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A CLASS OF MODULAR FUNCTIONS

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Abstract: In this paper, we relax the condition in Theorem 2 of [1] to include a larger class of N. We then prove similar results on congruence subgroups other than $\Gamma_0(N)$.
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1. Introduction

Let τ be in the upper half plane $\mathbb{H}$ and $n \in \mathbb{Z}$. The Dedekind eta function is defined by

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$$

and the Generalized Dedekind eta function [4, 2] is defined by

$$\eta_{\delta,g}(\tau) = e^{\pi i P_2(\frac{g}{\delta})\delta\tau} \prod_{\substack{m>0 \\ m \equiv g \ (mod\ \delta)}} (1 - x^m) \prod_{\substack{m>0 \\ m \equiv -g \ (mod\ \delta)}} (1 - x^m),$$

where $x = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$, $g \in \mathbb{Z}$ such that $0 \leq g < \delta$,

$$P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$$

is the second Bernoulli function, and $\{t\} = t - [t]$ is the fractional part of t. Note that

$$\eta_{\delta,0}(\tau) = \eta(\delta\tau)^2$$

and that

$$\eta_{\delta,\delta/2}(\tau) = \frac{\eta^2((\delta/2)\tau)}{\eta^2(\delta\tau)}.$$

Consider

$$SL_2(Z) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

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Consider also

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \mod N \right\},$$
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \mod N, a \equiv d \equiv 1 \mod N \right\},$$

and

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv 1 \mod N, b \equiv c \equiv 0 \mod N \right\}.$$

Definition 1.1. *A modular form of weight k on a subgroup Γ of the full modular group of finite index is a function $f(\tau)$ meromorphic throughout the complex upper plane $\mathbb{H}$, which is also meromorphic at the cusps and satisfies the transformation law*

$$f(M\tau) = \nu(M)(c\tau + d)^k f(\tau),$$

for all $M \in \Gamma$ and $\mid \nu(M) \mid = 1$ where ν is a character of the group Γ.

Similar to any complex-valued, periodic function on $\mathbb{H} \cup \{\infty\}$ that is not identically zero, f has the product expansion

$$f(z) = cq_M^h \prod_{n \geq 1} (1 - q_M^n)^{c(n)} \tag{1.1}$$

where the infinite product is convergent in $\mid q_M \mid < \epsilon$ for some $\epsilon > 0$. Here c is a non-zero constant, M is the least positive integer with

$$\begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix} \in \Gamma, q_M = e^{2\pi i z/M}$$

for $z \in \mathbb{H}$, h is the order of f at infinity and the $c(n)$ are uniquely determined complex numbers.

2. A Result on $\Gamma_0(N)$

In [1, Theorem 2], Kohnen shows that if a modular form f of integral weight on $\Gamma_0(N)$, where N is square free, has no zeroes or poles in $\mathbb{H}$ then $c(n)$ defined in (1.1) depends only on the greatest common divisor (n, N). The converse of Theorem 2 in [1] was proved in Theorem 1 in the same paper.

In this paper, we relax the condition on N and we prove similar results on $\Gamma_0(N)$ where N belongs to a bigger class than being square free and for which f can have rational weight. We then use the same technique to prove similar results on the subgroups $\Gamma_1(N)$ and $\Gamma(N)$.

Theorem 2.1. *Let f be a modular form of rational weight k' on $\Gamma_0(N)$ for which the rank of*

$$\left((\delta, c)^2 P_2(ag/(\delta, c))\right)_{(\delta|N, 0 \leq g < \delta), (c|N, a)} \tag{2.1}$$

is equal to the number of cusps, where the columns of the matrix corresponds to the cusps a/c of $\Gamma_0(N)$. Then f has no zeroes or poles in $\mathbb{H}$ if and only if $c(n)$ depends on the greatest common divisor (n,N) and on a polynomial of the fractional part of residue class modulo n and N.

Note The condition on the rank of the matrix defined above is a condition on N which is automatically satisfied when N is square free. See [3].

Proof. We need only to prove that if f has no zeroes or poles in $\mathbb{H}$ then $c(n)$ has the above property because the converse was proved in [1, Theorem 1]. To prove the asserted result, we follow closely the proof of [3, Theorem 3]. For given integers $a_{\delta,g}$ put

$$F(\tau) = \prod_{\delta|N} \prod_{0\le g<\delta} \eta_{\delta,g}(\tau)^{a_{\delta,g}}. \tag{2.2}$$

$F(\tau)$ is a modular form on $\Gamma_1(N)$ of weight $k = \sum a_{\delta,0}$ and by [4],

$$\mathrm{ord}_{a/c} F = \frac{w_{a/c}}{2} \sum_{\delta|N} \sum_{0\le g<\delta} \frac{(\delta,c)^2}{\delta} P_2\left(\frac{ag}{(\delta,c)}\right) a_{\delta,g}. \tag{2.3}$$

As in [3, p. 44], we apply $H(F)$, the operator that lifts the modular form F from a form on $\Gamma_1(N)$ to a form on $\Gamma_0(N)$. Thus

$$H(F) = \prod_j F \mid_k \beta_j \tag{2.4}$$

where $\{\beta_j\}$ are coset representatives of $\Gamma_1(N)$ in $\Gamma_0(N)$. As a result, we get as in $[3, p.46]$ that

$$H(F) = \nu \prod_{\delta|N} \prod_{0\le g<\delta} \eta_{\delta,g}(\tau)^{a'_{\delta,g}},$$

where again $a'_{\delta,g} = \sum_j a_{\delta,a_j g}$ are new exponents and ν is a constant depending on β_j for all j. Using the condition on matrix (2.1), we have to solve now for $a'_{\delta,g}$. Also we have

$$\mathrm{ord}_{a/c} H(F) = \frac{w_{a/c}}{2} \sum_{\delta|N} \sum_{0\le g<\delta} \frac{(\delta,c)^2}{\delta} P_2\left(\frac{ag}{(\delta,c)}\right) a'_{\delta,g}.$$

We still want to show that $a'_{\delta,g}$ can be chosen so that

$$\mathrm{ord}_{a/c} H(F) = m h_{a/c}, \tag{2.5}$$

for all cusps a/c of $\Gamma_0(N)$. Here $h_{a/c}$ is the order of f at a/c and m is an appropriate non-zero integer depending only on f. By hypothesis, the rank of

$$\left((\delta,c)^2 P_2(ag/(\delta,c))\right)_{(\delta|N, 0\le g<\delta),(c|N,a)}$$

is equal to the number of cusps. Therefore we can choose $a'_{\delta,g}$ so that (2.5) is satisfied, with an appropriate m. Since η does not vanish on $\mathbb{H}$, we find from the valence formula applied to $H(F)$ that the sum of the orders of $H(F)$ at the different cusps of $\Gamma_0(N)$ is equal to

$$\frac{k_1}{12}[\Gamma(1):\Gamma_0(N)].$$

where k_1 is the weight of $H(F)$. We then deduce from (2.5) that

$$k_1 = mk'.$$

We see that $f^m/H(F)$ is a modular form of weight 0 that has no zeroes or poles in $\mathbb{H}$ or at the cusps and therefore is constant. We conclude that $f^m = cH(F)$. ∎

3. Results on $\Gamma_1(N)$ and on $\Gamma(N)$

We now present similar results for the congruence subgroup $\Gamma_1(N)$ and on $\Gamma(N)$. In this case, we impose a condition on the order of the function at the cusps. Note that the cusps of $\Gamma_1(N)$ are given by

$$\frac{a}{c} \tag{3.1}$$

where c is taken modulo N and a is taken modulo $d = (N, c)$ and $(a, d) = 1$. The width of the every cusp in (3.1) is given by

$$w_{a/c} = \frac{N}{(c, N)}.$$

Note also that the cusps of $\Gamma(N)$ are given by a/c where c is taken modulo N and a is taken modulo N and $(a, \nu) = 1$ where $\nu = (c, N)$. The width of the every cusp a/c here is given by

$$w_{a/c} = N.$$

Theorem 3.1. *Let f be modular form of integral weight k on $\Gamma_1(N)$ or on $\Gamma(N)$. Suppose further that the order of the function f at each cusp of $\Gamma_1(N)$ or of $\Gamma(N)$ is independent of a and for the cusps a_1/c_1 whose denominator does not divide N, the function will have the same order at a/c as at those cusps whose denominators are (c, N). Then f has no zeroes or poles in $\mathbb{H}$ if and only if $c(n)$ depends only on the greatest common divisor (n, N).*

Proof. As in Theorem 1, we only have to prove that if f has no zeroes or poles in $\mathbb{H}$ then $c(n)$ has the above property. Because of the condition that is imposed on the order of the function at the cusps, we will see that the proof follows exactly the lines of the proof in [1, Theorem 2]. We have

$$\Delta(\tau) = q\prod_{n\geq 1}(1-q^n)^{24}.$$

For given integers a_d put

$$F(\tau) = \prod_{d|N} \Delta(d\tau)^{a_d}.$$

Then F is a modular form on $\Gamma_1(N)$ and

$$\mathrm{ord}_{a/c} F = w_{a/c} \left(\sum_{d|N} \frac{(d,c)^2}{d} a_d \right).$$

Note that the order at every cusp a/c is independent of a and hence F itself satisfies the order condition given by Theorem 2. Moreover the conditions imposed in the theorem are important since $\sum_{d|N}$ in the above expression for $\mathrm{ord}_{a/c}$ runs only over the divisors of N. So similarly as in [1, Theorem 2], we want to show that a_d can be chosen so that

$$\mathrm{ord}_{a/c} F = m h_{a/c} \tag{3.2}$$

for all cusps a/c of $\Gamma_1(N)$. Here $h_{a/c}$ is the order of f^{12} at a/c and m is an appropriate non-zero integer depending only on f. The proof proceeds exactly as in [1, Theorem 2] and hence we get the desired result as we see that f^m/F has weight 0 with no zeroes or poles in $\mathbb{H}$ or at the cusps and hence constant. ■

We now relax the condition on N and we get a theorem on $\Gamma_1(N)$ and $\Gamma(N)$ similar to the one on $\Gamma_0(N)$.

Theorem 3.2. *Let f be a modular form of rational weight k on $\Gamma_1(N)$ or on $\Gamma(N)$ for which the rank of*

$$\big((\delta,c)^2 P_2(ag/(\delta,c))\big)_{(\delta|N, 0\le g<\delta),(c|N,a)} \tag{3.3}$$

is equal to the number of cusps whose denominator divides N. Suppose further that the order of the function f at each cusp a/c of $\Gamma_1(N)$ or of $\Gamma(N)$ whose denominator does not divide N, the function will have the same order as at those cusps whose denominators are (c,N). Then f has no zeroes or poles in $\mathbb{H}$ if and only if $c(n)$ depends on the greatest common divisor (n,N) and on a polynomial of the fractional part of residue class modulo n and N.

Note For $N = 16$, the above matrix has a rank equal to the number of cusps whose denominators divide N. This is also mentioned in [3].

Proof. For given integers $a_{\delta,g}$ put

$$F(\tau) = \prod_{\delta|N} \prod_{0\le g<\delta} \eta_{\delta,g}(\tau)^{a_{\delta,g}},$$

We want to find $a_{\delta,g}$ such that $f^m = cF$ for some constant c. F is a modular form on $\Gamma_1(N)$ and hence on $\Gamma(N)$ of weight $k_1 = \sum a_{\delta,0}$ and by [4],

$$\mathrm{ord}_{a/c} F = \frac{w_{a/c}}{2} \sum_{\delta|N} \sum_{0\le g<\delta} \frac{(\delta,c)^2}{\delta} P_2\left(\frac{ag}{(\delta,c)}\right) a_{\delta,g}.$$

We still want to show that $a_{\delta,g}$ can be chosen so that

$$\mathrm{ord}_{a/c}F = mh_{a/c}, \tag{3.4}$$

for all cusps a/c of $\Gamma_0(N)$. Here $h_{a/c}$ is the order of f at a/c and m is an appropriate non-zero integer depending only on f. It is given that the rank of (3.3) is equal to the number of cusps whose denominator divides N. Thus we have a non trivial solution. Following the same steps as in Theorem 1, we see that f^m/F is a again a modular form of weight 0 that has no zeroes or poles in $\mathbb{H}$ or at the cusps. We conclude that $f^m = cF$, as required. ■

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