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EXPLICIT ESTIMATES OF SOLUTIONS OF SOME DIOPHANTINE EQUATIONS

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Abstract: Let k be a fixed non-zero integer, and let x and y be integers such that

$$y^2 = x^3 + k.$$

We show that

$$\log \max\{|x|, |y|\} < \min_{(c,d)\in S}\{c|k|(\log |k|)^d\},$$

where

$$S = \{(10^{181}, 4), (10^{23}, 5), (10^{19}, 6)\}.$$

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1. Introduction

Baker [1] established the first explicit estimate for the solutions of the Mordell equation. In particular, Baker showed for any non-zero integer k, all solutions of the Mordell equation

$$Y^2 - X^3 = k,$$

in integers x and y satisfy

$$\log \max\{|x|, |y|\} < 10^{10^5}|k|^{10^4}.$$

Following Baker's work, Hall Jr. [12] advanced the following conjectures based on extensive computations.

Conjecture 1.1 (Hall Jr.)**.** For every $\epsilon > 0$, there exists a constant $c(\epsilon) > 0$ such that

$$0 < |y^2 - x^3| < c(\epsilon)|x|^{1/2+\epsilon}$$

for infinitely many integers x and y.

Conjecture 1.2 (Hall Jr.)**.** For all integers x and y such that $y^2 \neq x^3$,

$$|y^2 - x^3| > (1/5)\sqrt{|x|}.$$

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Conjecture 1.3 (Hall Jr.)**.** For all integers x and y such that $y^2 \neq x^3$, and for some absolute constant $c > 0$,

$$|y^2 - x^3| > c\sqrt{|x|}.$$

Danilov [8] showed that

$$0 < |y^2 - x^3| < (432\sqrt{2})\sqrt{|x|}$$

for infinitely many integers x and y, and thereby resolved Conjecture 1.1. Despite more computational evidence in favour of Conjecture 1.2 (see [11]), Elkies [9] discovered the equation

$$y^2 - x^3 = -1641843,$$

with solution

$$(x, y) = (5853886516781223, 447884928428402042307918),$$

where

$$\frac{|y^2 - x^3|}{\sqrt{x}} = (0.0214\ldots) < \frac{1}{5},$$

and thereby resolved Conjecture 1.2. Conjecture 1.3 remains open.

Stark [20] made the first giant step in the direction of Conjecture 1.3. In particular, Stark showed that given any $\epsilon > 0$, there exists a constant $c(\epsilon) > 0$ such that

$$\log \max\{|x|, |y|\} < c(\epsilon)|k|^{1+\epsilon}.$$

Sprindzuk [19] later showed that Baker's [2] estimate implies there exists an absolute constant $c > 0$ such that

$$\log \max\{|x|, |y|\} < c|k|(\log |k|)^6.$$

We show that

$$\log \max\{|x|, |y|\} < \min_{(c,d)\in S}\{c|k|(\log |k|)^d\},$$

where

$$S = \{(10^{181}, 4), (10^{23}, 5), (10^{19}, 6)\}.$$

The main new ingredient in establishing our result is a lemma due to Bombieri and Cohen [5] which enables us to apply sharp estimates for linear forms in 2 logarithms obtained by Laurent, Mignotte, and Nesterenko [13]. We note that Bilu and Bugeaud [4] have made use of this lemma for another purpose.

The article is organised as follows. In section 2, we establish an explicit lower bound for $\max\{|x|, |y|\}$ satisfied by infinitely many integers x and y. In section 3, we establish an explicit upper bound for $\max\{|x|, |y|\}$ assuming the truth of a conjecture due to Baker. Finally, in section 4, we establish our main stated explicit upper bound for $\log \max\{|x|, |y|\}$. We remark that the notation of every conjecture, definition, lemma, and theorem is self contained.

2. Explicit Lower Bound

In this section we establish an explicit lower bound for the naive height of the integer solutions of the Mordell equation satisfied by infinitely many integers. We follow Danilov's approach using Elkies' identity.

Theorem 2.1. *Let $k \in \mathbb{Z}$, $k \neq 0$. There exist infinitely many integers x and y such that $y^2 - x^3 = k$ and*

$$\max\{|x|, |y|\} > (0.1671\ldots)|k|^2.$$

Proof. Since $125v^2 - 114v + 26 = u^2$ has an integer solution $(u, v) = (61, -5)$, it has infinitely many integer solutions from the theory of Pell's equation, which we label (u_n, v_n), $n \geq 1$. Let

$$x_n = 3125v_n^2 - 3000v_n + 719,$$

and let

$$y_n = u_n(15625v_n^2 - 15375v_n + 3781).$$

Using Elkies [9] identity

$$\begin{aligned}&(125t^2 - 114t + 26)(15625t^2 - 15375t + 3781)^2\\&= (3125t^2 - 3000t + 719)^3 - 27(2t - 1),\end{aligned}$$

we deduce that

$$0 < |y_n^2 - x_n^3| = |54v_n - 27|.$$

Since

$$v_n = \pm\left(\frac{x_n + 1}{3125}\right)^{1/2} + \frac{12}{25},$$

it follows by the triangle inequality that

$$\begin{aligned}&|54v_n - 27|\\&= \left|\pm\frac{54}{\sqrt{3125}}(x_n + 1)^{1/2} + \frac{648}{25} - 27\right|\\&< \left(54\sqrt{\frac{2}{3125}} + \frac{27}{25}\right)\sqrt{|x_n|}.\end{aligned}$$

■

3. Explicit Conjectured Upper Bound

Recently, Baker [3] explored the connections between the *abc* conjecture, and the theory of linear forms in logarithms, and due to some computations completed at ETH Zurich using known *abc* examples, formulated a version of the *abc* conjecture with an explicit constant. In this section we make use of Baker's explicit *abc* conjecture in order to establish an explicit conditional upper bound on the naive height of the integer solutions of the Mordell equation. In order to apply Baker's conjecture, we need a technical result established by Pethö and de Weger.

Conjecture 3.1 (Baker [3])**.** If $a, b, c \in \mathbb{Z}$ such that $a+b+c=0$ and $\gcd(a,b,c) = 1$, then

$$\max\{|a|,|b|,|c|\} < \frac{6}{5}\frac{G(\log G)^w}{w!},$$

where $G = \prod_{p|abc} p$ and w is the number of distinct prime factors of abc.

Lemma 3.1. *Let $a \geq 0$, $h \geq 1$, $b > (e^2/h)^h$, and $x \in \mathbb{R}$ be the largest solution of $x = a + b(\log x)^h$. Then $x < 2^h(a^{1/h} + b^{1/h}\log(h^h b))^h$.*

Proof. See Pethö and de Weger [18], Lemma 2.2. ■

Theorem 3.1. *Assume the truth of Conjecture 3.1. Let k be a non-zero integer and let x and y be integers such that $y^2 - x^3 = k$. Then*

$$\max\{|x|,|y|\} < c_1(|k|(\log|k|)^w)^3,$$

where

$$c_1 = (2c_2\log(\sqrt[w]{e}c_2))^{2w} + 1/2,$$

$$c_2 = 10w\left(\frac{6}{5w!}\right)^{1/w},$$

and w is the number of distinct prime factors of

$$\frac{x^3y^2(x^3-y^2)}{(\gcd(x^3,y^2))^3}.$$

Proof. Assume first that $x \geq 3$, $y \geq 3$, $|k| \geq 3$, and $|y - x^{3/2}| < 1/2$. Let

$$\begin{aligned} d &= \gcd(x^3, y^2), \\ a &= y^2/d, \\ b &= -x^3/d, \\ c &= (x^3-y^2)/d, \\ G &= \prod_{p|abc} p, \end{aligned}$$

and let w be the number of distinct prime factors of abc. Note that

$$a + b + c = 0$$

and

$$\gcd(a,b,c) = 1.$$

By Conjecture 3.1, and the inequality

$$G \leq \frac{|xy||k|}{d},$$

it follows that

$$\max\{|a|,|b|,|c|\} < \frac{6}{5w!}\left(\frac{|xy||k|}{d}\right)\left(\log\left(\frac{|xy||k|}{d}\right)\right)^{w}.$$

Multiplying inequalities for $|a|$ and $|b|$ together, we obtain

$$x < \left(\frac{6}{5w!}\right)^{2}|k|^{2}\left(\log\left(\frac{|xy||k|}{d}\right)\right)^{2w}. \tag{3.1}$$

Since $|y - x^{3/2}| < 1/2$, and $|k| = |y - x^{3/2}||2y - (y - x^{3/2})|$, it follows that $|k| < |y| + 1/4$. We deduce from $y < x^{3/2} + 1/2$ and $|k| \leq |y|$ that

$$\log\left(\frac{|xy||k|}{d}\right) < 5\log x. \tag{3.2}$$

Substituting (3.2) in (3.1), we deduce

$$x < 5^{2w}\left(\frac{6}{5w!}\right)^{2}|k|^{2}(\log x)^{2w}.$$

It remains to apply Lemma 3.1 and the inequality $\max\{|x|,|y|\} < |x|^{3/2} + 1/2$. Finally, we let $M = \max\{|x|,|y|\}$, and note that in case

1. $x \geq 3$, $y \geq 3$, and $|y^2 - x^3| > 1/2$, then $M < 2|k|$.
2. $|k| < 3$, then, by classical results of Gauss, Euler and Wantzel, $M < 6$.
3. $0 < x < 3$, then $M < \max\{3, \sqrt{|27+k|}\}$.
4. $0 < y < 3$, then $M < \max\{\sqrt[3]{|9-k|}, 3\}$.
5. $y \leq 0$, then by symmetry can use results for $y > 0$.
6. $x \leq 0$, then $M < |k|$.

■

4. Explicit Upper Bound

In this last section, we explore three different trails branching out from Baker's garden with regards to establishing explicit upper bounds for the naive height of the integer solutions of the Mordell equation. Along one trail, we follow Baker's approach, and apply Matveev's recent refinements for linear forms in n logarithms. Along the other two trails, we explore Bombieri and Cohen's lead, already explored by Bugeaud, and Bilu and Bugeaud in another context, in order to obtain our explicit upper bounds unconditionally. We group this section into three subsections, the first of which consists of the preliminary lemmas and definitions, while the second consists of the three lemmas, corresponding to the three trails above, which we use in the last subsection in order to establish our main result.

4.1. Preliminary Lemmas and Definitions

Lemma 4.1. *Let $\mathbb{K}$ be an algebraic number field of degree $d = r_1 + 2r_2 = 3$, and let $R_{\mathbb{K}}$ and $D_{\mathbb{K}}$ denote the regulator and discriminant of $\mathbb{K}$, respectively. Then*

$$R_{\mathbb{K}} < (0.0736)\sqrt{|D_{\mathbb{K}}|}(\log|D_{\mathbb{K}}|)^2.$$

Furthermore, if $|D_{\mathbb{K}}| = 108|k|$, where $|k| \geq 3$ is an integer, then

$$R_{\mathbb{K}} < (24.7)\sqrt{|k|}(\log|k|)^2.$$

Proof. By Dirichlet's class number formula,

$$R_{\mathbb{K}} \leq h_{\mathbb{K}} R_{\mathbb{K}} = \frac{w_{\mathbb{K}}\sqrt{|D_{\mathbb{K}}|}}{2^{r_1}(2\pi)^{r_2}}\kappa_{\mathbb{K}}.$$

Suppose $r_1 = 3$ and $r_2 = 0$. Then $w_{\mathbb{K}} = 2$ and (see [15])

$$\kappa_{\mathbb{K}} \leq \frac{1}{8}(\log|D_{\mathbb{K}}|)^2.$$

It follows that

$$R_{\mathbb{K}} < (0.0314)\sqrt{|D_{\mathbb{K}}|}(\log|D_{\mathbb{K}}|)^2.$$

On the other hand, suppose that $r_1 = 1$ and $r_2 = 1$. Then $w_{\mathbb{K}} = 2$ and (see Louboutin [14])

$$\kappa_{\mathbb{K}} \leq \left(\frac{e\log|D_{\mathbb{K}}|}{2(d-1)}\right)^{d-1}.$$

It follows that

$$R_{\mathbb{K}} < (0.0736)\sqrt{|D_{\mathbb{K}}|}(\log|D_{\mathbb{K}}|)^2.$$

Let $|D_{\mathbb{K}}| = 108|k|$. Then

$$\begin{aligned}&(0.0736)\sqrt{|D_{\mathbb{K}}|}(\log|D_{\mathbb{K}}|)^2\\ &= (0.0736)\sqrt{108}(\log 108 + 1)^2|k|(\log|k|)^2\\ &< (24.7)|k|(\log|k|)^2.\end{aligned}$$

■

Definition 4.1. *Let $\mathbb{K}$ be an algebraic number field, $\alpha \in \mathbb{K}$. The minimal polynomial of α is*

$$f(X) = \sum_{j=0}^{d} a_j X^j = a_d \prod_{j=1}^{d}(X - \alpha^{(j)})$$

where $f(X)$ is non-zero and of smallest degree which has α as a root, has coprime coefficients, and has positive leading coefficient.

Definition 4.2. *The Mahler measure of α is*

$$M(\alpha) = |a_d|\prod_{j=1}^{d}\max\{1, |\alpha^{(j)}|\}.$$

Definition 4.3. *The absolute logarithmic Weil height of α is*

$$h(\alpha) = \frac{1}{d} \log M(\alpha).$$

Lemma 4.2. *Let $\mathbb{K}$ be an algebraic number field of degree $d \geq 2$. There exists in $\mathbb{K}$ a fundamental system of units $\eta_1, \ldots, \eta_r$ with the following properties:*

$$\prod_{j=1}^{r} h(\eta_j) \leq c_1 R_{\mathbb{K}}, \tag{4.1}$$

$$h(\eta_j) \leq c_2 R_{\mathbb{K}}, \tag{4.2}$$

$$|(e_{ij})_{1 \leq i,j \leq r}| \leq c_3, \tag{4.3}$$

where e_{ij} are the entries of the inverse matrix of $(\log|\eta_j^{(i)}|)_{1\leq i,j\leq r}$, and for every non-zero $\alpha \in O_{\mathbb{K}}$, not a root of unity, and every integer $m \geq 1$, there exists a unit η such that

$$M(\eta^m \alpha) \leq |N_{\mathbb{K}}(\alpha)|^{1/d} \exp(mc_4 R_{\mathbb{K}}), \tag{4.4}$$

where $N_{\mathbb{K}}(\alpha)$ is the norm of α, $c_1 = (r!)^2/(2^{r-1}d^r)$, $c_2 = c_1(\lambda(d))^{1-r}$, $c_3 = c_1 d^{r-1}/\lambda(d)$, $c_4 = r^{r+1}(d\lambda(d))^{-(r-1)}/2$, and where $\lambda(d) > 0$ is any of the existing functions of d which satisfy the inequality $h(\alpha) > \lambda(d)$. In particular, we use Voutier's estimate, and take $\lambda(d) = 2/(d(\log 3d)^3)$.

Proof. See Bugeaud and Györy, [6], and Voutier [21], Corollary 2, page 84. ■

Definition 4.4. *Let $\alpha_1, \ldots, \alpha_n$ denote $n \geq 2$ non-zero algebraic numbers, $\mathbb{K} = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$, $D = [\mathbb{K} : \mathbb{Q}]$, $b_1, \ldots, b_n$ denote rational integers, and let*

$$\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n,$$

where $\log \alpha_1, \ldots, \log \alpha_n$ are the principal values of the logarithms.

Lemma 4.3. *If $\Lambda \neq 0$, then*

$$\log |\Lambda| > -c(n, \chi) D^2 \log(eD) \Omega \log(eB),$$

where $B = \max_{1 \leq j \leq n}\{|b_j|\}$, $A_j, j = 1, \ldots, n$ are positive real numbers such that $\log A_j \geq \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\}$, $\Omega = \prod_{j=1}^{n} \log A_j$,

$$\chi = \begin{cases} 1 & \textit{if } \mathbb{K} \subseteq \mathbb{R}, \\ 2 & \textit{otherwise}, \end{cases}$$

and

$$c(n, \chi) = \min\{\frac{(en/2)^{\chi} 30^{n+3} n^{3.5}}{\chi}, 2^{6n+20}\}.$$

Proof. See Matveev [17], Corollary 2.3. ■

Lemma 4.4. *Suppose that* α_1 *and* α_2 *are multiplicatively independent algebraic numbers,* b_1, b_2 *are positive integers, and* $\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1$. *Further, let* $D' = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]/[\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}]$, A_1, A_2 *be real numbers greater than 1 such that* $\log A_i \geq \max\{h(\alpha_i), |\log \alpha_i|/D', 1/D'\}, i = 1, 2$, *and* $b' = \frac{b_1}{D' \log A_2} + \frac{b_2}{D' \log A_1}$. *If* $\Lambda \neq 0$, *then*

$$\log |\Lambda| \geq -30.9(D')^4 \log A_1 \log A_2 (\max\{\log b', 21/D', 1/2\})^2.$$

Proof. See Laurent, Mignotte, and Nesterenko [13], Corollary 1. ■

Lemma 4.5. *If* $0 \leq \theta < 1$ *and* $z \in \mathbb{C}$ *such that* $|z - 1| \leq \theta$, *then*

$$|\log z| \leq \frac{1}{1-\theta} |z - 1|,$$

where $\log z$ *denotes the principal part of the complex logarithm.*

Proof. This is Exercise 1.1 (b), [22]. ■

Definition 4.5. *Let* $A > e$ *be a positive real number. We define* $f(x, y) = \sum_{i=0}^{3} a_i x^i y^{3-i}$ *to be a monic irreducible binary cubic form with nonzero discriminant, and integer coefficients* $|a_i| \leq A$.

Lemma 4.6. *Let* n_i, d_i, g_i, $i = 1, \ldots, \tau$ *be rational integers, let* $T = |\sum_{i=1}^{\tau} d_i n_i|$, *and let* $\lambda_i, i = 1, \ldots, \tau$ *be positive real numbers with* $\prod_{i=1}^{\tau} \lambda_i = 1$. *Let* U, V, W *be positive integers with* $V > \max \lambda_i^{\tau}$, *and* $W \geq 2TUV$.
Define $\Delta = \sqrt{1 + \sum_{i=1}^{\tau} (d_i \lambda_i)^2 V^{-2/\tau}}$. *Further, let* q_1 *be a rational prime number. Then there are rational integers* $v^* \geq 2$, $\gcd(v^*, q_1) = 1$, $1 \leq p_0 < 2UV\Delta$, *and* $p_i, i = 1, \ldots, \tau$ *and a rational number* w *with* $|w| \leq 1$ *such that*

$$n_i - v^* p_i = v^* (p_0 \frac{n_i}{W} - p_i) + w \frac{n_i}{v^* + w},$$

$\sum_{i=1}^{\tau} d_i p_i = 0$, $\sum_{i=1}^{\tau} g_i p_i \equiv 0 \mod U$, $W/p_0 - 1 < v^* \leq W/p_0 + 1$, *and*

$$|p_0 \frac{n_i}{W} - p_i| \leq \lambda_i V^{-1/\tau}.$$

Proof. See Bombieri and Cohen [5], Lemma 6.1. ■

Lemma 4.7. *Let* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *be non-zero algebraic numbers in an algebraic number field* $\mathbb{K}$ *of degree* D *over the rationals. Let* $w(\mathbb{K})$ *denote the number of roots of unity in* $\mathbb{K}$ *and define* $\lambda(D)$ *as in Lemma 4.2. Suppose that there are rational integers* $b_1, b_2, \ldots, b_n$, *not all zero, such that*

$$\prod_{i=1}^{n} \alpha_i^{b_i} = 1.$$

Then there are integers $q_1, q_2, \dots, q_n$*, not all zero, such that*

$$\prod_{i=1}^{n} \alpha_i^{q_i} = 1,$$

and for $k = 1, \dots, n$

$$|q_k| \le (n-1)!w(\mathbb{K}) \prod_{j \ne k} (Dh(\alpha_j)/\lambda(D)).$$

Proof. See [16], Theorem 3A. ∎

4.2. Three Fundamental Lemmas

The first lemma corresponds to the Baker approach using Matveev's estimate for linear forms in n logarithms.

Lemma 4.8. *Let* $f(x,y) = \prod_{j=1}^{3}(x - \alpha^{(j)}y)$ *be defined by Definition 4.5,* $R' = \max_{1 \le j \le 3} R_{\mathbb{K}^{(j)}}$ *where* $R_{\mathbb{K}^{(j)}}$ *is the regulator of* $\mathbb{K}^{(j)} = \mathbb{Q}(\alpha^{(j)})$*, and let* $m \ne 0$ *be an integer. Then all solutions in integers* x *and* y *of the Thue equation*

$$f(x,y) = m \tag{4.5}$$

satisfy

$$\max\{|x|,|y|\} < (A + 1 + |m|)X_0(2X_2)^{X_1}(\log[X_0(X_1X_2)^{X_1}])^{X_1},$$

where

$$\begin{aligned} X_0 &= \sqrt[3]{2|m|}2^8(A+1)^7, \\ X_1 &= (14 \times 10^{15})R'(\log(4(A+1)^2) + (44/3)R'), \\ X_2 &= 22e(44R' + \log[(A+2)(A+1+|m|)] + 1). \end{aligned}$$

Proof. By changing signs if necessary, we may assume that x and y are non-negative. In case $xy = 0$ we obtain a stronger bound. Let

$$f(x,y) = \prod_{j=1}^{3} \beta^{(j)} = \mathrm{N}_{\mathbb{K}}(\beta^{(j)}),$$

where $\beta^{(j)} = x - \alpha^{(j)}y$, $\mathbb{K}^{(j)} = \mathbb{Q}(\alpha^{(j)})$, $[\mathbb{K}^{(j)} : \mathbb{Q}] = 3$, $\mathbb{K} = \mathbb{Q}(\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)})$, $[\mathbb{K} : \mathbb{Q}] \le 6$, and let $\eta_1^{(j)}, \dots, \eta_r^{(j)}$ denote a system of fundamental units of $\mathbb{K}^{(j)}$. Note that for all integer solutions of equation (4.5), the element $\beta^{(j)} \in O_{\mathbb{K}^{(j)}}$ has a fixed norm, for each $j = 1,2,3$. Therefore, by Lemma 4.2, multiplying $\beta^{(j)}$ by a suitable unit $\eta^{(j)}$ of $O_{\mathbb{K}^{(j)}}$ gives a number $\gamma^{(j)}$ whose Mahler measure is bounded in terms of $R_{\mathbb{K}^{(j)}}$. More precisely, Lemma 4.2 implies for $\gamma^{(j)} = \beta^{(j)}\eta^{(j)}$ that

$$\mathrm{M}(\gamma^{(j)}) < \exp(22R_{\mathbb{K}^{(j)}}). \tag{4.6}$$

We now consider the equations

$$\beta^{(j)} = x - \alpha^{(j)} y, j = 1, 2, 3. \tag{4.7}$$

Equations (4.7) with $j = 1, 2$ imply

$$y = \frac{\beta^{(2)} - \beta^{(1)}}{\alpha^{(1)} - \alpha^{(2)}}. \tag{4.8}$$

Substituting equation (4.8) into equations (4.7) with $j = 1, 3$, implies

$$\begin{aligned} \beta^{(3)} &= x - \alpha^{(3)} y \\ &= \beta^{(1)} + \frac{\alpha^{(1)}(\beta^{(2)} - \beta^{(1)})}{\alpha^{(1)} - \alpha^{(2)}} - \frac{\alpha^{(3)}(\beta^{(2)} - \beta^{(1)})}{\alpha^{(1)} - \alpha^{(2)}}, \end{aligned}$$

from which we easily deduce the linear dependence relation between the $\beta^{(j)}$'s used by Siegel, namely

$$(\alpha^{(1)} - \alpha^{(2)})\beta^{(3)} - (\alpha^{(3)} - \alpha^{(2)})\beta^{(1)} - (\alpha^{(1)} - \alpha^{(3)})\beta^{(2)} = 0. \tag{4.9}$$

Without loss of generality, we let

$$\beta^{(1)} = \min_{1 \le j \le 3} |\beta^{(j)}|.$$

Dividing equation (4.9) by $(\alpha^{(1)} - \alpha^{(3)})\beta^{(2)}$, we obtain

$$z - 1 = \frac{(\alpha^{(3)} - \alpha^{(2)})\beta^{(1)}}{(\alpha^{(1)} - \alpha^{(3)})\beta^{(2)}}. \tag{4.10}$$

where for some integers $b_1, \ldots, b_r$,

$$\begin{aligned} z &= \frac{(\alpha^{(1)} - \alpha^{(2)})\beta^{(3)}}{(\alpha^{(1)} - \alpha^{(3)})\beta^{(2)}} \\ &= \alpha_{r+1} \prod_{j=1}^{r} \alpha_j^{b_j}, \end{aligned} \tag{4.11}$$

$$\alpha_j = \eta_j^{(2)} / \eta_j^{(3)}, j = 1, \ldots, r, \tag{4.12}$$

and

$$\alpha_{r+1} = \pm \frac{(\alpha^{(1)} - \alpha^{(2)})\gamma^{(3)}}{(\alpha^{(1)} - \alpha^{(3)})\gamma^{(2)}}. \tag{4.13}$$

Since

$$\mathrm{N}_{\mathbb{K}}((\alpha^{(i)} - \alpha^{(j)})) \ge 1,$$

and

$$|\alpha^{(i)}| < \left(\frac{1}{1 - |\alpha^{(i)}|^{-1}}\right) A$$

provided $|\alpha^{(i)}| > 1$, we deduce by the triangle inequality for $i \neq j$, $1 \leq i, j \leq 3$, that

$$[2(A+1)]^{-5} < |\alpha^{(i)} - \alpha^{(j)}| < 2(A+1). \tag{4.14}$$

Furthermore, by the triangle inequality, we deduce for $j = 2, 3$ that

$$|\beta^{(j)}| > 2^{-6}(A+1)^{-5}|y|, \tag{4.15}$$

provided

$$|y| \geq 2^6(A+1)^5|m|, \tag{4.16}$$

and we observe that

$$|\beta^{(1)}| \leq \frac{|m|}{|\beta^{(2)}||\beta^{(3)}|}. \tag{4.17}$$

Substituting (4.14), (4.15), and (4.17) in (4.10), it follows that

$$|z-1| < 2^{24}(A+1)^{21}|y|^{-3}|m|, \tag{4.18}$$

provided (4.16), and hence that $|z-1| < 1/2$ provided

$$|y| \geq 256\sqrt[3]{2|m|}(A+1)^7. \tag{4.19}$$

By Lemma 4.5 we have that

$$|z-1| \geq \frac{1}{2}|\Lambda|, \tag{4.20}$$

where

$$\Lambda = \log z = \sum_{j=1}^{r} b_j \log \alpha_j + b_{r+1} \log \alpha_{r+1} + b_{r+2} \log \alpha_{r+2}, \tag{4.21}$$

$b_j, j = 1, \ldots, r$ are defined by (4.11), $\alpha_j, j = 1, \ldots, r$ are defined by (4.12), α_{r+1} is defined by (4.13), $b_{r+1} = 1$, $\alpha_{r+2} = -1$, and b_{r+2} is an even integer. Since $z \neq 1$, it follows that $\Lambda \neq 0$, and by Lemma 4.3 we deduce

$$\log|\Lambda| > -c(r+2,\chi)D^2 \log(eD)\Omega \log(eB), \tag{4.22}$$

where

$$B = \max_{1 \leq j \leq r+2} |b_j|,$$
$$D = [\mathbb{Q}(\alpha_1, \ldots, \alpha_{r+2}) : \mathbb{Q}],$$
$$\Omega = \prod_{j=1}^{r+2} \max\{D\mathrm{h}(\alpha_j), |\log|\alpha_j||, 0.16\},$$
$$c(r+2,\chi) = \min\{\frac{(e(r+2)/2)^{\chi} 30^{r+5}(r+2)^{3.5}}{\chi}, 2^{6r+32}\}.$$

Inequalities (4.20) and (4.22) together imply

$$|z-1| > \frac{1}{2}\exp(-c(r+2,\chi)D^2 \log(eD)\Omega \log(eB)). \tag{4.23}$$

Combining (4.18) with (4.23) implies

$$|y| < \sqrt[3]{2|m|}2^8(A+1)^7 \exp((c(r+2,\chi)/3)D^2 \log(eD)\Omega \log(eB)), \tag{4.24}$$

provided (4.19). By definition,

$$c(r+2,\chi) < 2 \times 10^{13}, \tag{4.25}$$

and

$$D \leq 6. \tag{4.26}$$

By the properties of the absolute logarithmic Weil height and Lemma 4.2, it follows that

$$\Omega \leq 30.37R'(2\log[2(A+1)] + (44/3)R'). \tag{4.27}$$

By (4.6), the inequalities

$$|\beta^{(i)}| \leq (A+2)\max\{|x|,|y|\},$$
$$\max\{|x|,|y|\} \leq (A+1+|m|)|y|,$$

and Lemma 4.2, it follows that

$$B < 22(44R' + \log[(A+2)(A+1+|m|)|y|]). \tag{4.28}$$

Substituting (4.25), (4.26), (4.27), and (4.28), in (4.24), we deduce that

$$|y| < X_0 \exp(X_1 \log X_1'),$$

provided (4.19), where X_0 and X_1 were defined at the outset, and

$$X_1' = 22e(44R' + \log[(A+2)(A+1+|m|)] + \log|y|).$$

Since $|y| \geq 3$, we note that

$$X_1' < X_2 \log|y|,$$

where X_2 was defined at the outset, and hence that

$$|y| < X_0 X_2^{X_1}(\log|y|)^{X_1}.$$

Since the smallest regulator of any number field is 0.2052 (see [10]), we may apply Lemma 3.1 in order to deduce

$$|y| < X_0(2X_2)^{X_1}(\log[X_0(X_1X_2)^{X_1}])^{X_1},$$

provided (4.19). It remains to note that in case (4.19) is false we obtain a stronger bound. ∎

The second lemma corresponds to the Bugeaud approach, following Cohen's direction.

Lemma 4.9. *Let $f(x,y)$ be defined by Definition 4.5, $f(\alpha,1)=0$, $\mathbb{K}=\mathbb{Q}(\alpha)$, $d=[\mathbb{K}:\mathbb{Q}]$, $R_{\mathbb{K}}$ be the regulator of $\mathbb{K}$, and let $m\neq 0$ be an integer. Then all solutions in integers x,y of the Thue equation*

$$f(x,y)=m$$

satisfy

$$\log\max\{|x|,|y|\}\leq c_1(A,m,d,R_{\mathbb{K}})+c_2(d,R_{\mathbb{K}}),$$

where

$$\begin{aligned} c_1(d,R_{\mathbb{K}},A,m) &= 10^{22d}d^{10d}d^{3d}(\log d)^{8d}R_{\mathbb{K}}\log[(A+1)|m|],\\ c_2(d,R_{\mathbb{K}}) &= 10^{45d}d^{20d}d^{8d}(\log d)^{16d}R_{\mathbb{K}}^2. \end{aligned}$$

Proof. This follows directly from Bugeaud [7], Theoréme 3, with $r\leq d$, the height of α less than $A+1$, and the height of m equal to $|m|$. ∎

Finally, the third lemma corresponds to the Baker, Bilu and Bugeaud approach, using Laurent, Mignotte, and Nesterenko's estimate for linear forms in 2 logarithms.

Lemma 4.10. *Let $f(x,y)=\prod_{j=1}^{3}(x-\alpha^{(j)}y)$ be defined by Definition 4.5, $R'=\max_{1\leq j\leq 3}R_{\mathbb{K}^{(j)}}$ where $R_{\mathbb{K}^{(j)}}$ is the regulator of $\mathbb{K}^{(j)}=\mathbb{Q}(\alpha^{(j)})$, and let $m\neq 0$ be an integer. Then all solutions in integers x and y of the Thue equation*

$$f(x,y)=m$$

satisfy

$$\log\max\{|x|,|y|\}<\log(A+1+|m|)+4(\sqrt{a}+\sqrt{b}\log(4b))^2,$$

where

$$\begin{aligned} a &= \log c_1,\\ b &= c_2c_3(\max\{1,6\log(2(A+1+|m|)),R'\})^2,\\ c_1 &= \sqrt[3]{2|m|}2^8(A+1)^7,\\ c_2 &= (30.9/3)6^4(3(3\sqrt{2}+1/160)(16/3+1+44)+\pi)\times\\ &\quad\ (3(3\sqrt{2}+1/32)(16/3+1+44)+\pi),\\ c_3 &= (\max\{1+\log c_4,21\})^2,\\ c_4 &= (c_5/c_6)+(1/c_7),\\ c_5 &= 176(44+2)+1,\\ c_6 &= 3(3\sqrt{2}+1/32)(16/3+1+44)+\pi,\\ c_7 &= (3(3\sqrt{2}+1/160)(16/3+1+44)+\pi)\max\{1,6\log(2(A+1+|m|)),R'\}. \end{aligned}$$

Proof. We recall the setup and notation of the proof of Lemma 4.8, and the definitions $\alpha_j = \frac{\eta_j^{(2)}}{\eta_j^{(3)}}, j = 1, \ldots, r$, $\alpha_{r+1} = \pm\frac{(\alpha^{(1)}-\alpha^{(2)})\gamma^{(3)}}{(\alpha^{(1)}-\alpha^{(3)})\gamma^{(2)}}$, $\alpha_{r+2} = -1$. Furthermore, with respect to the notation in Lemma 4.6, and the notation of the proof of Lemma 4.8, let

$$\begin{aligned}
n_i &= b_i, i = 1, \ldots, r+2,\\
d_i &= \begin{cases} 1 \text{ if } b_i \geq 0,\\ -1 \text{ otherwise,}\end{cases}\\
g_i &\in \mathbb{Z},\\
T &= \sum_{i=1}^{r+2} |b_i|,\\
\lambda_i &= \begin{cases} 1/T \; i = 1, \ldots, r+1,\\ T^{r+1} \; i = r+2,\end{cases}\\
U &= 1,\\
V &= 20T^{(r+1)(r+2)},\\
W &= 2TUV,\\
\Delta &= \sqrt{1 + \sum_{i=1}^{r+2} (d_i\lambda_i)^2 V^{-2/(r+2)}},\\
q_1 &\in \{2, 3, 5, 7, \ldots\}.
\end{aligned}$$

By Lemma 4.6, there are rational integers

$$v^*, p_0, p_1, p_2, \ldots, p_{r+2},$$

and a rational number w such that $v^* \geq 2$, $\gcd(v^*, q_1) = 1$, $1 \leq p_0 < 2UV\Delta$, $|w| \leq 1$,

$$n_i - v^* p_i = v^*(p_0 \frac{n_i}{w} - p_i) + w\frac{n_i}{v^* + w},$$

$\sum_{i=1}^{r+2} d_i p_i = 0$, $\sum_{i=1}^{r+2} g_i p_i \equiv 0 \mod U$,

$$\frac{W}{p_0} - 1 < v^* \leq \frac{W}{p_0} + 1,$$

and

$$|p_0 \frac{n_i}{W} - p_i| \leq \lambda_i V^{-1/(r+2)}.$$

We define

$$I = \alpha_1^{p_1} \cdots \alpha_{r+2}^{p_{r+2}},$$

$$J = \alpha_1^{b_1 - v^* p_1} \cdots \alpha_{r+2}^{b_{r+2} - v^* p_{r+2}},$$

and

$$\Lambda = \log J + v^* \log I.$$

We note that $\Lambda \neq 0$. We assume first that I and J are multiplicatively independent algebraic numbers. In particular, not all p_i are zero. It follows by Lemma 4.4 that

$$\begin{aligned}\log|\Lambda| &= \log|\log J - v^* \log I^{-1}| \\ &\geq -30.9(D')^4 \log A_1 \log A_2 (\max\{\log b', 21/D', 1/2\})^2,\end{aligned}$$

where $D' = [\mathbb{Q}(I^{-1}, J) : \mathbb{Q}]/[\mathbb{R}(I^{-1}, J) : \mathbb{R}]$, A_1, A_2 are positive real numbers such that

$$\begin{aligned}\log A_1 &\geq \max\{\mathrm{h}(I^{-1}), |\log I^{-1}|/D', 1/D'\}, \\ \log A_2 &\geq \max\{\mathrm{h}(J), |\log J||/D', 1/D'\},\end{aligned}$$

and

$$b' = \frac{v^*}{D' \log A_2} + \frac{1}{D' \log A_1}.$$

From the proof of Lemma 4.8, for sufficiently large y, we deduce that

$$\begin{aligned}&|y| \\ &< \sqrt[3]{2|m|}\,2^8 (A+1)^7 \exp((30.9/3)(D')^4 \log A_1 \log A_2 (\max\{\log b', 21/D', 1/2\})^2).\end{aligned} \tag{4.29}$$

Since $I^{-1}, J \in \mathbb{Q}(\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)})$, we have that

$$1 \leq D' \leq 6. \tag{4.30}$$

We proceed to determine $\log A_1$ and $\log A_2$. Since $\mathrm{h}(I) = \mathrm{h}(I^{-1})$,

$$\max\{\mathrm{h}(I^{-1}), |\log I^{-1}|/D', 1/D'\} = \max\{\mathrm{h}(I), |\log I|/D', 1/D'\}.$$

Moreover, it follows from well known inequalities between sizes and heights (see [22]) that

$$\begin{aligned}|\log I| &\leq |\log|I|| + \pi \\ &\leq |\log \max\{den(I), house(I)\}| + \pi \\ &= |s(I)| + \pi \\ &\leq deg(I)\mathrm{h}(I) + \pi,\end{aligned}$$

and a priori that

$$\frac{|\log I|}{D'} \leq \frac{den(I)\mathrm{h}(I)}{D'} + \frac{\pi}{D'} \leq \mathrm{h}(I) + \pi.$$

Hence,

$$\max\{\mathrm{h}(I), |\log I|/D', 1/D'\} \leq \max\{\mathrm{h}(I) + \pi, 1\} = \mathrm{h}(I) + \pi.$$

Similarly,

$$\max\{\mathrm{h}(J), |\log J|/D', 1/D'\} \leq \mathrm{h}(J) + \pi.$$

Plainly,

$$\mathrm{h}(I) \leq \sum_{i=1}^{r+2} |p_i| \mathrm{h}(\alpha_i),$$

and

$$\mathrm{h}(J) \leq \sum_{i=1}^{r+2} |b_i - v^* p_i| \mathrm{h}(\alpha_i).$$

Notice $\mathrm{h}(\alpha_{r+2}) = 0$. By Lemma 4.2, we deduce for $i = 1, \ldots, r$ that

$$\mathrm{h}(\alpha_i) \leq 2 \left(\frac{(r!)^2}{2^{r-1} d^r} \left(\frac{2}{d(\log(3d))^3} \right)^{1-r} \right) R',$$

from which it follows for $i = 1, \ldots, r$ that

$$\mathrm{h}(\alpha_i) \leq (8/3) R'.$$

By the properties of the logarithmic Weil height and the bound on the Mahler height of $\gamma^{(j)}$ implied by Lemma 4.2, we deduce

$$\begin{aligned}
\mathrm{h}(\alpha_{r+1}) &\leq 2(\max_{i \neq j} \mathrm{h}(\alpha^{(i)} - \alpha^{(j)}) + \max_{j} \mathrm{h}(\gamma^{(j)}) \\
\mathrm{h}(\alpha^{(i)} - \alpha^{(j)}) &\leq 3 \log[2(A+1)] \\
\mathrm{h}(\gamma^{(j)}) &\leq \log \exp(22R'),
\end{aligned}$$

and hence

$$\mathrm{h}(\alpha_{r+1}) \leq 2(3 \log[2(A+1)] + 22R').$$

Furthermore, for $i = 1, \ldots, r+1$,

$$\begin{aligned}
|b_i - v^* p_i| &\leq v^* |p_0 \frac{b_i}{W} - p_i| + \frac{|w||b_i|}{|v^* + w|} \\
&\leq v^* \lambda_i V^{-1/(r+2)} + \frac{|w||b_i|}{|v^* + w|},
\end{aligned}$$

and

$$\begin{aligned}
|p_i| &= \left| \frac{b_i - b_i + v^* p_i}{v^*} \right| \\
&\leq \frac{|b_i|}{v^*} + \lambda_i V^{-1/(r+2)} + \frac{|w||b_i|}{v^* |v^* + w|}.
\end{aligned}$$

We may assume $p_0 > V$, for otherwise

$$\begin{aligned}
|p_i| &\leq |p_0 \frac{b_i}{W} - p_i| + |p_0 \frac{b_i}{W}| \\
&\leq \lambda_i V^{-1/(r+2)} + 1/2 \\
&\leq 1/\sqrt[4]{20} + 1/2 \\
&< 1,
\end{aligned}$$

which implies $p_i = 0$ for all $i = 1, \ldots, r+2$, a contradiction to our assumption that I and J are multiplicatively independent. It follows that

$$\begin{aligned} v^* &< \frac{W}{p_0} + 1 \\ &< \frac{2TUV}{V} + 1 \\ &= 2T + 1. \end{aligned}$$

Moreover, since $T \geq 2$, for $i = 1, \ldots, r+1$,

$$\begin{aligned} v^* \lambda_i V^{-1/(r+2)} &< \frac{2T+1}{20T^3} \\ &\leq 1/32, \end{aligned}$$

while

$$\lambda_i V^{-1/(r+2)} \leq 1/160.$$

On the other hand,

$$\begin{aligned} \frac{|b_i|}{v^*} &< T/v^* \\ &< \Delta(1 + 1/v^*) \\ &\leq (3/2)\sqrt{2}, \end{aligned}$$

$$\begin{aligned} \frac{|w||b_i|}{|v^* + w|} &\leq \frac{T}{v^* - 1} \\ &< \left(\frac{v^*+1}{v^*-1}\right)\Delta \\ &\leq 3\sqrt{2}, \end{aligned}$$

and

$$\frac{|w||b_i|}{v^*|v^* + w|} \leq (3/2)\sqrt{2}.$$

We set

$$\begin{aligned} \log A_1 &= (3(3\sqrt{2} + 1/160)(16/3 + 1 + 44) + \pi) \max\{1, 6\log[2(A+1) + |m|], R'\}, \\ \log A_2 &= (3(3\sqrt{2} + 1/32)(16/3 + 1 + 44) + \pi) \max\{1, 6\log[2(A+1) + |m|], R'\}. \end{aligned}$$

Furthermore, we note that

$$b' \leq \frac{2T+1}{\log A_2} + \frac{1}{\log A_1},$$

and

$$T \leq 4B,$$

where we recall that

$$B < 22(44R' + \log(A+2) + \log(A+1+|m|) + \log|y|).$$

For y sufficiently large, it follows that

$$(\max\{\log b', 21/D', 1/2\})^2 < (\max\{1+\log c_4, 21\} \log\log|y|)^2,$$

where c_4 was defined in the statement of this Lemma. It follows from (4.29) that for y sufficiently large (quantified by (4.19)),

$$|y| < c_1 \exp(c_2 c_3 (\max\{1, 6\log[2(A+1)+|m|], R'\})^2 (\log\log|y|)^2),$$

where c_1, c_2, and c_3 were defined in the statement of this Lemma. It remains to apply Lemma 3.1 in order to deduce our result. As before, in case (4.19) is false, we obtain a stronger bound.

We are left to consider the case that I and J are multiplicatively dependent. In this case there exist integers s and t, not both zero, such that $I^s J^t = 1$. Suppose first that $t \neq 0$. Note that $s/t \neq v^*$, and $\log I \neq 0$, as otherwise we obtain a contradiction with $v^* \log I + \log J \neq 0$ on one hand, and $(s/t)\log I + \log J = 0$ on the other hand. Since $J = I^{-s/t}$, we deduce that

$$|y|^3 < 2^{25}(A+1)^{21}|m| \frac{|t|}{|\log I||v^* t - s|}.$$

By Lemma 4.7, we may bound $|t|$ in order to obtain a stronger bound in comparison to (4.29). Suppose now that $t = 0$. Then $s \neq 0$, so that $I = 1^{1/s} = 1$, and $|\log J| \neq 0$, from which we deduce the stronger bound

$$|y|^3 < 2^{25}(A+1)^{21}|m| \frac{1}{|\log J|}. \qquad \blacksquare$$

4.3. Main Theorem

In this section we put the pieces of the last section together in order to obtain our main result.

Theorem 4.1. *Let k be a fixed non-zero integer, and let x and y be integers such that $y^2 = x^3 + k$. Then*

$$\log\max\{|x|,|y|\} < \min_{(c,d)\in S}\{c|k|(\log|k|)^d\},$$

where

$$S = \{(10^{181}, 4), (10^{23}, 5), (10^{19}, 6)\}.$$

Proof. In case $|k| < 3$, it follows by classical arguments of Euler, Gauss, and Wantzel that $\log\max\{|x|,|y|\} \leq \log 5$. We assume $|k| \geq 3$. Baker [1] shows that the binary form

$$f(X,Y) = X^3 - 3xXY^2 - 2yY^3$$

of discriminant
$$D_f = -27(-2y)^2 - 4(-3x)^3 = -108k,$$
is equivalent ($f \sim F$ for some integers p, q, r, s, $ps - qr = \pm 1$) to a reduced binary cubic form
$$F(X, Y) = a_3X^3 + a_2X^2Y + a_1XY^2 + a_0Y^3,$$
in which $a_3 \neq 0$ and each coefficient has absolute value at most $\sqrt{108|k|}$. Using the identities
$$f(pX' + qY', rX' + sY') = F(X', Y'),$$
and
$$\pm f(X, Y) = F(sX - qY, -rX + pY),$$
Baker [1] obtains
$$a_3s^3 - a_2s^2r + a_1sr^2 - a_0r^3 = \pm 1, \tag{4.31}$$
$$\pm 3x = 3(a_0rp^2 - a_3sq^2) + 2pq(a_2s - a_1r) + a_2rq^2 - a_1sp^2,$$
and
$$\pm 2y = a_3q^3 - a_2pq^2 + a_1p^2q - a_0p^3,$$
from which he deduces that
$$3|x| \leq 12\sqrt{108|k|}\max\{|r|, |s|\}(2\sqrt{108|k|}\max\{|r|, |s|\}^2)^2, \tag{4.32}$$
and
$$2|y| \leq 4\sqrt{108|k|}(2\sqrt{108|k|}\max\{|r|, |s|\}^2)^3. \tag{4.33}$$
By inequalities (4.32) and (4.33), we deduce
$$\max\{|x|, |y|\} \leq 16(108|k|)^2\max\{|r|, |s|\}^6. \tag{4.34}$$
In case the left hand side of equation (4.31) is reducible, Baker obtains
$$\max\{|r|, |s|\} \leq 6(108|k|)^{7/2}. \tag{4.35}$$
Substituting (4.35) in (4.34), we obtain
$$\max\{|x|, |y|\} \leq 16(108|k|)^2(6(108|k|)^{7/2})^6 < 5 \times 10^{52}|k|^{23},$$
from which we deduce
$$\log\max\{|x|, |y|\} < 2791\log|k|. \tag{4.36}$$
On the other hand, in case the left hand side of equation (4.31) is irreducible, we apply Lemma 4.8, Lemma 4.9, and Lemma 4.10, to
$$(a_3s)^3 - a_2(a_3s)^2r + a_1a_3(a_3s)r^2 - (a_0a_3^2)r^3 = \pm a_3^2,$$
in order to deduce three bounds for
$$\log\max\{|a_3s|, |r|\}$$

from which we deduce our desired result from (4.34), or more precisely, since $|a_3| \geq 1$, from

$$\log\max\{|x|,|y|\} \leq \log 16 + 2\log(108|k|) + 6\log\max\{|a_3 s|,|r|\}. \tag{4.37}$$

It remains to establish our three bounds. We set $A = (108|k|)^{3/2}$, and note $|m| \leq 108|k|$, and $R' < (24.7)\sqrt{|k|}(\log|k|)^2$. By Lemma 4.8,

$$\max\{|a_3 s|,|r|\} < (A+1+|m|)X_0(2X_2)^{X_1}(\log[X_0(X_1X_2)^{X_1}])^{X_1},$$

where X_0, X_1, and X_2 are all defined in the statement of Lemma 4.8. Note

$$A+1+|m| < \exp(\xi_1 \log|k|),$$

where

$$\begin{aligned}\xi_1 &= \log[108^{3/2}+1+108]+3/2\\ &= 8.615\ldots,\end{aligned}$$

$$X_0 < \exp(\xi_2\log|k|),$$

where

$$\begin{aligned}\xi_2 &= \log[\sqrt[3]{2(108)}2^8((108)^{3/2}+1)^7]+(1/3+21/2)\\ &= 67.338\ldots,\end{aligned}$$

$$X_1 < \xi_3|k|(\log|k|)^4,$$

where

$$\begin{aligned}\xi_3 &= (14\times 10^{15})(24.7)(\log[4((108)^{3/2}+1)^2]+3+(44/3)(24.7))\\ &= (1.31\ldots)\times 10^{20},\end{aligned}$$

and

$$X_2 < \xi_4\sqrt{|k|}(\log|k|)^2,$$

where

$$\begin{aligned}\xi_4 &= 22e(44(24.7)+\log[(108^{3/2}+2)(108^{3/2}+1+108)]+3+1)\\ &= (6.60\ldots)\times 10^4.\end{aligned}$$

It follows that

$$(2X_2\log[X_0(X_1X_2)^{X_1}])^{X_1} < \exp(\xi_5|k|(\log|k|)^5),$$

where

$$\begin{aligned}\xi_5 &= \xi_3(\log 2+\log\xi_4+1/2+2+\log\xi_2+1+\log\xi_3+1+4+\\ &\quad \log(\log\xi_3+1+4+\log\xi_4+1/2+2)+1)\\ &= (1.00\ldots)\times 10^{22}.\end{aligned}$$

Hence,

$$\max\{|a_3 s|, |r|\} < \exp(\xi_6 |k| (\log |k|)^5),$$

where

$$\begin{aligned} \xi_6 &= \xi_1 + \xi_2 \xi_5 \\ &= (1.00\ldots) \times 10^{22}. \end{aligned}$$

From (4.37) we deduce

$$\log \max\{|x|, |y|\} < \xi_7 |k| (\log |k|)^5,$$

where

$$\begin{aligned} \xi_7 &= \log 16 + 2\log 108 + 2 + 6\xi_6 \\ &= (6.00\ldots) \times 10^{22}. \end{aligned}$$

On the other hand, by Lemma 4.9, we deduce that

$$\max\{|a_3 s|, |r|\} < \xi_{10} |k| (\log |k|)^4,$$

where

$$\begin{aligned} \xi_8 &= 10^{22(3)} 3^{10(3)} 3^{3(3)} (\log 3)^{8(3)} \\ &= (9.56\ldots) \times 10^{86}, \\ \xi_9 &= 10^{45(3)} 3^{20(3)} 3^{8(3)} (\log 3)^{16(3)}, \\ &= (6.66\ldots) \times 10^{179}, \\ \xi_{10} &= \xi_8 (\log[108(108^{3/2}+1)] + 5/2) + \xi_9, \\ &= (6.66\ldots) \times 10^{179}. \end{aligned}$$

From (4.37) we deduce

$$\log \max\{|x|, |y|\} < \xi_{11} |k| (\log |k|)^4,$$

where

$$\begin{aligned} \xi_{11} &= \log 16 + 2\log 108 + 2 + 6\xi_{10} \\ &= (3.99\ldots) \times 10^{180}. \end{aligned}$$

Finally, by Lemma 4.10,

$$\log \max\{|a_3 s|, |r|\} < \log[(108|k|)^{3/2} + 1 + 108|k|] + 4(\sqrt{a} + \sqrt{b}\log[4b])^2,$$

where a and b are defined in the statement of Lemma 4.10. We deduce

$$a < \xi_{12} \log |k|,$$

where

$$\begin{aligned}\xi_{12} &= \log(\sqrt[3]{2(108)}2^8((108)^{3/2}+1)^7) + (1/3 + 21/2)\\ &= 67.33\ldots,\end{aligned}$$

$$b < \xi_{13}|k|(\log|k|)^4,$$

where c_2 and c_3 are defined in the statement of Lemma (4.10), and

$$\xi_{13} = c_2 c_3 (24.7)^2,$$

and

$$\log[(108|k|)^{3/2} + 1 + 108|k|] < \xi_{14}\log|k|,$$

where

$$\begin{aligned}\xi_{14} &= \log((108)^{3/2} + 1 + 108) + 3/2\\ &= 8.61\ldots.\end{aligned}$$

It follows that

$$\log\{|a_3 s|, |r|\} < \xi_{15}|k|(\log|k|)^6,$$

where

$$\begin{aligned}\xi_{15} &= \xi_{14} + 4(\sqrt{\xi_{12}} + \sqrt{\xi_{13}}(\log(4\xi_{13}) + 1 + 4))^2\\ &= (8.37\ldots)\times 10^{17}.\end{aligned}$$

From (4.37) we deduce

$$\log\max\{|x|, |y|\} < \xi_{16}|k|(\log|k|)^6,$$

where

$$\begin{aligned}\xi_{16} &= \log 16 + 2\log 108 + 2 + 6\xi_{15}\\ &= (5.02\ldots)\times 10^{18}.\end{aligned}$$

■

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Bibliography

[1] A. Baker, *Contributions to the Theory of Diophantine Equations II; The Diophantine Equation $y^2 - x^3 = k$,* Philos. Trans. Roy. Soc. London Ser. A, **263** (1967), 193–208.

[2] A. Baker, *The theory of linear forms in logarithms,* in: Transcendence Theory: Advances and Applications, ed. by A. Baker and D. W. Masser, Academic Press, London (1977), 1–27.

[3] A. Baker, *Experiments on the abc conjecture*, Publ. Math. Debrecen, **65/3-4** (2004), 253-260.

[4] Y. Bilu and Y. Bugeaud, *Démonstration du théorème de Baker-Feldman via les formes linéaires en deux logarithmes,* J. Th. Nombres Bordeaux, **12** (2000) 13–23.

[5] E. Bombieri and P. Cohen, *Effective Diophantine Approximation on G_m, II,* Ann. Scuola Norm. Sup. Pisa Cl. Sci., **24** (1997), 205–225.

[6] Y. Bugeaud and K. Győry, *Bounds for the solutions of unit equations*, Acta Arithmetica, **74** (1996), 67-80.

[7] Y. Bugeaud, *Bornes effectives pour les solutions des equations en s-unites et des equations de Thue-Mahler,* J. Number Theory, **71** (1998), 227–244.

[8] L. V. Danilov, *Diophantine Equation $x^3-y^2=k$ and Hall's Conjecture,* Math Notes of the Academy of Sciences in USSR, **32** (1982), 617–618.

[9] N. D. Elkies, *Rational points near curves and small nonzero $|x^3-y^2|$ via lattice reduction,* LNCS 1838, (2000), 33–63.

[10] E. Friedman, *Analytic formulas for regulators of number fields,* Invent. Math., **98** (1989), 599–622.

[11] J. Gebel and A. Pethö and H. G. Zimmer, *On Mordell's Equation*, Compositio Mathematica, **110** (1998), 335–367.

[12] M. Hall Jr., *Computers in Number Theory,* Proc. Sci. Res. Council Atlas Symp. Oxford, **2** (1969), 173–198.

[13] M. Laurent and M. Mignotte and Y. Nesterenko, *Formes linéaires en deux logarithmes et d éterminants d'interpolation,* J. Number Theory, **55** (1995), 285–321.

[14] S. Louboutin, *Explicit Bounds for Residues of Dedekind Zeta Functions, Values of L- Functions at $s=1$, and Relative Class Numbers*, Journal of Number Theory, **85** (2000), 263-282.

[15] S. Louboutin, *Explicit Upper Bounds for Residues of Dedekind Zeta Functions and Values of L- Functions at $s=1$, and Explicit Lower Bounds for Relative Class Numbers of CM Fields,* Can. J. Math., **53(6)** (2001), 1194-1222.

[16] J. H. Loxten and A. J. van der Poorten, *Multiplicative dependence in number fields*, Acta Arithmetica, **XLII** (1983), 291–302.

[17] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers,* Izvestiya Mathematics, **64/6** (2000), 1217–1269.

[18] A. Pethö and B. M. M. de Weger, *Products of prime power in binary recurrence sequences, I, The hyperbolic case, with an application to the Ramanujan-Nagell equation,* Math. Comp., **47** (1986), 713–727.

[19] V. G. Sprindžuk, *Classical Diophantine Equations (Translation of a work published in the USSR, 1982)*, Springer LNM 1559, (1993).

[20] H. M. Stark, *Effective estimates of solutions of some Diophantine equations,* Acta Arithmetica, **XXIV** (1973), 251–259.

[21] P. Voutier, *An effective lower bound for the height of algebraic numbers,* Acta Arithmetica, **74** (1996), 81–95.

[22] M. Waldschmidt, *Diophantine Approximation on Linear Algebraic Groups*, Springer-Verlag, 2000.

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