|u|^2)^{q/2}}\right) = \frac{1}{(1+|u|^2)^{q/2}}(-\Delta p - \tilde{g}_1\nabla p - \tilde{g}_2 p + \tilde{g}_0)\,, \tag{5.4}$$

where $\tilde{g}_0, \tilde{g}_1, \tilde{g}_2$ are measurable functions such that

$$|\tilde{g}_1| \leqslant K|\nabla u|(1+|u|^2)^{-1/2}, \tag{5.5}$$
$$|\tilde{g}_2| \leqslant K|\nabla u|^2(1+|u|^2)^{-1} + K|\nabla^2 u|(1+|u|^2)^{-1/2}\,. \tag{5.6}$$

From (5.5) and (5.6) and the assumptions (5.2) and (5.3) we get the estimates

$$\int_0^T\int_{\Omega_0} |\tilde{g}_1|^{3+3\delta}\,dx\,dt \leqslant K_{\Omega_0}\,, \qquad \int_0^T\int_{\Omega_0} |\tilde{g}_2|^{3(1+\delta)/2}\,dx\,dt \leqslant K_{\Omega_0},$$
$$\int_0^T\int_{\Omega_0} |\nabla\tilde{g}_1|^{(3+3\delta)/2}dx\,dt \leqslant K_{\Omega_0}, \qquad \int_0^T\int_{\Omega_0} |\tilde{g}_0|^{3(1+\delta)/2}\,dx\,dt \leqslant K_{\Omega_0}.$$

With z as in (5.1) and

$$g_0 = \frac{\tilde{g}_0}{(1+|u|^2)} - \frac{\Delta p}{(1+|u|^2)^{q/2}},$$

equation (5.4) can be written as:

$$-\Delta z = g_1\cdot\nabla z + g_2 z + g_0, \tag{5.7}$$

where

$$g_1 \in L^{3+3\delta}, \quad g_0, g_2 \in L^{(3+3\delta)/2}, \quad \nabla g_1 \in L^{(3+3\delta)/2} \tag{5.8}$$

with some $\delta > 0$ and almost every t. A classical bootstrap argument leads to the interior regularity of $W^{1,1}_{\rm loc}$-solutions to equation (5.7). If $z \in W^{1,1}_{\rm loc}$ then $\nabla z \in L^{3/2}_{\rm loc}$, $z \in L^3_{\rm loc}$, hence Hölder's inequality and interior elliptic regularity lead to $z \in W^{2,q_1}_{\rm loc}$ with $1/q_1 = 2/3 + 1/(3+3\delta)$. Repeating this arguments m times gives

$$\|\Delta z\|_{q_m} \leqslant \|g_1\|_{3+3\delta}\|\nabla z\|_{p_{m-1}} + K_\Omega\left(\|g_2\|_{(3+3\delta)/2}\|z\|_{r_{m-1}} + \|g_0\|_{q_m}\right),$$

$$\|z\|_{r_m} + \|\nabla z\|_{p_m} + \|\nabla^2 z\|_{q_m} \leqslant K(\|\Delta z\|_{q_m} + \|z\|_{q_m})$$

for

$$\frac{1}{q_m} = \frac{1}{p_{m-1}} + \frac{1}{3+3\delta}, \quad \frac{1}{p_m} = \frac{1}{q_m} - \frac{1}{3}, \quad \frac{1}{r_m} = \frac{1}{p_m} - \frac{1}{3},$$

where $\|\cdot\|_p$ stands for local L^p-norms and we end up with $z \in W^{2,(3+3\delta)/2}$ and estimates in terms of z, g_0, g_1 and g_2 when m is the smallest integer greater or equal to $3+1/\delta$. Clearly the a.e. regularity itself is known for z defined by (5.1).

Unfortunately these reasonings do not lead to better integrability properties in time of the normalized pressure z or its spatial derivatives, unless we assume additional properties of z or the coefficient functions. Since p behaves approximately like u^2, it might look reasonable to assume $p(1+u^2)^{-q/2} \in L^r(L^s_{\mathrm{loc}})$ for large r, s, provided q is large. In this case we can clearly derive from the equation for the renormalized pressure that

$$\nabla^2 z \in L^{3/2+\delta}\big(L^{3/2+\delta}_{\mathrm{loc}}\big), \qquad \nabla z \in L^{3+\delta}\big(L^{3+\delta}_{\mathrm{loc}}\big).$$

Obviously these presumptions are too strong – in the renormalized Navier-Stokes equation one would expect C^α-regularity already under Morrey conditions (see (6.3) and (6.5)). Another interesting alternative obtaining new pressure estimates relies on the fact, that the solution φ of the equation

$$-\Delta\varphi = p \tag{5.9}$$

is locally semibounded from above. This leads to higher integrability of the negative part p_-, provided one divides by some power of $K-\varphi$.
The fact that the solution of (5.9) is semibounded follows from certain weighted integral identities used in [8], which are briefly presented.

Lemma 5.1. *Let u, p satisfy the the pressure equation* (1.5) *and the regularity properties* (1.4)*, and assume $p \in L^\infty(L^1)$ in addition. Then, for every $\tau \in C_0^\infty(\Omega)$*

$$\int \tau^2 p \, \frac{1}{|x-x_0|} \, dx \leqslant K$$
$$\int \tau^2 \Big(p + \frac{u^2}{2}\Big) \frac{1}{|x-x_0|} \, dx \geqslant -K\,.$$

From the representation formula of potential theory we obtain from Lemma 5.1

Lemma 5.2. *Let $\varphi, \psi \in H_0^1(\Omega_0)$, $\Omega_0 \subset\subset \Omega$ the solution of $-\Delta\varphi = p$, $-\Delta\psi = \frac{1}{2}u^2 + p$, respectively. Under the hypothesis of Lemma 5.9 there exists a constant $K = K(\Omega_{00})$ such that*

$$\varphi \leqslant K \quad \textit{and} \quad \psi \geqslant -K$$

on $\Omega_{00} \subset \Omega_0$.

Proof of Lemma 5.1. Since $\nabla^2 p \in L^1(L^1_{\mathrm{loc}})$ and $p \in L^{5/3}\big(L^{3/2}_{\mathrm{loc}}\big)$, we have $\nabla^2 p(t) \in L^1_{\mathrm{loc}}$ and $p(t) \in L^{2/3}_{\mathrm{loc}}$ for almost all t. In the equation

$$\Delta p(t) = -\sum_{j,k=1}^{3} D_j u_k(t) D_k u_j(t) + \operatorname{div} f(t) \tag{5.10}$$

we use the function $|x-x_0|\tau^2$ as a test function where $\tau \in C_0^2$, $\tau \geqslant 0$, $\tau = 1$ in $u(x_0)$. This yields

$$\int p(t)\big[\tau^2\Delta|x-x_0| + 2\nabla|x-x_0|\nabla\tau^2 + |x-x_0|\Delta\tau\big]\,dx \tag{5.11}$$
$$= \int -\sum_{j,k=1}^{3} u_k u_j \big[D_j D_k |x-x_0|^2\tau^2 - 2D_k|x-x_0|D_j\tau^2$$
$$- |x-x_0|D_j D_k\tau^2\,dx\big] + \text{pollution coming from } f\,.$$

With the obvious identities

$$\Delta|x-x_0| = \frac{2}{|x-x_0|}$$
$$\sum_{j,k=1}^{3} u_j u_k D_j D_k|x-x_0| = \frac{u^2}{|x-x_0|} - \frac{(u\cdot(x-x_0))^2}{|x-x_0|^3},$$

the inclusions $u \in L^\infty(L^2_{\mathrm{loc}})$ and $p \in L^\infty(L^1_{\mathrm{loc}})$, and the fact that $D_k|x-x_0|D_j\tau^2$ and $|x-x_0|D_jD_k\tau^2 \in L^\infty$, the identity (5.11) implies

$$\left|\int_\Omega \tau^2\left[\frac{1}{|x-x_0|}(2p+u^2) - \frac{(u\cdot(x-x_0))^2}{|x-x_0|^3}\right]\right| \leqslant K \tag{5.12}$$

uniformly in t and x_0, $K = K_\tau$. Since $u^2 - \frac{(u\cdot(x-x_0))^2}{|x-x_0|^2} \geqslant 0$ we obtain from (5.12)

$$\int_\Omega \tau^2 \frac{p}{|x-x_0|}\,dx \leqslant K,$$

and, by using $u \in L^\infty(L^2_{\mathrm{loc}})$ again,

$$\int_\Omega \tau^2\frac{1}{|x-x_0|}\Big(p+\frac{1}{2}u^2\Big)\,dx \geqslant -K$$

uniformly in t and x_0, where $K = K_\tau$. ■

Since φ is semibounded we obtain via a similar Bernis-type argument as in Section 4:

Lemma 5.3. *Let u, p comply with the requirements of Lemma 5.1. Let φ be a local solution of $-\Delta\varphi = p$ such that $\varphi - k_0 \leqslant -1$, where $p \in L^\infty(L^1_{\mathrm{loc}})$ satisfies the pressure equation* (5.10). *Then*

$$\int_0^T\int_{\Omega_0} \frac{|\nabla\varphi|^{10/3}}{(\varphi-k_0)^{5/3}}\,dx\,dt \leqslant K_{\Omega_0}.$$

Theorem 5.4. *If for u, p, φ as above the inclusion $|\nabla u|^2 \ln(1-p_-) \in L^1(L^1_{\text{loc}})$ is valid then there holds*

$$\int_0^T \int_{\Omega_0} \left[\frac{|\nabla \varphi|^2}{(\varphi - k_0)^2} |p_-| + \frac{p_-^2}{|\varphi - k_0|} \right] dx\, dt \leqslant K_{\Omega_0} \tag{5.13}$$

with some constant K_{Ω_0} depending on the data and $\| |\nabla u|^2 \ln(1-p_-)) \|_{L^1(L^1)}$, $\|p\|_{L^\infty(L^1)}$ and Ω_0. Under the additional assumption $\nabla u \in L^{2+2\varepsilon}(L^{2+2\varepsilon})$ the estimate (5.13) *can be refined to*

$$\int_0^T \int_{\Omega_0} \left[\frac{|\nabla \varphi|^2}{(\varphi - k_0)^2} |p_-|^\beta + \frac{|p_-|^{1+\beta}}{|\varphi - k_0|} \right] dx\, dt \leqslant K_{\Omega_0}$$

with $\beta = 1 + (5\varepsilon)/3$.

Proof. We argue somewhat formally by partial integration, however, the proof can be done rigorously working with the mollification $\omega * (\varphi - k_0)$ rather than $\varphi - k_0$. We denote by $p_- = \min\{p, 0\}$, $p_+ = \max\{p, 0\}$ the negative and positive part of p, respectively, then we have (with some localization function τ)

$$\begin{aligned} \iint \tau^2 \frac{|\nabla \varphi|^2}{(\varphi - k_0)^2} p_-\, dx\, dt = &- \iint \tau^2 \frac{(\varphi - k_0)\Delta\varphi}{(\varphi - k_0)^2} p_-\, dxdt \\ &+ 2 \iint \tau^2 \frac{(\varphi - k_0)|\nabla \varphi|^2}{(\varphi - k_0)^3} p_-\, dx\, dt \\ &- \iint \tau^2 \frac{1}{(\varphi - k_0)} \nabla\varphi \nabla p_- dx\, dt + \text{ pollution terms.} \end{aligned}$$

Using equation (5.9) and pointwise Young's inequality we obtain

$$\begin{aligned} - \iint \tau^2 \frac{|\nabla \varphi|^2}{(\varphi - k_0)^2} p_-\, dx\, dt - \iint \tau^2 \frac{p_-^2}{(\varphi - k_0)} dx\, dt \qquad& \\ \leqslant \frac{1}{2} \iint \tau^2 \frac{|\nabla \varphi|^2}{(\varphi - k_0)^2} (1 - p_-) dx\, dt& \\ + \frac{1}{2} \iint \tau^2 \frac{|\nabla p_-|^2}{(1 - p_-)} dx\, dt + \text{ pollution term}\,.& \end{aligned} \tag{5.14}$$

Observe that both summands on the left hand side of (5.14) are nonnegative. The first summand on the right hand side splits into a term which can be absorbed by the left hand side and a bounded integral due to the assumptions on φ, the pollution term is bounded due to the assumptions on p and Lemma 5.12. To show that $\iint \tau^2 \frac{|\nabla p_-|^2}{(1-p_-)}\, dx\, dt$ is finite we test the pressure equation with $\ln(1-p_-)\tau^2$.

If $\nabla u \in L^{2+2\varepsilon}(L^{2+2\varepsilon}_{\text{loc}})$, we have $\nabla^2 p \in L^{1+\varepsilon}(L^{1+\varepsilon}_{\text{loc}})$, which implies that $p \in L^{(3+3\varepsilon)/(1-2\varepsilon)}(L^{1+\varepsilon}_{\text{loc}})$ by Sobolev embedding. Since we have also $p \in L^\infty(L^1_{\text{loc}})$, the classical Ladyzenskaya–Uralcewa interpolation argument leads to $p \in L^{5(1+\varepsilon)/3}(L^{5(1+\varepsilon)/3}_{\mathrm{loc}})$. Then we may use $-(1-p_-)^\alpha \tau^2$, $\alpha = 5\varepsilon/3$, as a test function in the pressure equation (5.9) and we obtain

$$\iint \tau^2 \frac{|\nabla p_-|^2}{(1-p_-)^{1-\alpha}} dx\, dt \leqslant K. \tag{5.15}$$

Hence, we can modify the proof by starting with the term

$$\iint \tau^2 \frac{|\nabla \varphi|^2}{(\varphi - k_0)^2} |p_-|^\beta \, dx\, dt, \qquad \beta = 1 + \frac{5\varepsilon}{3},$$

and end up with an estimate

$$\int_0^T \int_{\Omega_0} \left[\frac{|\nabla \varphi|^2}{(\varphi - k_0)^2} |p_-|^\beta + \frac{|p_-|^{1+\beta}}{|\varphi - k_0|} \right] dx\, dt$$
$$\leqslant K \iint \tau^2 \frac{|\nabla p_-|^2}{(1-p_-)^{1-\alpha}} dx\, dt + \text{ poll. terms}$$

where we can use (5.15). ■

6. The Renormalized Navier-Stokes Equations

We fix $\frac{1}{2} \leqslant \alpha_i \leqslant 1, i = 1, \ldots, 3$, and define

$$|w|^2 = \sum_{i=1}^{3} \alpha_i |w_i|^2 \qquad \text{for } w \in \mathbb{R}^3.$$

Let $c \in \mathbb{R}^3$ be given, we choose $\omega_i = \alpha_i(u_i - c_i)(1 + |u - c|^2)^{-q/2}$. We test the Navier-Stokes equation with the function $\omega\varphi = (\omega_1, \omega_2, \omega_3)\varphi$, where $\varphi \in C^1\big[0, T; C_0^\infty(\Omega)\big]$, $\varphi|_T = 0$. Then it is easy to see that the function

$$y = \frac{1}{2-q}(1 + |u - c|^2)^{1-q/2}$$

satisfies the equation

$$(y_t, \varphi) + \mu(\nabla y, \nabla \varphi) = (F_0, \varphi) + (G_0, \varphi) + (uy, \nabla \varphi) + (f_0, \varphi) \tag{6.1}$$

where

$$F_0 = \mu \sum_{i=1}^{3} \frac{\alpha_i |\nabla u_i|^2}{(1 + |u - c|^2)^{q/2}} - \frac{q\mu}{4} \frac{|\nabla(|u - c|^2)|^2}{(1 + |u - c|)^{q/2+1}},$$

$$f_0 = \sum_{i=1}^{3} f_i \, \frac{\alpha_i (u_i - c_i)}{(1 + |u - c|^2)^{q/2}},$$

$$G_0 = -\sum_{i=1}^{3} D_i p \, \frac{\alpha_i (u_i - c_i)}{(1 + |u - c|^2)^{q/2}};$$

hence

$$(G_0, \varphi) = -\int \sum_{i=1}^{3} \alpha_i D_i \left(\frac{p(u_i - c_i)}{(1 + |u - c|^2)^{q/2}} \right) \varphi \, dx$$
$$+ \int \sum_{i=1}^{3} \alpha_i p D_i \left(\frac{(u_i - c_i)}{(1 + |u - c|^2)^{q/2}} \right) \varphi \, dx.$$

We may consider (6.1) as some type of renormalized Navier-Stokes equation. The term G_0 can be split into two terms $G_0 = \operatorname{div} G_1 + G_2$, where

$$|G_1| \leqslant \frac{|p|}{(1 + |u|^2)^{(q-1)/2}}, \quad |G_2| \leqslant \frac{|p||\nabla u|}{(1 + |u|^2)^{q/2}}.$$

Thus we have to discuss the parabolic equation

$$(y_t, \varphi) + \mu(\nabla y, \nabla \varphi) = -(G_1, \nabla \varphi) + (G_2, \varphi) + (uy, \nabla \varphi) + (F_0, \varphi) + (f_0, \varphi). \quad (6.2)$$

The theory of parabolic equations implies $y \in C^\alpha((0, t) \times \Omega)$ provided F_0, G_1, G_2 satisfy the Morrey conditions

$$\sup_{Q_R} \iint_{Q_R} |G_1| dx \, dt \leqslant K R^{4+\delta}, \quad (6.3)$$

$$\sup_{Q_R} \iint_{Q_R} |F_0| dx \, dt \leqslant K R^{3+\delta}, \quad (6.4)$$

$$\sup_{Q_R} \iint_{Q_R} |G_2| dx \, dt \leqslant K R^{3+\delta}, \quad (6.5)$$

where $Q_R \subset \Omega_0 \times [0, T]$, $\Omega_0 \subset\subset \Omega$. Since uy is bounded for $q \geqslant 2$ also the convective term satisfies (6.3). The property $y \in C^\alpha$ *would be the final aim to obtain for the solution in the renormalized setting,* however the conditions (6.3), (6.5) are too strong. For G_1 one might agree with Da Veiga's reasoning: As already mentioned, p, roughly speaking, behaves like u^2, hence a condition of the type $\frac{p}{(1+u^2)^{(q-1)/2}} \in L^{5+\delta}(L^{1+\delta}_{\text{loc}})$, q large, or weaker,

$$\iint_{Q_R} \frac{p}{(1 + u^2)^{(q-1)/2}} \leqslant K R^{4+\delta} \quad (6.6)$$

might be interesting. Condition (6.6) implies (6.3), but there are still (6.4) and (6.5). For a discrete set of constants α and c, a system of type (6.2) can be considered as a diagonal parabolic system with a "lower order" term which is quadratic in the gradients of the unknown functions, thereby the functions $y(\alpha_i, c)$ can be understood as normalized velocities. This follows since $u, \nabla u$ can be expressed by $y(\alpha_i, c), \nabla y(\alpha_i, c)$ and vice versa if sufficiently many α_i, c are used. However, since there occurs a degenerate factor depending on y, solutions to systems of this type may have singularities.

Thus we suggest to study a *scalar* equation which can be established in the case of semibounded velocities. We assume that the components u_ν of the velocity are locally semibounded. For each x, we can find an invertible matrix $\Lambda = (\lambda_{j\nu})_{j,\nu=1,2,3}$, such that on a suitable neighborhood of x the functions

$$U_j = \sum_{\nu=1}^{3} \lambda_{j\nu}(u_\nu + k_\nu) + 1\,, \qquad j = 1,2,3 \tag{6.7}$$

fulfill

$$U_j \geqslant 1, \quad k_0 + c|u| \leqslant U_j \leqslant K|u| + K, \qquad j = 1,2,3 \tag{6.8}$$

with local constants $c, K, k_0 > 0$.
By linearity, the function U_j satisfies the equation

$$(U_j)_t - \mu\Delta U_j + (u\cdot\nabla)U_j = -(\Lambda\nabla p)_j + (\Lambda f)_j \tag{6.9}$$

in the weak sense and a.e..

We divide (6.9) by U_j^q and see that the function $v = U_j^{1-q}$ satisfies the equation

$$\begin{aligned} v_t - \mu\Delta v + \frac{\mu q}{1-q}\frac{|\nabla v|^2}{v} + u\nabla v \\ = (1-q)(\Lambda\nabla)_j\left(\frac{p}{U_j^q}\right) + q(1-q)\frac{p}{U_j}\frac{(\Lambda\nabla)_j U_j}{U_j^q} + \frac{(1-q)(\Lambda f)_j}{U_j^q} \end{aligned} \tag{6.10}$$

which we consider as "renormalized Navier-Stokes equation".

In the case of the scalar renormalized Navier-Stokes equation we can prove a Morrey condition for the function $|\nabla u|^2(1+|u|^2)^{-q/2}$, provided that the renormalized pressure (5.1) satisfies the Morrey condition (6.6) for some $\delta \geqslant 0$.

Theorem 6.1. *Let (u,p) be a suitable solution of the Navier Stokes system, u semibounded and assume that the Morrey condition*

$$\iint_{Q_R} \frac{p^2}{(1+|u|^2)^{(q+1)/2}}dx\,dt \leqslant KR^3 \tag{6.11}$$

holds for some $q \geqslant 2$, then the velocity field fulfills the following renormalized Morrey condition

$$\iint_{Q_R} \frac{|\nabla u|^2}{(1+|u|^2)^{q/2}}\,dx\,dt \leqslant K_{\Omega_0}R^3 \tag{6.12}$$

for all parabolic cylinders $Q_R = [t_0 - R^2, t_0] \times B_R(x_0)$ with $B_R(x_0) \subset \Omega_0 \subset\subset \Omega$.

Proof. Let τ be a smooth nonnegative localization function with the property $\tau = 1$ on $[T_0 - R^2, T_0] \times B_R(x_0)$, $\operatorname{supp} \tau \subset [T_0 - 2R^2, T_0 + R^2] \times B_{2R}(x_0)$ and

$$|\nabla \tau| \leqslant KR^{-1}, \quad |\nabla^2 \tau| \leqslant KR^{-2}, \quad |\tau_t| \in KR^{-2}. \tag{6.13}$$

We test the renormalized Navier-Stokes equation (6.10) with this function τ. The leading term to be estimated is $\iint |\nabla v|^2 v^{-1} \tau dx\, dt$. Since v is bounded, $|Q_{2R}| = KR^5$, and (6.13) holds for τ we obtain

$$\left| \iint v_t \tau \; dx\, dt \right| = \left| \iint v \tau_t \; dx\, dt \right| \leqslant KR^3$$
$$\left| \iint \Delta v \tau \; dx\, dt \right| = \left| \iint v \Delta \tau \; dx\, dt \right| \leqslant KR^3\,.$$

Then the inclusion $u \in L^{10/3}(L^{10/3})$ leads to

$$\left| \iint (u \cdot \nabla) v\, \tau dx\, dt \right| = \left| \iint uv \nabla \tau \; dx\, dt \right| \leqslant KR^3$$

The term with f is obviously bounded by KR^5 since $f \in L^\infty(L^\infty)$. Furthermore,

$$\left| \iint (\Lambda \nabla)_j \left(\frac{p}{U_j^q} \right) \tau \; dx\, dt \right| = \left| \iint \frac{p}{U_j^q} (\Lambda \nabla)_j \tau \; dx\, dt \right| \leqslant KR^3$$

since $\nabla \tau \sim R^{-1}$ and (6.11) holds. Finally we estimate

$$\left| \iint \frac{p}{U_j^q} \frac{(\Lambda \nabla)_j U_j}{U_j} \tau \; dx\, dy \right| \leqslant \varepsilon \iint \frac{|\nabla U_j|^2}{U_j^{q+1}} \tau dx\, dt + K_\varepsilon \iint \frac{p^2}{U_j^{q+1}} \tau \; dx\, dt.$$

The first term on the right-hand side is the leading term again use the assumption (6.11). This gives (6.12) for U_j while summing over $j = 1, 2, 3$ ∎

Remark. From Theorem 6.1 and (6.11) we obtain that the term

$$\Pi_0 = \frac{p\, (\Lambda \nabla)_j U_j}{U_j^{q+1}}$$

satisfies the Morrey-condition

$$\iint_{Q_R} |\Pi_0| \; dx\, dt \leqslant KR^{3+\delta/2}$$

provided $\iint_{Q_R} p^2 U_j^{-(q+1)} \, dx\, dt \leqslant KR^{3+\delta}$. Thus the slightly improved Morrey condition on $p^2 U_j^{-(q+1)}$ leads to the supercritical case where one can expect C^α-regularity for the solution to the parabolic equation (6.10).

We note that the results of Theorem 6.1 can be refined by replacing the test function τ by $\tau \Gamma$ in the proof, where Γ is the fundamental solution to the backward heat equation.

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