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CORRIGENDUM TO "REFINEMENTS OF GOLDBACH'S CONJECTURE, AND THE GENERALIZED RIEMANN HYPOTHESIS"

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Abstract: Karin Halupczok [4] pointed out that we have stated an estimate in [3] that does not follow as easily as claimed. Although we are unable to obtain the claimed estimate, we prove a good enough estimate to (mostly) recover the theorems claimed in [3].
Keywords: Goldbach conjecture, explicit formula for the number of primes.

An explicit version of the prime number theorem states that if x is an integer and $1 \leq T \leq x$ then

$$\sum_{p\leq x} \log p = x - \sum_{\substack{\rho \\ |\mathrm{Im}\rho|\leq T}} \frac{x^\rho}{\rho} + O\left(\frac{x(\log x)^2}{T}\right), \tag{1}$$

where the sum is over zeros ρ of $\zeta(\rho) = 0$ with $\mathrm{Re}(\rho) > 0$. Let $B = \sup\{\mathrm{Re}\,\rho\colon \zeta(\rho) = 0\}$ (note that $1 \geq B \geq 1/2$). We claimed [3, (5.1)] that by partial summation with $T = x$ it is not hard to show that

$$\sum_{2N\leq x} G(2N) = \sum_{p+q\leq x} \log p \log q = \frac{x^2}{2} - 2 \sum_{\substack{\rho \\ |\mathrm{Im}\rho|\leq x}} \frac{x^{1+\rho}}{\rho(1+\rho)} + O(x^{2B+o(1)});$$

however we have not been able to repeat the argument and Karen Halupczok [4] pointed out the references [1,2] where this issue has been investigated in some detail for $B = \frac{1}{2}$, and nothing so strong has been proved. Here we sketch a simple argument to prove that

$$\sum_{2N\leq x} G(2N) = \sum_{p+q\leq x} \log p \log q = \frac{x^2}{2} - 2 \sum_{\substack{\rho \\ |\mathrm{Im}\rho|\leq x}} \frac{x^{1+\rho}}{\rho(1+\rho)} + O(x^{\frac{2+4B}{3}}(\log x)^2). \tag{2}$$

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Thanks are due to Karin Halupczok for finding the mistake and for discussing this correction.

Note that $2B < \frac{2+4B}{3} < 1+B$ as $B<1$, so our error term is not quite as good as was claimed in [3], but it is comfortably strong enough to recover Theorem 1A of [3]. Using a zero-density estimate one can improve the error term to $\ll x^{2B}(\log x)^{O(1)}$, as claimed, when $B \geq \frac{3}{4}$, and to an exponent between $2B$ and $\frac{2+4B}{3}$ when $\frac{1}{2} \leq B \leq \frac{3}{4}$.

This mistake is repeated in all four parts of Theorem 1 in [3], so corrections are needed throughout: Replacing $2B$ by $\frac{2+4B}{3}$ on the fourth line of page 171 allows us to recover Theorem 1B. Similarly replacing $2B$ by $\frac{2+4B}{3}$ in (1.3) allows us to recover Theorem 1D. There is a mistake in the proof of Theorem 1C, two lines above (5.5), where complex variables ρ and σ are treated as if they are real variables. If a similar correction is made there we do not quite recover Theorem 1C. Instead we can prove that if (1.2) holds then the Riemann Hypothesis for Dirichlet L-functions mod q holds; and if the Riemann Hypothesis for Dirichlet L-functions mod q holds then we obtain (1.2) with error term $O(x^{\frac{4}{3}}(\log x)^2)$.

Sketch of proof of (2). What does follow from (1) by partial summation (and noting that $\sum_{T<|\mathrm{Im}\rho|\leq x} |x^{1+\rho}/\rho(1+\rho)| \ll x^2 \log T/T$), is

$$\sum_{p+q\leq x} \log p \log q = \frac{x^2}{2} - 2\sum_{\substack{\rho \\ |\mathrm{Im}\rho|\leq x}} \frac{x^{1+\rho}}{\rho(1+\rho)} + \sum_{\substack{\rho,\rho' \\ |\mathrm{Im}\rho|,|\mathrm{Im}\rho'|\leq T}} \frac{\Gamma(\rho)\Gamma(\rho')}{\Gamma(\rho+\rho')}\cdot\frac{x^{\rho+\rho'}}{\rho+\rho'} + O\left(\frac{x^2(\log x)^2}{T}\right).$$

Stirling's formula implies that $|e^\rho\Gamma(\rho)| \asymp |\rho^{\rho-1/2}| = |\rho|^{\mathrm{Re}(\rho)-\frac{1}{2}}e^{-\arg(\rho)\mathrm{Im}(\rho)}$ so that if $\rho=\beta+i\gamma$ with $\beta\in(0,1)$ and $|\gamma|\gg 1$ then $|e^\rho\Gamma(\rho)| \asymp |\gamma|^{\beta-\frac{1}{2}}e^{-\frac{\pi}{2}|\gamma|}$, since $\arg(\rho) = \pm(\frac{\pi}{2}+O(\frac{1}{|\gamma|}))$ when $\mathrm{Im}(\rho)=\pm|\gamma|$. Let $\rho'=\beta'+i\gamma'$ with $|\gamma|\geq|\gamma'|$. Therefore if γ and γ' have the same sign then

$$\Gamma(\rho)\Gamma(\rho')/(\rho+\rho')\Gamma(\rho+\rho') \asymp |\gamma|^{\beta-\frac{1}{2}}|\gamma'|^{\beta'-\frac{1}{2}}/|\gamma+\gamma'|^{\beta+\beta'+\frac{1}{2}} \asymp |\gamma'|^{\beta'-\frac{1}{2}}/|\gamma|^{\beta'+1}.$$

If γ and γ' have opposite signs then

$$\begin{aligned}\Gamma(\rho)\Gamma(\rho')/(\rho+\rho')\Gamma(\rho+\rho') &\asymp |\gamma|^{\beta-\frac{1}{2}}|\gamma'|^{\beta'-\frac{1}{2}}e^{-\pi|\gamma'|}/(1+|\gamma+\gamma'|)^{\beta+\beta'+\frac{1}{2}} \\ &\ll |\gamma'|^{\beta'-\frac{1}{2}}/|\gamma|^{\beta'+1},\end{aligned}$$

since $|\gamma|^{\beta+\beta'+\frac{1}{2}}e^{-\pi|\gamma'|} \ll (1+|\gamma+\gamma'|)^{\beta+\beta'+\frac{1}{2}}$. We have $(|\gamma'|/|\gamma|)^{\beta'} \leq 1$ which implies that $|\gamma'|^{\beta'-\frac{1}{2}}/|\gamma|^{\beta'+1} \leq 1/|\gamma'|^{\frac{1}{2}}|\gamma|$. Hence the final sum in the last displayed equation is $\ll x^{2B}\sum_{|\gamma'|\leq|\gamma|\leq T} 1/|\gamma'|^{\frac{1}{2}}|\gamma| \ll x^{2B}\sum_{|\gamma|\leq T}(\log|\gamma|)/|\gamma|^{\frac{1}{2}} \ll x^{2B}T^{1/2}(\log T)^2$; and (2) follows by selecting $T = x^{\frac{4}{3}(1-B)}$.

Improvement using a zero-density estimate. In the bound above the contribution is majorized by those terms with $\beta,\beta'\geq\frac{1}{2}$ and $\gamma,\gamma'\geq 0$ (using the symmetries of the zeros). By using Carlson's zero-density estimate $\#\{\rho:\ \zeta(\rho)=0$ and $\beta\geq\sigma,\ |\gamma|\leq T\}\ll T^{4\sigma(1-\sigma)}(\log T)^{O(1)}$, we can improve our bound (we will select $T\leq x^{1/(8B-4)}$ below, which simplifies several steps, since then $B\leq\frac{1}{2}+\frac{\log x}{8\log\gamma}$): throughout we sum over the zeros arranged by height, in dyadic intervals, and obtain that the final sum in the displayed equation is

$$\begin{aligned}
&\ll \sum_{1\leq\gamma'\leq\gamma\leq T}\frac{(\gamma')^{\beta'-\frac{1}{2}}}{\gamma^{\beta'+1}}\cdot x^{\beta+\beta'} \ll \mathcal{L}\sum_{\gamma\leq T}\max_{1\leq t\leq\gamma}\int_{\sigma=1/2}^{B}\frac{t^{\sigma-\frac{1}{2}}}{\gamma^{\sigma+1}}\cdot x^{\beta+\sigma}t^{4\sigma(1-\sigma)}d\sigma\\
&\ll \mathcal{L}\sum_{\gamma\leq T}\max_{1/2\leq\sigma\leq B}x^{\beta+\sigma}\gamma^{4\sigma(1-\sigma)-\frac{3}{2}} \ll \mathcal{L}\sum_{\gamma\leq T}x^{\beta+B}\gamma^{4B(1-B)-\frac{3}{2}}\\
&\ll \mathcal{L}\max_{u\leq T}\max_{1/2\leq\tau\leq B}x^{\tau+B}u^{4B(1-B)+4\tau(1-\tau)-\frac{3}{2}} \ll \mathcal{L}\max_{u\leq T}x^{2B}u^{8B(1-B)-\frac{3}{2}}\\
&\ll x^{2B}(1+T^{8B(1-B)-\frac{3}{2}})(\log x)^{O(1)}
\end{aligned}$$

where $\mathcal{L}=(\log x)^{O(1)}$. If $B\geq\frac{3}{4}$ then this is $\ll\mathcal{L}x^{2B}$; selecting $T=x^{1/(8B-4)}$ we get an error term $\ll x^{2B}(\log x)^{O(1)}$, which is as good as can be hoped for. If $B\leq\frac{3}{4}$ then the above error term is $\ll\mathcal{L}x^{2B}T^{8B(1-B)-\frac{3}{2}}$; to minimize we select $T=x^{4(1-B)/(16B(1-B)-1)}$, which leads to an error term of $x^{\frac{2+4B}{3}-\theta_B}(\log x)^{O(1)}$ where $\theta_B=\frac{16(1-B)(1-2B)^2}{3(16B(1-B)-1)}$.

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