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SUMS OF FOURTH POWERS OF POLYNOMIALS OVER A FINITE FIELD OF CHARACTERISTIC 3

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Abstract: Let F be a finite field with q elements and characteristic 3. A sum

$$M = M_1^4 + \ldots + M_s^4$$

of fourth powers of polynomials $M_1, \ldots, M_s$ is a strict one if $4 \deg M_i < 4 + \deg M$ for each $i = 1, \ldots, s$. Our main results are: Let $P \in F[T]$ of degree ≥ 329. If $q > 81$ is congruent to 1 (mod. 4), then P is the strict sum of 9 fourth powers; if $q = 81$ or if $q > 3$ is congruent to 3 (mod 4), then P is the strict sum of 10 fourth powers. If $q = 3$, every $P \in F[T]$ which is a sum of fourth powers is a strict sum of 12 fourth powers, if $q = 9$, every $P \in F[T]$ which is a sum of fourth powers and whose degree is not divisible by 4 is a strict sum of 8 fourth powers; every $P \in F[T]$ which is a sum of fourth powers, whose degree is divisible by 4 and whose leading coefficient is a fourth power is a strict sum of 7 fourth powers.
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1. Introduction

Let F be a finite field of characteristic p with q elements and let $\mathcal{S}(q, k)$ be the set of polynomials in $F[T]$ which are sums of k-th powers. Let $g(q, k)$, respectively, $G(q, k)$ denote the least integer s, if it exists, such that every polynomial $M \in \mathcal{S}(q, k)$, respectively, every polynomial $M \in \mathcal{S}(q, k)$ of sufficiently large degree, may be written as a sum

$$M = M_1^k + \ldots + M_s^k$$

with $M_1, \ldots, M_s$ polynomials satisfying the degree condition: $k \deg M_i < k + \deg M$. Such a representation is called a *strict representation* in opposition to representations without degree conditions. Waring's problem consists in determining or, at least, bounding the numbers $g(q, k)$ and $G(q, k)$. Bounds for $g(q, k)$ and $G(q, k)$ were given in [3] where the author described a process introduced in [8] and performed in [4] to deal with the polynomial Waring's problem for cubes.

Some notations and definitions are necessary before stating the main results proved in [3].

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If every $a \in F$ is a sum of k-th powers, the field F is called a Waring field for the exponent k or briefly, a k-Waring field. If F is a k-Waring field, let $\ell(q,k)$ denote the the least integer ℓ such that every element of F is the sum of ℓ k-th powers. Let $\lambda(q,k)$ denote the least integer s such that -1 is the sum of s k-th powers. Let $d(q,k) = \gcd(q-1,k)$.

Let $v(q,k)$ denote the least integer v, if it exists, such that T may be written as a sum $(a_1T+b_1)^k + \ldots + (a_vT+b_v)^k$ with $a_i, b_i \in F$. Otherwise, let $v(q,k) = \infty$. If $v(q,k)$ is finite, every $P \in F[T]$ may be written as a sum

$$P = (a_1P+b_1)^k + \ldots + (a_{v(q,k)}P + b_{v(q,k)})^k$$

so that $\mathcal{S}(q,k) = F[T]$ and F is a k-Waring field. If x is a real number, $[x]$ is defined as the integral part of x and $\lceil x \rceil$ is defined as the least integer $\geq x$.

In what follows, unless otherwise stated we agree that, if R is a ring, the statement $a \in R$ is a sum of fourth powers means that a is a sum $b_1^4 + \ldots + b_s^4$ with $b_1, \ldots, b_s \in R$.

In [3], the bounds for $G(q,k)$ and $g(q,k)$ arise from the two following propositions.

Proposition 1.1. ([3]) (I) *If F is a k-Waring field and if $q > k$, then*

$$v(q,k) \leq k/d(q,k) + \ell(q,k)(k - k/d(q,k))).$$

(II) *Assume that one of the following conditions is satisfied:* (1) $p > k$, (2) *F is a k-Waring field, $q > k$, $k = hp^s - 1$ with $1 \leq h \leq p, s \geq 1$. Then every $A \in F[T]$ of degree $\leq kn$ is the strict sum of $\ell(q,k)(kn+1)$ k-th powers.*

Proposition 1.2. ([3]) *Assume that F is a k-Waring field and that $q > k$. Put*

$$r = \log\left(k/(k-1)\right).$$

(I) *Let $m \geq [\frac{\log k}{r}]$. Then, every $P \in F[T]$ of degree at least equal to $n(m,k) = k\lceil \frac{k^2-2k-k^2(1-\frac{1}{k})^{m+1}}{1-k(1-\frac{1}{k})^{m+1}} \rceil - k + 1$ is the strict sum of $m + v(q,k) + \max(\ell(q,k), 1+\lambda(q,k))$ k-th powers. Moreover, if $m \geq \frac{\log k}{r}$, then, $n(m,k) \leq k^4 - 3k^3 + 2k^2 - 2k + 1$.*

(II) *Let $m \geq \frac{\log(k(k-1)/2)}{r}$. Then, every $P \in F[T]$ of degree $\geq k^3 - 3k + 1$ is the strict sum of $m + v(q,k) + \max(\ell(q,k), 1 + \lambda(q,k))$ k-th powers.*

(III) *Let $m \geq \frac{3\log k}{r} - 1$. Then, every $P \in F[T]$ such that $k^3 - 2k^2 - k + 1 \leq \deg P \leq k^3 - 3k$ is the strict sum of $m + v(q,k) + \max(\ell(q,k), 1 + \lambda(q,k))$ k-th powers.*

Roughly speaking, the object of this paper is the study of the Waring problem in the particular case $k = 4, p = 3$. It can be viewed as a continuation of the work in [5] where it was proved that $G(q,4) \leq 11$ for $q \notin \{3,9,5,13,17,25,29\}$ and that $G(q,4) \leq 10$ for $q \notin \{17,25\}$ and congruent to 1 (mod 8). This case does not fall in the scope of the second part of Proposition 1.1, and the study of the numbers $g(3^m,4)$ has not be done. In the special case $k = 4, p = 3$, it is possible to compute the exact value of $v(3^m, 4)$. This involves an improvement for the bounds given in [3] and [5]. Since the numbers $g(3^m, 4)$ and $G(3^m, 4)$ are not sufficient to describe every possible case, we introduce new parameters. Let $\mathcal{S}^\star(q, 4)$ denote the set of polynomials in $F[T]$ which are strict sums of fourth powers. Let $g^\star(q, 4)$, respectively, $G^\star(q, 4)$ denote the least integer s, if it exists, such that every polynomial $M \in \mathcal{S}^\star(q, 4)$ respectively, every polynomial $M \in \mathcal{S}^\star(q, 4)$ of sufficiently large degree, may be written as a strict sum

$$M = M_1^4 + \ldots + M_s^4.$$

The main results proved in this work are summarized in the following theorem.

Theorem 1.1. *Assume that F is a finite field with $q = 3^N$ elements.*

(I) *For $N \geq 3$, $\mathcal{S}(3^N, 4)$ is equal to the whole ring $F[T]$ and $\mathcal{S}^\star(3^N, 4)$ is the union of the set $\{A \in F[T] \mid \deg A > 4\}$ and the set of polynomials*

$$A = aT^4 + bT^3 + cT + d$$

with $a, b, c, d \in F$. For $N \in \{1, 2\}$, $\mathcal{S}(3^N, 4)$ is the subset of $F[T]$ formed by the polynomials A such that $T^9 - T$ divides $A^3 - A$. Moreover, $\mathcal{S}^\star(3, 4) = \mathcal{S}(3, 4)$ and $\mathcal{S}^\star(9, 4)$ is the set formed by the polynomials $A \in \mathcal{S}(9, 4)$ satisfying one of the two following conditions:

(i) 4 does not divide $\deg A$, (ii) 4 divides $\deg A$ and the leading coefficient of A is in the prime field $\mathbb{F}_3$.

(II) *We have $g(3^N, 4) = \infty$ for $N \geq 2$ and $g(3, 4) \leq 12$. We have $g^\star(3^N, 4) \leq 19$ for even $N > 4$, $g^\star(81, 4) \leq 21$ and $g^\star(9, 4) \leq 8$. We have $g^\star(3^N, 4) \leq 20$ for odd $N > 1$ and $g^\star(3, 4) \leq 12$.*

(III) *We have $G(3^N, 4) \leq 9$ for even $N > 4$, $G(81, 4) \leq 10$ and $G(9, 4) = \infty$. We have $G(3^N, 4) \leq 10$ for odd $N > 1$ and $G(3, 4) \leq 12$. We have $G^\star(3^N, 4) \leq 9$ for even $N > 4$, $G^\star(81, 4) \leq 10$ and $G^\star(9, 4) \leq 8$. We have $G^\star(3^N, 4) \leq 10$ for odd $N > 1$ and $G^\star(3, 4) \leq 12$.*

Observe that for the classical Waring's problem we have $G(4) = 16$ and $g(4) = 19$, see [6], [1] and [7].

The paper is organized as follows. In order to prove that T is a strict sum of fourth powers, we have to prove that some algebraic equations have solutions in F. This is done in Section 2. In Section 3, we prove that for $q = 27$ or $q > 81$, $v(q, 4) = 3$ and that $v(81, 4) = 4$ and we deduce bounds for the numbers $G(q, 4)$. In Section 4, we prove some identities and we show that, with the exception $q = 3$, the sets $\mathcal{S}(3^N, 4)$ and $\mathcal{S}^\star(3^N, 4)$ are different. In section 5, we describe a new descent process and we end the proof.

Choosing an algebraic closure $\bar{F}$ of F, we shall denote by $\mathbb{F}_Q$ the unique subfield of $\bar{\mathbb{F}}$ with Q elements, so that $F = \mathbb{F}_q$. Let $\alpha \in \mathbb{F}_9$ be such that

$$\alpha^2 = -1 \tag{1.1}$$

and let

$$\beta = 1 - \alpha. \tag{1.2}$$

Then,

$$\beta^2 = \alpha, \beta^4 = -1. \tag{1.3}$$

2. Equations

Although it is very simple, the following lemma is very useful to obtain representations of polynomials as sums of fourth powers.

Lemma 2.1. *Let $(u, v) \in F^2$ be such that $uv \neq 0$ and $u^8 \neq v^8$. Then, for each ordered pair $(a, b) \in F^2$, the system $(\mathcal{E}(u, v, a, b))$:*

$$\begin{cases} a = u^3x + v^3y \\ b = ux^3 + vy^3 \end{cases} \tag{2.1}$$

has a unique solution in F^2.

Proof. If $(x, y) \in F^2$ is a solution of $\mathcal{E}(u, v, a, b)$, then,

$$y = \frac{(a - u^3x)}{v^3}, \tag{2.2}$$

so that

$$x^3 = \frac{bv^8 - a^3}{u(v^8 - u^8)}. \tag{2.3}$$

Conversely, there is one and only one $x \in F$ satisfying (2.3) and, for that x, there is one and only one $y \in F$ satisfying (2.2). ■

When $q \equiv 3 \pmod 4$, the set of fourth powers in F is the set of squares in F, so that the numbers $\nu_i(a)$ of representations of $a \in F$ as a sum of i fourth powers are well known. (See e.g. [2]). It remains to compute these numbers in the case when q is congruent to 1 (mod 4). For that, we have to introduce some character sums

2.1. Character sums

In this subsection we suppose $q \equiv 1 \pmod 4$. Then, $\mathbb{F}_9 \subset F$. Let $\alpha \in F$ be defined by (1.1). Let tr denote the absolute trace on F and let ψ be the character of the additive group of F defined by

$$\psi(x) = \exp(\frac{2\pi i \mathrm{tr}(x)}{3}). \tag{2.4}$$

Then, ψ is not the trivial character. For a and b elements of F let

$$\sigma(a, b) = \sum_{x \in F} \psi(ax^3 + bx). \tag{2.5}$$

Proposition 2.1. *Let $a, b \in F$. Then,*

(i) $\sigma(a,b) \in \{0, q\}$.

(ii) $\sigma(a,b) = q$ if and only if $a + b^3 = 0$.

Proof. The map $\gamma : x \mapsto (ax^3 + bx)$ is additive so that $\psi \circ \gamma$ is a character of the additive group of F. This proves (i). Let $b \in F$. Then,

$$\sum_{a\in F} \sigma(a,b) = \sum_{a\in F}\sum_{x\in F} \psi(ax^3 + bx).$$

Inverting the order of summation gives

$$\sum_{a\in F} \sigma(a,b) = \sum_{x\in F} \psi(bx) \sum_{a\in F} \psi(ax^3).$$

Since ψ is not trivial, the last inner sum is 0 if $x \neq 0$ and q if $x = 0$. Thus,

$$\sum_{a\in F} \sigma(a,b) = q.$$

Since $\sigma(a,b) \in \{0, q\}$ for each $a \in F$, there exists one and only one $a \in F$ such that $\sigma(a,b) = q$. For every $x \in F$, $\mathrm{tr}((bx)^3) = \mathrm{tr}(bx)$ and $\psi(b^3x^3 - bx) = 1$. Thus, $\sigma(-b^3, b) = q$ so that $-b^3$ is the unique $a \in F$ such that $\sigma(a,b) = q$. ∎

Let B denote the set of non-zero fourth powers in F. Observe that

$$|B| = \frac{q-1}{4}. \tag{2.6}$$

For $t \in F$ let

$$f(t) = \sum_{x\in F} \psi(tx^4). \tag{2.7}$$

Remark 2.1. For every $t \in F$,

$$f(t) = f(-t) = \overline{f(t)}, \tag{2.8}$$

so that f takes real values.

Proof. Observe that q is congruent to 1 (mod 8), so that -1 is a fourth power in F, say $-1 = \beta^4$ with β defined by (1.2). ∎

Proposition 2.2. (I) *We have $f(0) = q$.*

(II) *Let $t \in F^\star$.*

(i) If $t/\alpha \notin B$, then $f(t)^2 = q$.

(ii) If $t/\alpha \in B$, then $f(t) = f(\alpha)$ and $f(t)^2 = 9q$.

(iii) If $t/\alpha \notin B$, then $3f(t) + f(\alpha) = 0$.

(iv) If $q = 9$, then $f(1) = f(-1) = -3$ and $f(\alpha) = f(-\alpha) = 9$.

(v) If $q = 81$, then $f(1) = f(-1) = f(\alpha) = f(-\alpha) = -27$.

Proof. (I) is obvious. From (2.8), for every $t \in F$, $f(t)^2 = |f(t)|^2$. Let $t \in F^\star$. Then, with (2.5),

$$f(t)^2 = \sum_{x\in F}\sum_{y\in F} \psi(t((x+y)^4 - y^4)) = \sum_{x\in F} \psi(tx^4)\sigma(tx, tx^3).$$

From Proposition 2.1, $\sigma(tx, tx^3) = 0$ or q and is equal to q if and only $x \in X(t)$ where

$$X(t) = \{x \in F \mid tx + t^3x^9 = 0\} = \{x \in F \mid x + t^2x^9 = 0\}.$$

If $X(t)$ contains a non-zero element x, then $t^2x^8 = -1$ so that $tx^4 = \pm\alpha$, and t/α is a 4-th power. Thus, if $t/\alpha \notin B$, then $f(t)^2 = q$. Suppose that $t/\alpha = u^4$ with $u \in F$. Then, $1/u \in X(t)$. Thus, $X(t) = \{z/u \mid z \in \mathbb{F}_9\}$, so that

$$f(t)^2 = q \sum_{z\in\mathbb{F}_9} \psi(\alpha z^4).$$

If $z \in \mathbb{F}_9$, then $z^4 \in \mathbb{F}_3$, so that $\mathrm{tr}(\alpha z^4) = z^4\mathrm{tr}(\alpha) = 0$ and $f(t)^2 = 9q$. Moreover, if $t/\alpha = u^4$ with $u \in F$, the change of the variable $y = ux$ in the sum (2.7) gives $f(t) = f(\alpha)$.

Let B' denote the set of $x \in F$ which are not fourth powers. Then,

$$|B'| = \frac{3(q-1)}{4}. \tag{2.9}$$

Let $b \in B'$. If $t \notin \alpha B$, then $t \in \alpha B'$, so that $|f(t)| = |f(b\alpha)|$. Set $f(t) = \varepsilon_t f(b\alpha)$. Observe that $\varepsilon_t = \pm 1$.We compute the sum

$$\Sigma = \sum_{t\in F^\star} f(t). \tag{2.10}$$

Firstly,

$$\Sigma = \sum_{t\in F} f(t) - q = \sum_{t\in F}\sum_{x\in F} \psi(tx^4) - q.$$

Inverting the order of summation gives

$$\Sigma = 0. \tag{2.11}$$

On the other hand,

$$\Sigma = \sum_{t\in\alpha B} f(t) + \sum_{t\in\alpha B'} f(t).$$

Thus,

$$\Sigma = |B|f(\alpha) + f(b\alpha)\sum_{t\in\alpha B'} \varepsilon_t. \tag{2.12}$$

From (2.9) and (2.11),

$$|f(b\alpha)\sum_{t\in\alpha B'} \varepsilon_t| = \frac{q-1}{4}|f(\alpha)|.$$

From II.(i) and II.(ii),

$$|\sum_{t\in\alpha B'}\varepsilon_t| = \frac{3(q-1)}{4} = |B'|.$$

Hence, for each $t \in \alpha B'$, we have $\varepsilon_t = \varepsilon_{b\alpha}$ and $f(t) = f(b\alpha)$. From (2.11) and (2.12),

$$\frac{q-1}{4}f(\alpha) + \frac{3(q-1)}{4}f(b\alpha) = 0.$$

Therefore, for every $(t, u) \in B \times B'$,

$$\frac{q-1}{4}f(t\alpha) + \frac{3(q-1)}{4}f(u\alpha) = 0,$$

proving II.(iii).

In the case when $F = \mathbb{F}_9$, we shall use f_1, ψ_1, in the place of f, resp. ψ, and similarly, we shall write f_2 and ψ_2 for f and ψ in the case $F = \mathbb{F}_{81}$. Denote by t_1 the absolute trace map from $\mathbb{F}_9$ to $\mathbb{F}_3$ and by τ the relative trace map from $\mathbb{F}_{81}$ to $\mathbb{F}_9$. If $x \in \mathbb{F}_9$, then $x^4 \in \mathbb{F}_{3.}$ Thus

$$t_1(\alpha x^4) = x^4 t_1(\alpha) = 0, t_1(x^4) = -x^4.$$

From (2.4) and (2.7),

$$f_1(\alpha) = 9,$$

$$f_1(1) = 1 + 4(\exp(\frac{2\pi i}{3}) + \exp(\frac{-2\pi i}{3})) = -3.$$

Let $\omega \in \mathbb{F}_{81}$ be such that $\omega^2 = 1 + \alpha$. Then, $\omega^4 = -\alpha$, so that α is a fourth power and $f_2(1) = f_2(\alpha)$. Now,

$$f_2(\alpha) = \sum_{x\in\mathbb{F}_9}\sum_{y\in\mathbb{F}_9}\psi_2\left(\alpha(x+y\omega)^4\right) = \sum_{x\in\mathbb{F}_9}\sum_{y\in\mathbb{F}_9}\psi_1\left(\tau\left(\alpha(x+y\omega)^4\right)\right) =$$

$$\sum_{x\in\mathbb{F}_9}\sum_{y\in\mathbb{F}_9}\psi_1(-\alpha x^4 - y^4) = f_1(-\alpha)f_1(-1) = -27.$$

■

2.2. Sums of fourth powers in F

Let i be a positive integer. For $a \in F$, let $\nu_i(a)$ denote the number of solutions $(x_1, \ldots, x_i) \in F^i$ of the equation

$$a = x_1^4 + \ldots + x_i^4. \tag{2.13}$$

Proposition 2.3. *If* $q \equiv 3 \pmod 4$, *then*

$$\nu_2(0) = 1,$$

$$\nu_3(0) = q^2$$

and for $a \in F^\star$, we have

$$\nu_2(a) = q+1,$$

$$\nu_3(a) = \begin{cases} q^2 - q & \text{if} \quad a \in B, \\ q^2 + q & \text{if} \quad a \notin B. \end{cases}$$

Proof. Observe that $a \in F$ is a fourth power if and only if a is a square. Apply the well-known results on sums of squares in a finite field, [2, exercise 5, p.175-176]. ■

Proposition 2.4. *If $q \equiv 1 \pmod 4$, then*

$$\nu_2(0) = 4q - 3;$$
$$\nu_3(0) = q^2 + 2f(\alpha)(q-1)$$

and for $a \in F^\star$, we have

$$\nu_2(a) = q - 3 + 2f(a\alpha);$$
$$\nu_3(a) = q^2 - q + q\nu_1(a) - 2f(\alpha) + 2f(\alpha)f(a\alpha).$$

Proof. By orthogonality, for $i = 1, 2, 3$,

$$\nu_i(a) = \sum_{x_1 \in F} \dots \sum_{x_i \in F} \frac{1}{q} \sum_{t \in F} \psi(t(x_1^4 + \ldots + x_i^4 - a)).$$

After inverting the order of summation, we get with (2.7),

$$q\nu_i(a) = \sum_{t \in F} \psi(-at) f(t)^i. \tag{2.14}$$

Let $i = 2, 3$. From Proposition 2.2,

$$q\nu_i(a) = q^i + 9q \sum_{t \in \alpha B} \psi(-at) f(t)^{i-2} + q \sum_{\substack{t \in F^\star \\ t \notin \alpha B}} \psi(-at) f(t)^{i-2}.$$

Hence,

$$q\nu_i(a) = q^i - q^{i-1} + 8q \sum_{t \in \alpha B} \psi(-at) f(t)^{i-2} + q \sum_{t \in F} \psi(-at) f(t)^{i-2}. \tag{2.15}$$

Suppose $i = 2$. Then, from (2.6)

$$\nu_2(0) = q - 1 + 2(q-1) + q.$$

Let $a \in F^\star$. With (2.15),

$$\nu_2(a) = q - 1 + 8 \sum_{t \in \alpha B} \psi(-at).$$

If $t \in \alpha B$, the equation $t/\alpha = u^4$ has exactly 4 solutions in F. Thus,

$$\nu_2(a) = q - 1 + 2 \sum_{u \in F^\star} \psi(-a\alpha u^4) = q - 3 + 2 \sum_{u \in F} \psi(-a\alpha u^4)$$

so that with (2.7) and (2.8),

$$\nu_2(a) = q - 3 + 2f(-a\alpha) = q - 3 + 2f(a\alpha).$$

Suppose $i = 3$. Then, from (2.15) and (2.14),

$$\nu_3(a) = q^2 - q + 8 \sum_{t \in \alpha B} \psi(-at)f(t) + q\nu_1(a),$$

so that

$$\nu_3(a) = q^2 - q + 2 \sum_{u \in F^\star} \psi(-a\alpha u^4)f(\alpha u^4) + q\nu_1(a).$$

From Proposition 2.2-(ii),

$$\nu_3(a) = q^2 - q + q\nu_1(a) + 2f(\alpha) \sum_{u \in F^\star} \psi(-a\alpha u^4).$$

With (2.7),

$$\nu_3(a) = q^2 - q + q\nu_1(a) - 2f(\alpha) + 2f(\alpha)f(-a\alpha).$$

Thus,

$$\nu_3(0) = q^2 - 2f(\alpha) + 2qf(\alpha).$$ ∎

Proposition 2.5. (I) *F is a 4-Waring field if and only if $q \neq 9$.*
(II) *If $q \neq 9$, then $\ell(q, 4) = 2$.*

Proof. (I) is given by [10, lemma 3.6, p. 181]. We suppose $q \neq 9$. From [9], if $q > 81$, then $\ell(q, 4) \leq 2$. Let $a \in F^\star$. From Proposition 2.3, if $q \equiv 3 \pmod 4$, then $\nu_2(a) = q - 1 > 0$; from Proposition 2.4, if $q \equiv 1 \pmod 4$, then $\nu_2(a) = q - 3 + 2f(a\alpha)$ and in view of Proposition 2.2, $\nu_2(a) \geq q - 3 - 6q^{1/2} \geq 24$. In any case, a is a sum of two 4−th powers. Therefore, F is a 4-Waring field with $\ell(q, 4) \leq 2$. We have $d(q, 4) \geq 2$, so that, from [3, Proposition 3.1], $\ell(q, k) \geq 2$. ∎

Proposition 2.6. *For $a \in F$, let $N_3(a)$ denote the number of $(x, y, z) \in F^3$ such that*

$$\begin{cases} x^4 + y^4 + z^4 = a, & (e_1) \\ xy \neq 0, & (e_2) \\ x^8 \neq y^8 & (e_3). \end{cases} \qquad (\mathcal{F}(a))$$

(I) *If $q \equiv 1 \pmod 4$, then*

$$N_3(0) = q^2 - 28q + 27 + 2(q - 1)f(\alpha)$$

and for every $a \in F^\star$, *we have*

$$N_3(a) = \begin{cases} q^2 - q + 54 - 14f(\alpha) & \text{if} \quad a \in B, \\ q^2 - 13q + 18 + 2f(\alpha) & \text{if} \quad a \notin B. \end{cases}$$

(II) *If* $q \equiv 3 \pmod 4$, *then*

$$N_3(0) = q^2 - 4q + 3$$

and for every $a \in F^\star$, *we have*

$$N_3(a) = \begin{cases} q^2 - 5q + 6 & \text{if} \quad a \in B, \\ q^2 - 3q & \text{if} \quad a \notin B. \end{cases}$$

Proof. Let $\mathcal{A}(a)$ denote the set formed by the $(x, y, z) \in F^3$ satisfying conditions $(e_1), (e_2)$ and (e_3). Then,

$$N_3(a) = |\mathcal{A}(a)| \tag{2.16}$$

Let

$$\mathcal{B}_0(a) = \{(x, y, z) \in F^3 \mid x^4 + y^4 + z^4 = a, xy = 0\}, \tag{2.17}$$

$$\mathcal{B}_1(a) = \{(x, y, z) \in F^3 \mid x^4 + y^4 + z^4 = a, xy \neq 0, x^8 = y^8\}. \tag{2.18}$$

Then

$$\nu_3(a) = |\mathcal{A}(a)| + |\mathcal{B}_0(a)| + |\mathcal{B}_1(a)|. \tag{2.19}$$

Firstly, we deal with $\mathcal{B}_0(a)$. We have

$$\mathcal{B}_0(a) = \mathcal{B}_{0,0}(a) \cup \mathcal{B}_{0,1}(a) \cup \mathcal{B}_{1,0}(a), \tag{2.20}$$

with the $\mathcal{B}_{i,j}(a)$ defined as follows. For $(x, y, z) \in \mathcal{B}_0(a)$,

$$\begin{aligned} (x, y, z) \in \mathcal{B}_{0,0}(a) &\Leftrightarrow (x, y) = (0, 0), \\ (x, y, z) \in \mathcal{B}_{0,1}(a) &\Leftrightarrow y \neq 0, \\ (x, y, z) \in \mathcal{B}_{1,0}(a) &\Leftrightarrow x \neq 0. \end{aligned}$$

Now, $(0, 0, z) \in \mathcal{B}_{0,0}(a) \Leftrightarrow a = z^4$, so that

$$|\mathcal{B}_{0,0}(a)| = \nu_1(a); \tag{2.21}$$

and $(0, y, z) \in \mathcal{B}_{0,1}(a) \Leftrightarrow a = y^4 + z^4$ with $y \neq 0$, so that

$$|\mathcal{B}_{0,1}(a)| = \nu_2(a) - \nu_1(a). \tag{2.22}$$

By symmetry, with (2.20), (2.21) and (2.22),

$$|\mathcal{B}_0(a)| = 2\nu_2(a) - \nu_1(a). \tag{2.23}$$

Now, we deal with $\mathcal{B}_1(a)$. Let $(x,y) \in F^\star \times F^\star$. Then $x^8 = y^8 \Leftrightarrow y = ux$ with $u^8 = 1$. Thus,

$$|\mathcal{B}_1(a)| = \sum_{u^8=1} n_u(a), \tag{2.24}$$

where $n_u(a)$ is the number of $(x,z) \in F^\star \times F$ such that

$$a = x^4(1+u^4) + z^4. \tag{2.25}$$

We have to distinguish two cases accordingly as -1 is or is not a fourth power. Suppose $\mathbb{F}_9 \subset F$, so that $-1 = \beta^4$. There are exactly 8 elements $u \in F$ such that $u^8 = 1$, for four of them $u^4 = 1$, and for the others, $u^4 = -1 = \beta^4$. Thus, by (2.24),

$$|\mathcal{B}_1(a)| = 4(n_1(a) + n_\beta(a)). \tag{2.26}$$

Now, $n_\beta(a)$ is the number of $(x,z) \in F^\star \times F$ such that $a = z^4$, so that

$$n_\beta(a) = (q-1)\nu_1(a), \tag{2.27}$$

and $n_1(a)$ is the number of $(x,z) \in F^\star \times F$ such that $a = -x^4 + z^4$, so that

$$n_1(a) = \nu_2(a) - \nu_1(a). \tag{2.28}$$

From (2.26), (2.27) and (2.28),

$$|\mathcal{B}_1(a)| = 4(\nu_2(a) + (q-2)\nu_1(a)). \tag{2.29}$$

Suppose now that $\mathbb{F}_9 \not\subset F$, so that -1 is not a fourth power. Then, for $u \in F$, $u^8 = 1 \Leftrightarrow u = \pm 1$, and in this case $u^4 = 1$. By (2.24) and (2.25), $|\mathcal{B}_1(a)| = 2\mu(a)$, where $\mu(a)$ denotes the number of $(x,z) \in F^\star \times F$ such that

$$a = -x^4 + z^4.$$

We have $\mu(a) = \rho(a) - \nu_1(a)$, where $\rho(a)$ denotes the number of $(x,z) \in F^2$ such that

$$a = -x^2 + z^2.$$

From [2, exercise 4, p.175],

$$\mu(a) = \begin{cases} 2q-2 & \text{if} \quad a = 0, \\ q-1-\nu_1(a) & \text{if} \quad a \neq 0. \end{cases}$$

Thus,

$$|\mathcal{B}_1(a)| = \begin{cases} 4(q-1) & \text{if} \quad a = 0, \\ 2(q-1-\nu_1(a)) & \text{if} \quad a \neq 0. \end{cases} \tag{2.30}$$

We are ready to conclude. Firstly, we suppose $q \equiv 1 \pmod 4$. Combining (2.19), (2.23) and (2.29) gives

$$\begin{aligned} |\mathcal{A}(a)| &= \nu_3(a) - (2\nu_2(a) - \nu_1(a) - (4(\nu_2(a) + (q-2)\nu_1(a))) \\ &= \nu_3(a) - 6\nu_2(a) - (4q-9)\nu_1(a). \end{aligned}$$

We end the proof, using results given by Proposition 2.4. For brevity, we only give the proof in the case $a \neq 0$. From Proposition 2.4,

$$|\mathcal{A}(a)| = q^2 - 7q + 18 - (3q-9)\nu_1(a) - 2f(\alpha) + 2f(\alpha)f(-a\alpha) - 12f(-a\alpha).$$

If $a \in B$, then from Proposition 2.2, $f(-a\alpha) = f(\alpha)$ and $f(\alpha)f(-a\alpha) = 9q$, so that

$$|\mathcal{A}(a)| = q^2 - q + 54 - 14f(\alpha).$$

If $a \notin B$, from Proposition 2.2, $f(-a\alpha) = -f(\alpha)/3$ and $f(\alpha)f(-a\alpha) = -3q$, so that

$$|\mathcal{A}(a)| = q^2 - 13q + 18 + 2f(\alpha).$$

Now, we suppose $q \equiv 3 \pmod 4$. Combining (2.19), (2.23) and (2.30) gives

$$|\mathcal{A}(a)| = \nu_3(a) - (2\nu_2(a) + 3\nu_1(a)) - 2(q-1)$$

for $a \in F^\star$ and

$$|\mathcal{A}(0)| = \nu_3(a) - (2\nu_2(0) - \nu_1(0)) - 4(q-1).$$

We conclude using Proposition 2.3. ■

Corollary 2.1. (I) *Let $a \in F$. If $a \neq 0$ and $q \notin \{3,9\}$, or if $a = 0$ and $q \notin \{3,9,81\}$, then $(\mathcal{F}(a))$ has a solution in F^3. If $q \in \{3,9,81\}$, then $(\mathcal{F}(0))$ has zero solutions in F^3.*

(II) *Let $a \in \mathbb{F}_{81}$. Then there exists $(x,y,z,u) \in \mathbb{F}^4$ such that*

$$\begin{cases} x^4 + y^4 + z^4 + u^4 = a, & (\epsilon_1) \\ xy \neq 0, & (e_2) \\ x^8 \neq y^8 & (e_3). \end{cases} \qquad (\mathcal{G}(a))$$

Proof. (I) Suppose $q > 9$ and $\neq 81$. From the previous proposition, for each $a \in F$, $N_3(a) > 0$. so that $(\mathcal{F}(a))$ has a solution. If $q \leq 9$, there is no pair $(x,y) \in F^2$ satisfying (e_2) and (e_3). If $q = 81$, then $N_3(a) > 0$ for $a \neq 0$.

(II) Let $a \in \mathbb{F}_{81}$. If $a \neq 0$, for every (x,y,z) solution of $(\mathcal{F}(a))$, $(x,y,z,0)$ is a solution of $(\mathcal{G}(a))$, if $a = 0$, for every (x,y,z) solution of $(\mathcal{F}(-1))$, $(x,y,z,1)$ is a solution of $(\mathcal{G}(a))$. ■

3. The numbers $v(q,4)$

Remark 3.1. We have $v(q,4) \geq 3$.

Proof. Suppose $v(q,4) \leq 2$. Then, there is $(x,y,u,v) \in F^4$ such that

$$T = (xT+y)^4 + (uT+v)^4,$$

so that,

$$0 = x^4 + u^4, \tag{3.1}$$
$$0 = x^3 y + u^3 v, \tag{3.2}$$
$$1 = xy^3 + uv^3, \tag{3.3}$$
$$0 = y^4 + v^4. \tag{3.4}$$

By (3.1), if $xu = 0$, then $(x, u) = (0, 0)$ and (3.3) is not satisfied, so that $xu \neq 0$. Thus, from (3.1), -1 is a 4−th power and q is congruent to 1 (mod 4). Now, by (3.1), $u = xz$ with $z^4 = -1$, thus, with (3.2), $v = zy$ so that from (3.3), $1 = xy^3(1 + z^4) = 0$, leading to a contradiction. ∎

Proposition 3.1. (I) *If* $q \in \{3, 9\}$, *then* $v(q, 4) = \infty$.
(II) *If* $q = 27$ *or if* $q > 81$, *then* $v(q, 4) = 3$.
(III) *If* $q = 81$, *then* $v(q, 4) = 4$.

Proof. Suppose $v(q, 4) = s$. Then, there exists $(u_1, v_1, \ldots, u_s, v_s) \in F^{2s}$ such that

$$T = \sum_{i=1}^{s} (u_i T + v_i)^4,$$

so that

$$0 = \sum_{i=1}^{s} u_i^3 v_i \tag{3.5}$$

and

$$1 = \sum_{i=1}^{s} u_i {v_i}^3. \tag{3.6}$$

Raising (3.5) to the power 3 gives

$$0 = \sum_{i=1}^{s} {u_i}^9 {v_i}^3.$$

If $F \subset \mathbb{F}_9$, then for all i's, ${u_i}^9 = u_i$ leading to $0 = 1$, a contradiction. We suppose $q = 27$ or $q > 81$. From Corollary 2.1, there exists $(a_1, a_2, a_3) \in F^3$ such that

$$\begin{cases} (a_1)^4 + (a_2)^4 + (a_3)^4 = 0, \quad (e_1) \\ a_1 a_2 \neq 0, \quad (e_2) \\ (a_1)^8 \neq (a_2)^8 \quad (e_3). \end{cases}$$

Let $(b_1, b_2) \in F^2$ be solution of $(\mathcal{E}(a_1, a_2, 0, 1))$ with $(\mathcal{E}(u, v, w, t))$ defined at Lemma 2.1. Then,

$$(a_1)^3 b_1 + (a_2)^3 b_2 = 0,$$
$$a_1 (b_1)^3 + a_2 (b_2)^3 = 1,$$

so that

$$(a_1T+b_1)^4+(a_2T+b_2)^4+(a_3T)^4=T+(b_1)^4+(b_2)^4$$

and T is sum of three 4−th powers of linear polynomials. Therefore, $v(q,4)\le 3$ and by Remark 3.1 we get $v(q,4)=3$. Suppose $q=81$. From [5, Corollary 3.3], $v(q,4)\le 4$. We prove that $v(q,4)\ge 4$. Suppose $v(q,4)=3$. Then, there exists $(u_1,v_1u_2,v_2,u_3,v_3)\in F^6$ such that

$$T=\sum_{i=1}^{3}(u_iT+v_i)^4.$$

If $u_3=0$, the change $U=T-v_3^4$ shows that $v(q,4)=2$ and leads to a contradiction. Thus, $u_3\ne 0$. Now, the change $U=T+v_3u_3^{-1}$ shows that there exists $(a_1,a_2,b_1,b_2,a_3)\in F^5$ such that

$$T=\sum_{i=1}^{2}(a_iT+b_i)^4+(a_3T)^4,$$

so that $(\mathcal{F}(0))$ admits a solution in contradiction with Corollary 2.1. ■

Corollary 3.1. *If $q\notin\{3,9\}$, then $\mathcal{S}(q,4)=F[T]$. More precisely, if $q=27$ or if $q>81$, then, every $A\in F[T]$ is a sum of 3 fourth powers, and if $q=81$, then, every $A\in F[T]$ is a sum of 4 fourth powers.*

We are ready to present our first result.

Proposition 3.2. (I) *Suppose $q>81$ and congruent to 1 (mod 4). Then,*

(i) every $P\in F[T]$ of degree ≥ 329 is the strict sum of 9 fourth powers;

(ii) every $P\in F[T]$ of degree ≥ 89 is the strict sum of 10 fourth powers;

(iii) every $P\in F[T]$ of degree ≥ 53 is the strict sum of 12 fourth powers; (iv) every $P\in F[T]$ such that $29\le\deg P\le 52$ is the strict sum of 19 fourth powers.

(II) *Suppose $q=81$ or $q\ge 27$ congruent to 3 (mod 4). Then,*

(i) every $P\in F[T]$ of degree ≥ 329 is the strict sum of 10 fourth powers;

(ii) every $P\in F[T]$ of degree ≥ 89 is the strict sum of 11 fourth powers;

(iii) every $P\in F[T]$ of degree ≥ 53 is the strict sum of 13 fourth powers;

(iv) every $P\in F[T]$ such that $29\le\deg P\le 52$ is the strict sum of 20 fourth powers.

Proof. From the first part of Proposition 1.2, if $m\ge[\frac{\log 4}{\log(4/3)}]=4$, then, every $P\in F[T]$ of degree $\ge n(m,4)=4\lceil\frac{8-16(1-\frac{1}{4})^{m+1}}{1-4(1-\frac{1}{4})^{m+1}}\rceil-3$ is the strict sum of $m+v(q,4)+\max(\ell(q,4),1+\lambda(q,4))$ fourth powers.
Moreover, if $m\ge\frac{\log 4}{\log(4/3)}$, then, $n(m,4)\le 89$. Suppose $q>81$ congruent to 1 (mod 4). From Propositions 2.5 and 3.1, $v(q,4)+\max(\ell(q,4),1+\lambda(q,4))=5$. Then, every $P\in F[T]$ of degree $\ge n(4,4)=329$ is the strict

sum of 9 fourth powers and every $P\in F[T]$ of degree ≥ 89 is the strict sum of 10 fourth powers. We get the other points using parts II and III of Proposition 1.1. When $q=81$, or when $q\ne 3$ is congruent to 3 (mod 4), then

$v(q,4) + \max(\ell(q,4), 1 + \lambda(q,4) = 6$ so that an additional fourth power is necessary. ■

Corollary 3.2. *If $q \notin \{9, 81\}$ is congruent to* 1 (mod 4)*, then $G(q,4) \leq 9$.*
If $q = 81$, then $G(q,4) \leq 10$.
If $q \neq 3$ is congruent to 3 (mod 4)*, then $G(q,4) \leq 10$.*

Proof. Given by the first part of the previous proposition. ■

We end this section with the following proposition which is the case $p = 3$ of Proposition 4.4 in [3].

Proposition 3.3. *For every integer $n \geq 3$, there exists $B_n \in \mathbb{F}_9[T]$ of degree $4n$ which is a sum of 3 fourth powers and which is not a strict sum of fourth powers, so that $G(9,4) = \infty$.*

4. Identities and strict sums of small degree

Proposition 4.1. (I) *Suppose $q \geq 27$. Let $A \in F[T]$ with $\deg A \leq 4$. Then, A is a strict sum of fourth powers if and only if*

$$A = aT^4 + bT^3 + cT + d$$

with $a, b, c, d \in F$. Moreover, such a polynomial is a strict sum of 5 fourth powers if $q \neq 81$ and a strict sum of 6 fourth powers if $q = 81$.

(II) *If $q \geq 27$, then $\mathcal{S}(q,4) \neq \mathcal{S}^\star(q,4)$ and $g(q,4) = \infty$.*

Proof. Let $A \in F[T]$ be a strict sum of fourth powers and suppose that $\deg A \leq 4$. Then A is a sum of polynomials $A_i = (x_i T + y_i)^4$ with $x_i, y_i \in F$. Now, $A_i = x_i^4 T^4 + x_i^3 y_i T^3 + x_i y_i^3 T + y_i^4$ so that $A = aT^4 + bT^3 + cT + d$ with $a, b, c, d \in F$. We note that T^2 is not a strict sum of 4-th powers.

We suppose $q \geq 27$. From Corollary 3.1, every $P \in F[T]$ is a sum of 4-th powers. This proves the second part of the proposition. Let $(a, b, c, d) \in F^4$. From Corollary 2.1, if $q \neq 81$, then $(\mathcal{F}(a))$ has a solution, say (x_1, x_2, x_3), if $q = 81$, then $(\mathcal{G}(a))$ has a solution, say (x_1, x_2, x_3, x_4). Let $(y_1, y_2) \in F^2$ be solution of $(\mathcal{E}(x_1, x_2, b, c))$ with $(\mathcal{E}(u, v, w, t))$ defined at Lemma 2.1, that is

$$b = x_1^3 y_1 + x_2^3 y_2,$$
$$c = x_1 y_1^3 + x_2 y_2^3$$

According to Proposition 2.5, $d - y_1^4 - y_2^4$ is a sum of 2 fourth powers, say

$$d = y_1^4 + y_2^4 + z_1^4 + z_2^4$$

Then, if $q \neq 81$,

$$A = (x_1 T + y_1)^4 + (x_2 T + y_2)^4 + (x_3 T)^4 + (z_1)^4 + (z_2)^4,$$

so that A is a strict sum of 5 fourth powers and if $q = 81$,

$$A = (x_1T + y_1)^4 + (x_2T + y_2)^4 + (x_3T)^4 + (x_4T)^4 + (z_1)^4 + (z_2)^4,$$

so that A is a strict sum of 6 fourth powers. ■

The following very simple proposition is the key of the method.

Proposition 4.2. *For $r \in \{0, 1, 2\}$ and $X \in F[T]$ let*

$$L_r(X) = X^3T^r + XT^{3r}. \tag{4.1}$$

Then, L_r is additive,

$$L_r(X) = (X - T^r)^4 - (X + T^r)^4 = (X - T^r)^4 + (X + T^r)^4 + (X + T^r)^4, \tag{4.2}$$

$$L_r(X) + T^{4r} = (X + T^r)^4 - X^4, \tag{4.3}$$

$$L_r(X) - T^{4r} = X^4 - (X - T^r)^4, \tag{4.4}$$

and for every $b \in F$,

$$L_r(X + bT^r) = L_r(X) + (b^3 + b)T^{4r}. \tag{4.5}$$

Proof. Immediate. ■

Proposition 4.3. *Suppose that $q \notin \{3, 9, 81\}$. Let $A \in F[T]$ be such that $4 < \deg A \leq 8$. Then, A is the strict sum of 8 fourth powers. Let $A \in \mathbb{F}_{81}[T]$ be such that $4 < \deg A \leq 8$. Then A is the strict sum of 10 fourth powers.*

Proof. Let

$$A = \sum_{n=0}^{8} a_n T^n$$

be a polynomial of $F[T]$ of degree ≤ 8. We want to prove that there exists a positive integer s and, for $i = 1, \ldots, s$, polynomials

$$X_i = \sum_{n=0}^{2} x_{i,n} T^n$$

such that

$$A = \sum_{i=0}^{s} (X_i)^4.$$

In other words, we want to prove that there exists a positive integer s such that the system $((\epsilon_8), (\epsilon_7), \ldots, (\epsilon_1), (\epsilon_0))$ is solvable in F^{3s}, (ϵ_n) denoting the equation

$$a_n = \sum_{r=1}^{s} \sum_{\substack{n=3u+v \\ 0\leq u\leq 2 \\ 0\leq v\leq 2}} (x_{r,u})^3 x_{r,v}. \tag{ϵ_n}$$

We suppose $q \neq 81$.

First step: Corollary 2.1 implies the existence of a solution $(x_{1,2}, x_{2,2}, x_{3,2}) \in F^3$ of $(\mathcal{F}(a_8))$. Then, $x_{1,2}x_{2,2} \neq 0$ and $x_{1,2}^8 \neq x_{2,2}^8$. Let $(x_{1,1}, x_{2,1})$ be solution of $(\mathcal{E}(x_{1,2}, x_{2,2}, a_7, a_5))$ and let $(x_{1,0}, x_{2,0})$ be solution of $(\mathcal{E}(x_{1,2}, x_{2,2}, a_6, a_2))$, with $(\mathcal{E}(u, v, w, t))$ defined at Lemma 2.1. Let $x_{3,1} = x_{3,0} = 0$. Then, with $s = 3$, equations $(\epsilon_8), (\epsilon_7), (\epsilon_6), (\epsilon_5), (\epsilon_2)$ are satisfied.

Second step: Let $x_{4,2} = x_{5,2} = x_{6,2} = 0$. Corollary 2.1 implies the existence of a solution $(x_{4,1}, x_{5,1}, x_{6,1}) \in F^3$ of $(\mathcal{F}(a_4 - x_{1,0}^4 - x_{2,0}^4))$. Let $(x_{4,0}, x_{5,0})$ be solution of $(\mathcal{E}(x_{4,1}, x_{5,1}, a_3 - (x_{1,1})^3 x_{1,0} - (x_{2,1})^3 x_{2,0}, a_1 - x_{1,1}(x_{1,0})^3 - x_{2,1}(x_{2,0})^3))$. Then, with $s = 6$, equations $(\epsilon_8), (\epsilon_7), \ldots, (\epsilon_2), (\epsilon_1)$ are satisfied.

For $\nu = 1, 2, 3$, let

$$X_\nu = \sum_{j=0}^{2} x_{\nu,j} T^j.$$

Then,

$$A - \sum_{\nu=1}^{6} (X_\nu)^4 = a_0 - x_{1,0}^4 - x_{2,0}^4 - x_{4,0}^4 - x_{5,0}^4 = b$$

with $b \in F$.

Last step: Since F is a 4-Waring field, b is the sum of $\ell(q, 4) = 2$ fourth powers, so that A is the sum of 8 fourth powers.

The proof is similar when $q = 81$. The first and second steps need 4 fourth powers. ■

Lemma 4.1. *Suppose $F \subset \mathbb{F}_9$. Let $A \in F[T]$ be a sum of fourth powers. Then, $T^9 - T$ divides $A^3 - A$.*

Proof. Let $x \in \mathbb{F}_9$. Since $A \in \mathbb{F}_9[T]$, $A(x)$ is a sum of fourth powers in $\mathbb{F}_9$ Thus, $A(x) \in \mathbb{F}_3$, so that $A(x)^3 - A(x) = 0$. Therefore, $A^3 - A$ is divisible by $(T + x)$ for every $x \in \mathbb{F}_9$, so that, $T^9 - T = \prod_{x \in \mathbb{F}_9} (T + x)$ divides $A^3 - A$. ■

Proposition 4.4. *Suppose $F \subset \mathbb{F}_9$. Let*

$$A = \sum_{n=0}^{8} a_n T^n$$

be a polynomial of $F[T]$ of degree ≤ 8 such that $T^9 - T$ divides $A^3 - A$. Then,

(I) *for $n = 3j + i$ with $0 \leq j < 3, 0 \leq i < 3$, we have*

$$a_n = (a_{\bar{n}})^3$$

where, $\bar{n} = 3i + j$,

(II-1) *if $F = \mathbb{F}_3$ and $\deg A \leq 4$, then A is a strict sum of 3 fourth powers,*

(II-2) *if $F = \mathbb{F}_3$ and $4 < \deg A \leq 8$, then A is a strict sum of 6 fourth powers,*

(III-1) *if $F = \mathbb{F}_9$ and $\deg A \leq 4$, then A is a strict sum of 3 fourth powers,*

(III-2) *if $F = \mathbb{F}_9$ and $4 < \deg A \leq 8$, then A is a strict sum of 5 fourth powers.*

Proof. (I) Let

$$A = A_0 + A_1 T^3 + A_2 T^6$$

be the expansion of A in base T^3. Thus, for $j = 0, 1, 2$,

$$A_j = a_{3j} + a_{3j+1} T + a_{3j+2} T^2.$$

Then,

$$A^3 = \sum_{j=0}^{2} (A_j)^3 \left(T^{9j} - T^j\right) + \sum_{j=0}^{2} (A_j)^3 T^j.$$

For $j = 0, 1, 2,\quad T^{9j} - T^j$ is congruent to 0 (mod $T^9 - T$). Thus,

$$A^3 \equiv \sum_{j=0}^{2} (A_j)^3 T^j \pmod{(T^9 - T)}$$

and

$$A^3 - A \equiv \sum_{j=0}^{2} (A_j)^3 T^j - \sum_{j=0}^{2} A_j T^{3j} \pmod{(T^9 - T)}. \tag{4.6}$$

For $j = 0, 1, 2,\quad \deg((A_j)^3 T^j) \le 8$ and $\deg(A_j T^{3j}) \le 8$. Hence, by (4.6),

$$\sum_{j=0}^{2} \left((A_j)^3 T^j - A_j T^{3j}\right) = 0$$

that is

$$\sum_{j=0}^{2} \sum_{k=0}^{2} (a_{3j+k})^3 T^{3k+j} - \sum_{j=0}^{2} \sum_{k=0}^{2} a_{3j+k} T^{3j+k} = 0. \tag{4.7}$$

Let $n \in \{0, \ldots, 8\}$. By euclidean division, n is uniquely written as $n = 3u + v$, with $u, v < 3$. Set $\bar{n} = 3v + u$. By (4.7),

$$a_n = a_{3u+v} = (a_{3v+u})^3 = (a_{\bar{n}})^3. \tag{4.8}$$

this proves (I).

Let $n \in \{1, \ldots, 7\}$ be non divisible by 4. If $n = 3j + k$ with $0 \le j < 3, 0 \le k < 3$, then

$$a_n T^n + a_{\bar{n}} T^{\bar{n}} = (a_{3k+j})^3 T^{3j+k} + (a_{3k+j}) T^{3k+j}.$$

By (4.1),

$$a_n T^n + a_{\bar{n}} T^{\bar{n}} = L_k(a_{3k+j} T^j).$$

For n divisible by 4, equality (4.8) gives $a_n = (a_n)^3$, proving that a_n belongs to the prime field $\mathbb{F}_3$, this fact being obvious when $F = \mathbb{F}_3$.

(II) Suppose $F = \mathbb{F}_3$. Firstly, suppose $\deg A \le 4$. The result is obvious for the constants. Up to the changes $T \mapsto -T$, $T \mapsto T \pm 1$, $T \mapsto -T \pm 1$, we have to consider the following polynomials:

(i) $T^3 + T = (T+1)^4 + (T+1)^4 + (T-1)^4$,
(ii) $T^4, T^4+1, T^4-1 = T^4+1+1$,
(iii) $-T^4 = T^4 + T^4, -T^4+1 = T^4+T^4+1, -T^4-1 = (T+1)^4 + (T-1)^4$.
Each of them is a strict sum of 3 fourth powers.

Suppose now $\deg A \leq 8$. If $a_8 \neq 0$, we write

$$A = a_0 + L_0(a_1T) + a_4T^4 + L_2(a_6 + a_7T) + a_8T^8.$$

We have seen above that $a_0 + L_0(a_1T) + a_4T^4$ is a sum of 3 fourth powers of polynomials ≤ 1. By (4.3) and (4.4), $L_2(a_2 + a_5T) + a_8T^8$ is a sum of 3 fourth powers of polynomials of degree ≤ 2, so that A is a strict sum of 6 fourth powers. If $a_8 = 0$, we write

$$A = a_0 + L_0(a_1T + a_2T^2) + a_4T^4 + L_1(a_5T^2),$$

and by (4.3) an (4.4), A is a strict sum of 6 fourth powers.

(III) Suppose $F = \mathbb{F}_9$. The trace map $y \mapsto y^3 + y$ from F to $\mathbb{F}_3$ is onto. For every $k = 0, 1, 2$, there is $u_k \in F$ such that

$$a_{4k} = u_k^3 + u_k.$$

Moreover, since $a_{4k} \in \mathbb{F}_3$, we have $a_{4k} = v_k^3$ with $v_k \in F$.

If $\deg A \leq 4$, then, $a_2 = a_5 = 0$, and

$$A = (v_1T)^4 + u_0^3 + u_0 + L_0(a_1T),$$

so that by (4.5), then (4.3) and (4.4), A is a sum of 3 fourth powers of polynomials whose degrees are ≤ 1 and A is a strict sum of 3 fourth powers.

Now, suppose $\deg A > 4$. Proceeding as in the $\mathbb{F}_3$ case, we get that A is a strict sum of 5 fourth powers. ■

5. The descent

In this section, we describe a new descent process which works for exponent 4 and characteristic 3.

Proposition 5.1. *Let $n \geq 3$ be an integer and let $X \in F[T]$ be such that* $\deg X < 3n$. *Then, there exist $Y_0, Y_1, Y_2, R \in F[T]$ such that*

$$X = \sum_{r=0}^{2} L_r(Y_r) + R, \tag{5.1}$$

$$\deg(Y_r) < n \qquad if \qquad 0 \leq r \leq 2, \tag{5.2}$$

$$\deg R < 9, \tag{5.3}$$

$$R = \sum_{r=0}^{2}\sum_{j=0}^{r} a_{3j+r}T^{3j+r}, \tag{5.4}$$

with $a_{3j+r} \in F$.

Proof. Set

$$X = \sum_{j=0}^{3n-1} x_j T^j \tag{5.5}$$

with $x_j \in F$ for $j = 0, \ldots, 3n-1$. For $j = 0, \ldots, 3n-1$, let $\xi_j \in F$ be defined by

$${\xi_j}^3 = x_j. \tag{5.6}$$

(I) Suppose $n = 3$. Then,

$$X = (\xi_3 T + \xi_6 T^2)^3 + T(\xi_7 T^2)^3 + \sum_{r=0}^{2} T^r \left(\sum_{j=0}^{r} x_{3j+r} T^{3j} \right)$$

and by (4.1),

$$X = \sum_{r=0}^{1} L_r\Big(\sum_{j=r+1}^{2} \xi_{3j+r} T^j \Big) - \xi_3 T - \xi_6 T^2 - \xi_7 T^5 + \sum_{r=0}^{2} T^r \left(\sum_{j=0}^{r} x_{3j+r} T^{3j} \right).$$

Thus,

$$X = \sum_{r=0}^{2} L_r(Y_r(X)) + R(X)$$

with $Y_2 = 0$,

$$Y_r(X) = \sum_{j=r+1}^{2} \xi_{3j+r} T^j$$

for $r = 0, 1$ and

$$R(X) = \sum_{r=0}^{2} \sum_{j=0}^{r} a_{3j+r} T^{3j+r},$$

that is $R(X)$ of the form (5.4). We note that $\deg(Y_r(X)) < 3$.

(II) Suppose $n = 4$. Then,

$$X = L_2(\xi_{11} T^3) + L_1(\xi_{10} T^3) + (x_9 - \xi_{11}) T^9 + X'$$

with

$$\deg X' < 9.$$

Set $x_9 - \xi_{11} = \eta^3$. Then,

$$(x_9 - \xi_{11}) T^9 = L_0(\eta T^3) - \eta T^3,$$

so that

$$X = L_2(\xi_{11} T^3) + L_1(\xi_{10} T^3) + L_0(\eta T^3) + Y$$

with $\deg Y < 9$. From the case $n = 3$,

$$X = \sum_{r=0}^{2} L_r(Y_r(X)) + R(X)$$

with $R(X)$ of the required form (5.4) and $\deg Y_r(X) \leq 3$ for $r = 0, 1, 2$.

(III) Suppose now $n > 4$. We proceed inductively. Set

$$Z_r(X) = \sum_{j=0}^{n-1} \xi_{3j+r} T^j \tag{5.7}$$

and

$$\Phi(X) = -\sum_{r=0}^{2} Z_r T^{3r}, \tag{5.8}$$

so that

$$\deg Z_r(X) < n; \quad \deg \Phi(X) \leq n + 5 \tag{5.9}$$

and

$$X = \sum_{r=0}^{2} L_r(Z_r(X)) + \Phi(X). \tag{5.10}$$

(i) Step 0. Set

$$X = X_0, n = n_0, \tag{5.11}$$

so that

$$\deg X_0 < 3n_0. \tag{5.12}$$

(ii) Steps $1, \ldots, k, \ldots$ For $k \geq 1$, let

$$n_k = \lceil \frac{n_{k-1}}{3} \rceil + 2, \tag{5.13}$$

$$X_k = \Phi(X_{k-1}) \tag{5.14}$$

$$Y_{r,k} = Z_r(X_{k-1}) \tag{5.15}$$

for $r = 0, 1, 2$. Then, by (5.10), (5.11), (5.14) and (5.15),

$$X = \sum_{r=0}^{2} L_r(\sum_{i=1}^{k} Y_{r,i}) + X_k. \tag{5.16}$$

By (5.9) and (5.13),

$$\deg Y_{r,k} < n_{k-1}, \quad \deg X_k < 3n_k.$$

If $n_i > 4$, then $n_{i+1} < n_i$; if $n_i = 3, 4$, then $n_{i+1} = n_i$. Let k be the least integer such that $n_k \leq 4$. From (5.16), using results given by parts (I) or (II), we get

$$X = \sum_{r=0}^{2} L_r \left(\sum_{i=1}^{k} Y_{r,i} + Y_r(X_k) \right) + R(X_k).$$

The degree conditions (5.9), (5.11) and (5.13) imply

$$\deg\left(\sum_{i=1}^{k} Y_{r,i} + Y_r(X_k)\right) < n.$$

■

Corollary 5.1. *Suppose $F \subset \mathbb{F}_9$. Then, $\mathcal{S}(q,4)$ is the subset of $F[T]$ formed by the polynomials A such that $A^3 - A$ is multiple of $T^9 - T$.*

Proof. From Lemma 4.1,

$$\mathcal{S}(q,4) \subset \left\{A \in F[T] \mid (T^9 - T)|A^3 - A\right\}.$$

Conversely, let $X \in F[T]$ be such that $T^9 - T$ divides $X^3 - X$. By (5.1) and (5.3), X may be written as a sum

$$X = \sum_{r=0}^{2} L_r(Y_r) + R$$

with $Y_1, Y_2, Y_3, R \in F[T]$ and $\deg R < 9$. By (4.2), for $r = 0, 1, 2$, $L_r(Y_r) \in \mathcal{S}(q,4)$ so that from Lemma 4.1, $(L_r(Y_r))^3 - L_r(Y_r)$ is multiple of $T^9 - T$. Thus, $R^3 - R$ is multiple of $T^9 - T$. From Proposition 4.4, R is a sum of 4-th powers so that, using Proposition 4.2, we get that X is a sum of 4−th powers. ■

We are now ready to present our second result.

Proposition 5.2. (I) *Suppose $q > 81$ and q congruent to* 1 (mod 4). *Then,*

(i) *every $H \in F[T]$ of degree ≥ 29 is the strict sum of* 14 *fourth powers.*

(ii) *every $H \in F[T]$ of degree ≥ 9 is the strict sum of* 19 *fourth powers.*

(iii) *every $H \in F[T]$ such that $5 \leq \deg P \leq 8$ is the strict sum of* 8 *fourth powers.*

(II) *Suppose that $q = 81$. Then,*

(i) *every $H \in F[T]$ of degree ≥ 29 is the strict sum of* 15 *fourth powers.*

(ii) *every $H \in F[T]$ of degree ≥ 9 is the strict sum of* 21 *fourth powers.*

(iii) every $H \in F[T]$ such that $5 \leq \deg P \leq 8$ is the strict sum of 10 *fourth powers.*

(III) *Suppose q congruent to* 3 (mod 4) *and $q \geq 27$. Then,*

(i) *every $H \in F[T]$ with degree ≥ 29 is the strict sum of* 15 *fourth powers*

(ii) *every $H \in F[T]$ of degree ≥ 9 is the strict sum of* 20 *fourth powers.*

(iii) every $H \in F[T]$ such that $5 \leq \deg P \leq 8$ is the strict sum of 8 *fourth powers.*

(IV) *Suppose $F = \mathbb{F}_3$. Then*

(i) *every $H \in \mathcal{S}(3,4)$ is a strict sum of* 12 *fourth powers.*

(ii) *every $H \in F[T]$ with degree multiple of* 4 *is a strict sum of* 11 *fourth powers.*

(V.i) *Every $H \in \mathcal{S}(9,4)$ with degree non multiple of* 4 *is a strict sum of* 8 *fourth powers.*

(V.ii) *Every $H \in \mathcal{S}(9,4)$ of degree multiple of* 4 *and whose leading coefficient belongs to $\mathbb{F}_3$ is a strict sum of* 7 *fourth powers.*

Proof. The claims (I.(iii))-(III.(iii)) are given by the second part of Proposition 4.3. We prove the other ones. Let $H \in F[T]$ and let n be the integer defined by

$$4(n-1) < \deg H \leq 4n. \tag{5.17}$$

If $n \leq 2$, we conclude using Proposition 4.4. We suppose $n \geq 3$. According to [3, Lemma 5.1], there exist $B_1, \ldots, B_\lambda, P \in F[T]$ such that

$$H = B_1^4 + B_\lambda^4 + P \tag{5.18}$$

with

$$\lambda = \lambda(q,k),$$
$$\deg B_1 \leq n, \ldots, \deg B_\lambda \leq n, \deg P = 4n, \tag{5.19}$$

the leading coefficient of P being a fourth power.

According to [3, Lemma 5.2], there exist $X, Y \in F[T]$ such that:

$$P = Y^4 + X, \tag{5.20}$$

$$\deg X < 3n, \deg Y = n. \tag{5.21}$$

From Proposition 5.1, there exist $Y_0, Y_1, Y_2, R \in F[T]$ such that

$$X = \sum_{r=0}^{2} L_r(Y_r) + R, \tag{5.1}$$

$$\deg(Y_r) < n$$

for $0 \leq r < 3$ and

$$\deg R < 9. \tag{5.3}$$

(A) We suppose $q \notin \{3, 9\}$. By (4.2),

$$L_r(Y_r) = (Y_r - T^r)^4 + 2(Y_r + T^r)^4.$$

Thus,

$$X = \sum_{r=0}^{2} \left((Z_{r,1})^4 + (Z_{r,2})^4 + (Z_{r,3})^4\right) + R, \tag{5.22}$$

where for $j = 1, 2, 3$, $Z_{r,j}$ is a polynomial such that

$$\deg Z_{r,j} \leq \max(r, n-1). \tag{5.23}$$

Set $v = v(q, 4)$. Then, there exist $a_1, b_1, \ldots, a_v, b_v$ in F such that

$$R = (a_1 R + b_1)^4 + \ldots + (a_v R + b_v)^4. \tag{5.24}$$

From (5.18), (5.20), (5.22) and (5.24),

$$H = B_1^4 + B_\lambda^4 + Y^4 + \sum_{r=0}^{2} \sum_{j=1}^{3} (Z_{r,j})^4 + \sum_{i=1}^{v} (a_i R + b_i)^4, \tag{5.26}$$

so that H is written as a sum of $\lambda + v + 10$ fourth powers of polynomials. From (5.19), (5.21) and (5.23), these polynomials have their degrees bounded by $\max(n, 8)$. By (5.17), if $n \geq 8$, the above sum is a strict one.

On the other hand, in view of Proposition 4.3, since $\deg R < 9$, R may be written as a sum

$$R = \sum_{r=1}^{s(q)} (R_s)^4 \tag{5.27}$$

where $R_1, \ldots, R_{s(q)}$ are polynomials of degree ≤ 2 and where $s(q) = 8$ if $q \neq 81$ and $s(q) = 10$ if $q = 81$. Thus, by (5.18), (5.20) and (5.22),

$$H = B_1^4 + B_\lambda^4 + Y^4 + \sum_{r=0}^{2} \sum_{j=1}^{3} (Z_{r,j})^4 + \sum_{r=1}^{s(q)} (R_r)^4, \tag{5.28}$$

so that H is sum of $\lambda + 10 + s(q)$ fourth powers. From (5.17), if $n \geq 2$, then (5.28) is a strict representation.

The proof of the three first parts is complete after observing that in the case (I) we have $v(q,4) = 3, \lambda(q,4) = 1$, in the case (II), we have $v(q,4) = 4, \lambda(q,4) = 1$, and in the case (III), we have $v(q,4) = 3, \lambda(q,4) = 2$.

(B) We suppose $F \subset \mathbb{F}_9$. In addition, in the case when $q = 9$ and $\deg H = 4n$, we suppose that the leading coefficient of H is in $\mathbb{F}_3$. When $F = \mathbb{F}_9$, since the leading coefficient of H is a 4-th power, we take $P = H$ in (5.18) so that $B_1 = B_\lambda = 0$.

At this point, observe that R satisfies

$$R = \sum_{r=0}^{2} \sum_{j=0}^{r} a_{3j+r} T^{3j+r}. \tag{5.4}$$

In view of (5.1) and (4.2), $X - R$ is a sum of fourth powers. From (5.18) and (5.20), $H - R$ is a sum of fourth powers. Since $H \in \mathcal{S}(q,4)$, R is also a sum of fourth powers. From Lemma 4.1 and Proposition 4.4-(I), all coefficients a_{3j+r} of R with $j > r$ are equal to 0 and all coefficients a_{4r} are in the prime field $\mathbb{F}_3$. Therefore,

$$R = \sum_{r=0}^{2} y_r T^{4r},$$

with $y_1, y_1, y_2 \in \mathbb{F}_3$. By (5.1),

$$X = \sum_{r=0}^{2} \left(L_r(Y_r) + y_r T^{4r} \right).$$

(B.1) Suppose $F = \mathbb{F}_3$ Then, $\lambda = 2$.

In view of (4.2), (4.3) and (4.4), X is sum of 9 fourth powers. By (5.18) and (5.20), H is a sum of $\lambda + 10 = 12$ fourth powers.

Since $n \geq 3$, this sum is a strict one. Suppose in addition that $\deg H = 4n$. The leading coefficient of H is a sum of at most 2 fourth powers, say $b^4 + c^4$. In (5.18), we can take $B_\lambda = 0$ and $P = H - b^{4n}T^{4n}$, so that H is a sum of 11 fourth powers.

(B.2) Suppose $F = \mathbb{F}_9$. Then -1 is a fourth power and $\lambda = 1$.

For $r = 0, 1, 2$, $y_r = -(y_r)^3 - y_r$, so that

$$L_r(Y_r) + y_r T^{4r} = L_r(Y_r) - ((y_r)^3 + y_r)T^{4r}.$$

By (4.5), then (4.2), X is the sum of 6 fourth powers. From (5.18) and (5.20), H is the sum of 8 fourth powers. Moreover, if $\deg H = 4n$, we have $H = P$ so that H is the sum of 7 fourth powers. As above, this sum is a strict one. ■

Corollary 5.2. (I) *Suppose $q \geq 27$. Then, $\mathcal{S}^\star(q,4)$ is the union of the set $\{A \in F[T] \mid \deg A > 4\}$ and the set of polynomials*

$$A = aT^4 + bT^3 + cT + d$$

with $a, b, c, d \in F$. Moreover,

(i) if $q > 81$ is congruent to 1 (mod 4)*, then*

$$G(q,4) = G^\star(q,4) \leq 9,$$

$$g(q,4) = \infty, g^\star(q,4) \leq 19;$$

(ii)

$$G(81,4) = G^\star(81,4) \leq 10,$$

$$g(81,4) = \infty, g^\star(81,4) \leq 21;$$

(iii) if $q \geq 27$ is congruent to 3 (mod 4)*, then*

$$G(q,4) = G^\star(q,4) \leq 10,$$

$$g(q,4) = \infty, g^\star(q,4) \leq 20.$$

(II) $\mathcal{S}^\star(3,4) = \mathcal{S}(3,4) = \left\{A \in F[T] \mid A^3 - A \equiv 0 \pmod{(T^9 - T)}\right\}$,

$$G(3,4) \leq g(3,4) \leq 12, G^\star(q,4) \leq g^\star(3,4) \leq 12.$$

(III) $\mathcal{S}(9,4) = \left\{A \in F[T] \mid A^3 - A \equiv 0 \pmod{(T^9 - T)}\right\}$, $\mathcal{S}^\star(9,4)$ *is the set of $A \in \mathcal{S}(9,4)$ such that either* $\deg A$ *is not multiple of* 4*, or* $\deg A$ *is multiple of* 4 *and the leading coefficient of A is in the prime field* $\mathbb{F}_3$*;*

$$G(9,4) = g(9,4) = \infty, G^\star(9,4) \leq g^\star(9,4) \leq 8.$$

Proof. Apply Propositions 3.2, 3.3, 4.1, 4.3 and 4.4. ■

Bibliography

[1] R. Balasubramanian, J.-M. Deshouillers, F. Dress, *Problème de Waring pour les bicarrés. I: schéma de la solution*, C.R. Acad. Sci., Paris, Sér. I 303 (1986), 85-88.

[2] N. Bourbaki, *Eléments de mathématique,* Fascicule XI, chap. 5, Hermann, Paris (3rd. Ed), 1973.

[3] M. Car, *New bounds on some parameters in the Waring problem for polynomials over a finite field*, Contemporary Math, **461**, (2008) 59-77.

[4] M. Car, L. Gallardo, *Sums of cubes of polynomials*, Acta Arith. **112** (2004), 41-50

[5] M. Car, L. Gallardo, *Waring's problem for biquadrates over a finite field of odd characteristic*, Funct. Approx. Comment. Math., **37.1** (2007), 39-50.

[6] H. Davenport, *On Waring's problem for fourth powers*, Annals of Math. (2) 40, (1939), 731-747

[7] J.-M. Deshouillers; K. Kawada and T. Wooley, *On sums of sixteen biquadrates*, Mem. Soc. Math. Fr (N.S) n° **100** (2005).

[8] L. Gallardo, *On the restricted Waring problem over* $\mathbb{F}_{2^n}[t]$, Acta Arith. **42** (2000), 109-113.

[9] C. Small, *Sums of powers in large finite fields*, Proc. Amer. Math. Soc. **65** (1977), 35-36.

[10] M. R. Stein, *Surjective stability in dimension* 0 *for* K_2 *and related functors*, Trans. Amer. Math. Soc. **178** (1973), 165-191.

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