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# SUMS OF TWO SQUARES AND ONE BIQUADRATE

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**Abstract:** There are no nontrivial integer solutions of $x^2 + y^2 + z^4 = p^2$ for primes $p \equiv 7 \pmod 8$, even though there are no congruence obstructions.
**Keywords:** Sums of squares, Waring's problem for mixed powers

A classical theorem of Legendre and Gauß asserts that a positive integer $n$ is a sum of three integer squares if and only if $n$ is not of the form $4^a(8k+7)$. Davenport and Heilbronn [2] considered the more difficult problem of representing $n$ in the form $n = x^2 + y^2 + z^k$, solving the problem in the case of odd $k \geq 3$, for 'almost all' positive integers $n$. Extending their results Brüdern ([1], Satz 4.2) has shown that there are at most $O(N^{1-\frac{1}{k}+\epsilon})$ positive integers $n \leq N$ with no solutions of $n = x^2 + y^2 + z^k$ in positive integers, where $n$ is not in a residue class excluded by congruence obstructions. More recently, Jagy and Kaplansky [3] proved that for $k = 9$ and some $c_1 > 0$ there are $c_1 N^{1/3}/\log N$ positive integers $n \leq N$ that are not sums of two squares and one $k$-th power, showing that 'almost all' cannot be replaced by 'sufficiently large'. In this note we show that even for $k = 4$, for some $c_2 > 0$ there are $c_2 N^{1/2}/\log N$ exceptional positive integers $n \leq N$ that are not of the form $x^2 + y^2 + z^4$ for positive integers $x, y, z$, even though there are no congruence obstructions for those $n$.

**Theorem 0.1.** *Let $p$ be a prime with $p \equiv 7 \bmod 8$. Then there are no positive integers $x, y, z$ with $x^2 + y^2 + z^4 = p^2$.*

**Proof.** Assume there are solutions, then $x^2 + y^2 = (p - z^2)(p + z^2)$. If $z$ is even, then $p - z^2 \equiv 3 \bmod 4$. If $z$ is odd, then $p - z^2 \equiv 6 \bmod 8$. In both cases $p - z^2$ contains a prime divisor $q \equiv 3 \bmod 4$ of odd multiplicity. Therefore by the Two Squares Theorem both $p - z^2$ and $p + z^2$ are divisible by $q$. Hence their sum $2p$ and their difference $-2z^2$ are also divisible by $q$. Since $p$ is prime: $p = q$, and since $z \neq 0$: $q$ divides $z$. But this gives a contradiction: $x^2 + y^2 + z^4 > q^4 > q^2 = p^2$. ■

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## Bibliography

[1] Brüdern, J. Iterationsmethoden in der additiven Zahlentheorie, Dissertation, Universität Göttingen (1988).

[2] Davenport, H. & Heilbronn, H. Note on a result in the additive theory of numbers, *Proc. London Math. Soc.* **43** (1937), 142–151.

[3] Jagy, W.C. & Kaplansky, I. Sums of squares, cubes, and higher powers, *Experiment. Math.* **4** (1995), no. 3, 169–173.

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