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MOMENTS FOR GENERALIZED FAREY-BROCOT PARTITIONS

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Abstract: We prove new asymptotic formulas for some two-dimensional Farey-Brocot nets.
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1. Introduction

1.1. Brocot partitions

The Brocot sequences F_n, $n = 0, 1, 2, \dots$ are defined as follows. $F_0 = \{0, 1\} = \{\frac{0}{1}, \frac{1}{1}\}$, and if elements of F_n are ordered in absolute value

$$0 = x_{0,n} < x_{1,n} < \cdots < x_{N(n),n} = 1, N(n) = 2^n, \tag{1}$$

then

$$F_{n+1} = F_n \cup Q_{n+1},$$

where Q_{n+1} is defined by the rule

$$Q_{n+1} = \{x_{i,n} \oplus x_{i-1,n}, i = 1, \dots, N(n)\}, \quad \frac{p}{q} \oplus \frac{p'}{q'} = \frac{p+p'}{q+q'}.$$

Brocot sequences were first introduced in [1], [2], [3] (in fact, both Stern and Brocot started with $SB_0 = \left(\frac{0}{1}, \frac{1}{0}\right)$, then $SB_1 = \left(\frac{0}{1}, \frac{1}{1}, \frac{1}{0}\right)$, so we treat "half" of the original sequence here ; the notation $\frac{1}{0}$ above is used formally to define $\frac{a}{b} \oplus \frac{1}{0} = \frac{a+1}{b}$, particularly $\frac{a}{1} \oplus \frac{1}{0} = \frac{a+1}{1}$).

We consider the partition of the unit interval $[0, 1]$ generated by the points of (1). Let $p_{i,n} = x_{i,n} - x_{i-1,n}, i = 1, \dots, N(n)$ be the lengths of the intervals $[x_{i-1,n}, x_{i,n})$. For $\beta \geq 1$ we look for the value

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$$\sigma\left(F_n\right) = \sum_{i=1}^{N(n)} p_{i,n}^{\beta},$$

the moment of order β. The asymptotics for this sum were obtained by Moshchevitin and Zhigljavsky in [4]. They proved that for $\beta > 1$ the following asymptotic formula is valid

$$\sigma_\beta\left(F_n\right) = \frac{2}{n^\beta}\frac{\zeta\left(2\beta-1\right)}{\zeta\left(2\beta\right)}\left(1 + O\left(\frac{\log n}{n^{\frac{\beta-1}{2\beta}}}\right)\right), n \to \infty. \tag{2}$$

Here $\zeta(s)$ denotes the Riemann zeta function.

The aim of the present paper is to generalize formula (2) to two special multidimensional Farey-Brocot algorithms (algorithms $\mathfrak{A}$ and $\mathfrak{B}$).

We point out that recently a better asymptotics was obtained by Dushistova [5]. She proved that with some positive constants $C_k = C_k(\beta), C_k^* = C_k^*(\beta)$ we have

$$\sigma_\beta\left(F_n\right) = \frac{1}{n^\beta}\frac{2\zeta\left(2\beta-1\right)}{\zeta\left(2\beta\right)} + \sum_{1\le k<2\beta-2} C_k\frac{1}{n^{\beta+k}} + \sum_{0\le k<\beta-2} C_k^*\frac{1}{n^{2\beta+k}} + R_\beta(n), \tag{3}$$

where the remainder is $R_\beta(n) = O\left(\frac{\log^{3\beta} n}{n^{3\beta-2}}\right)$. In general all summands involved have different orders when $n \to \infty$. However, in the case $\beta \in \mathbb{N}$ this formula may be simplified.

1.2. Multidimensional generalized Farey-Brocot algorithms

There exist various multidimensional generalizations of Farey sequences. The history starts with Hurwitz' paper [10]. As for general concepts and basic results we refer to Grabiner [11]. The simplest construction of multidimensional Farey nets is due to Mönkemeyer [12]. Here we introduce a general approach to generalize the Brocot sequence, which in some sense is similar to the construction of Farey nets. In the next sections we give an analog to formula (2) for some special two-dimensional algorithms. Let $\mathcal{E} = \{g_1, ..., g_{d+1}\}$ be a basis of the lattice $\mathbb{Z}^{d+1}$. For such a basis $\mathcal{E}$ we define the cone

$$\mathcal{C}(\mathcal{E}) = \{x \in \mathbb{R}^{d+1} \ : \ x = \sum_{j=1}^{d+1} a_j g_j, \ \ a_1, ..., a_{d+1} \ge 0\}.$$

For a given basis $\mathcal{E} = \{g_1, ..., g_{d+1}\}$ we can consider a natural number $K \ge 2$ and a set of integer vectors $\mathcal{E}_*^k = \{g_1^k, ..., g_{d+1}^k\}$, $1 \le k \le K$ with the following properties.

(i) Each $\mathcal{E}_*^k$ is a basis for $\mathbb{Z}^{d+1}$. (4)

(ii) The set of cones $\mathcal{C}(\mathcal{E}_*^k)$, $1 \le k \le K$ forms a regular partition of the cone $\mathcal{C}(\mathcal{E})$.

It means that $\mathcal{C}(\mathcal{E}) = \bigcup_{1\le k\le K} \mathcal{C}(\mathcal{E}_*^k)$ and the intersection of every two cones $\mathcal{C}(\mathcal{E}_*^{k_1}), \mathcal{C}(\mathcal{E}_*^{k_2})$ from this union is a whole l-dimensional facet (for some $0 \le l \le d$) for both cones $\mathcal{C}(\mathcal{E}_*^{k_1})$ and $\mathcal{C}(\mathcal{E}_*^{k_2})$.

We shall work in Euclidean space $\mathbb{R}^{d+1}$ with coordinates $(x, y_1, ..., y_d)$.

Let the unit cube $\{z = (x, y_1, ..., y_d) \,:\, x = 1, y_j \in [0,1]\}$ be partitioned into K_0 simplices $\Delta_k, 1 \le k \le K_0$, in such a way that the vertices of the simplices are among the cube's vertices. Moreover let the set of vertices of each simplex Δ_k from this partition form a basis $\mathcal{E}^{0,k}$ of the lattice $\mathbb{Z}^{d+1}$. A *generalized Farey-Brocot algorithm* (GFBA) is a sequence of *rules* $R_\nu(\mathcal{E}^{\nu-1,1}, \ldots, \mathcal{E}^{\nu-1,K_{\nu-1}})$ of choosing a set of bases $\left(\mathcal{E}^{\nu,k}, 1 \le k \le K_\nu(\mathcal{E}^{\nu-1,1}, \ldots, \mathcal{E}^{\nu-1,K_{\nu-1}})\right)$ for each set $(\mathcal{E}^{\nu-1,k}, 1 \le k \le K_{\nu-1})$ from a previous step of the algorithm in such a way that every basis $\mathcal{E}^{\nu-1,k}$ is decomposed into some bases from the set $(\mathcal{E}^{\nu,k}, 1 \le k \le K_\nu)$ in such a way that the conditions $(4)(i), (ii)$ above are satisfied. For given rules $R_1, ..., R_{\nu-1}$ we can construct an infinite set of admissible rules R_ν. So we can speak about a "tree" of algorithms (compare with [7]). We shall use the gothic letter $\mathfrak{F}$ for an individual algorithm (a precisely described set of rules). For the ν-th set of bases we shall use the notation $\mathcal{E}^{\nu,k} = \{g_1^{\nu,k}, ..., g_{d+1}^{\nu,k}\}$ and for coordinates of each vector $g_j^{\nu,k}$ we put $g_j^{\nu,k} = (x_j^{\nu,k}, y_{j,1}^{\nu,k}, ..., y_{j,d}^{\nu,k})$.

A GFB algorithm $\mathfrak{F}$ is called *complete* if any integer vector $(x, y_1, ..., y_d) \in \mathbb{Z}^{d+1}, x \ge 1, 0 \le y_j \le x,$ g.c.d.$(x, y_1, ..., y_d) = 1$ occurs as a vector from some basis $\mathcal{E}^{\nu,k}$ of the considered algorithm (there are GFBAs, which are not complete).

Let $\Theta = (1, \theta_1, ..., \theta_d), \theta_l \in [0,1]$ be a real vector and a GFB algorithm $\mathfrak{F}$ be given. To every algorithm $\mathfrak{F}$ we can construct a multidimensional continued fraction algorithm by the following procedure. At each step ν of the algorithm $\mathfrak{F}$ we choose a basis $\mathcal{E}^{\nu,k_\nu}$ in such a way that that Θ can be expressed in the form $\Theta = \sum_{j=1}^{d+1} a_j g_j^{\nu,k_\nu}$ and all *coefficients* a_j of the vector Θ with respect to the basis $\mathcal{E}^{\nu,k_\nu}$ are nonnegative: $a_j \ge 0, j = 1, ..., d+1$. In other words, $\Theta \in \mathcal{C}(\mathcal{E}^{\nu,k_\nu})$. (One may note that in general such a sequence of bases may not be unique, for example when the coordinates of the vector Θ are linearly dependent over $\mathbb{Z}$. Hence sometimes the corresponding multidimensional continued fraction decomposition of the vector Θ may be not unique.) A multidimensional continued fraction algorithm is called *weakly convergent* in Θ [9], if for all j, l from the intervals $1 \le j \le d+1,\ 1 \le l \le d$ the sequence $y_{j,l}^{\nu,k_\nu}/x_j^{\nu,k_\nu}$ converges to θ_l. There are classical examples of algorithms which are not weakly convergent (see [9], [7], [6]) such as Poincaré's algorithm.

Lemma 1.1. *Let the multidimensional continued fraction algorithm corresponding to the GFB algorithm $\mathfrak{F}$ be weakly convergent everywhere. Then the GFBA $\mathfrak{F}$ is complete.*

(We note that it is sufficient to suppose weak convergence of the corresponding multidimensional continued fraction algorithm only for vectors Θ with rational coordinates. Also, completeness of a GFBA does not necessarily lead to the convergence of the corresponding multidimensional continued fraction algorithm, and it is easy to construct the corresponding example.)

Proof. Let $z = (x, y_1, ..., y_d), x \geq 1, 0 \leq y_j \leq x$, be a primitive integer point. Suppose it does not occur in algorithm $\mathfrak{F}$ as an element of a basis. If the described multidimensional continued fraction algorithm is weakly convergent in the point $\Theta = (1, y_1/x, ..., y_d/x)$ then for some sequence of basis $\mathcal{E}^{\nu,k_\nu}$ we have $y_{j,l}^{\nu,k_\nu}/x_j^{\nu,k_\nu} \to \theta_l$ as $\nu \to \infty$ for all j, l. As we have supposed $x \cdot \Theta \neq g_j^{\nu,k_\nu}$ for all j from the interval $1 \leq j \leq d+1$ and for all natural ν, this means that for every j we have $x_j^{\nu,k_\nu} \to +\infty$ as $\nu \to \infty$. But as $\mathcal{E}^{\nu,k_\nu}$ is a basis and the coefficients for Θ are nonnegative we have $z = x \cdot \Theta = \sum_{j=1}^{d+1} a_j g_j^{\nu,k_\nu}$ with nonnegative integers a_j and at least one of them (say a_m) is ≥ 1. Then $x \geq a_m \cdot x_m^{\nu,k_\nu} \to +\infty$ and this is a contradiction. ■

Each GFB algorithm $\mathfrak{F}$ generates a sequence of partitions ("tilings") $\mathrm{Til}_\nu(\mathfrak{F})$ of the unit cube $[0,1]^d$ and a sequence of graphs $T_\nu(\mathfrak{F})$ as follows:

Let a GFB algorithm $\mathfrak{F}$ be given. We look for the set of all bases $\mathcal{E}$ at the ν-th step of our algorithm. The number of such bases may vary according to $\mathfrak{F}$, but the corresponding cones $\mathcal{C}(\mathcal{E})$ form a regular partition of the cubic cone $\{z = (x, y_1, ..., y_d) \ : \ x \geq 0, y_j \in [0,1]\}$. We restrict this partition on the set $\{z = (x, y_1, ..., y_d) \ : \ x = 1, y_j \in [0,1]\}$ and obtain the partition $\mathrm{Til}_\nu(\mathfrak{F})$ of the unit cube $[0,1]^d$ into simplices $\Delta = \mathcal{C}(\mathcal{E}) \cap \{z = (x, y_1, ..., y_d) \ : \ x = 1, y_j \in [0,1]\}$. The main object of the paper is the sum

$$\sigma_{n,\beta}(\mathfrak{F}) = \sum_{\Delta \in \mathrm{Til}_n(\mathfrak{F})} (\mathrm{mes}\,\Delta)^\beta,$$

the moment of order β.

Obviously for any GFBA and for any natural n we have

$$\sigma_{n,1}(\mathfrak{F}) = 1. \tag{5}$$

The following simple statement is well-known (see, for example [8]).

Lemma 1.2. *Let the simplex Δ correspond to the basis $\mathcal{E} = \{g_1, ..., g_{d+1}\}$, and the vector g_j from this basis have coordinates $g_j = (x_j, y_{j,1}, ..., y_{j,d})$. Then*

$$\mathrm{mes}\,\Delta = \frac{1}{d! x_1 \cdots x_{d+1}},$$

where mes $(\cdot)$ *means the d-dimensional Lebesgue measure.*

Proof. See [8, Thm. 9]. ■

The graph $T_\nu(\mathfrak{F})$ is defined as follows. The set $V_\nu(\mathfrak{F})$ of its vertices is the set of all vectors from all bases of the ν-th step of the algorithm, and we have an edge between vertices u and v if the integer vectors u, v belong to the same basis $\mathcal{E}$. We also consider the graph $T(\mathfrak{F})$ whose vertices $V(\mathfrak{F})$ are the vectors of *all* bases Ȩ appearing in the algorithm $\mathfrak{F}$ and there exists an edge between vertices u and v if and only if vectors u, v belong to the same basis $\mathcal{E}$. We emphasize that if $\mathfrak{F}$ is complete, $\left\{\left(\frac{y_1}{x},\ldots,\frac{y_d}{x}\right) \mid (x,y_1,\ldots,y_d)\in V(\mathfrak{F})\right\} = \mathbb{Q}^d\cap[0,1]^d$. Clearly $T_n(\mathfrak{F})$ is a subgraph of $T(\mathfrak{F})$.

We define a GFB algorithm to be *finite* if there is a positive constant $M(\mathfrak{F})$ such that for any vertex $v\in V(\mathfrak{F})$ of the graph $T(\mathfrak{F})$ its degree $\deg(v)$ (the number of edges with the endpoint in this vertex) is bounded by $M(\mathfrak{F})$.

Lemma 1.3. *If the GFBA $\mathfrak{F}$ is finite then it is complete.*

Proof. Let $z=(x,y_1,...,y_d), x\geq 1, 0\leq y_j\leq x$, be a primitive integer point and $\Theta=(1,y_1/x,...,y_d/x)$. Suppose z does not occur in algorithm $\mathfrak{F}$ as an element of a basis, $z\notin V(\mathfrak{F})$. As in Lemma 1.1, we define for every ν a k_ν such that $z\in\mathcal{C}(\mathcal{E}^{\nu,k_\nu})$, and look for the basis $\mathcal{E}^{\nu,k_\nu}=\{g_1^{\nu,k_\nu},...,g_{d+1}^{\nu,k_\nu}\}$ and for the corresponding simplex Δ_ν from partition Til_ν. We have $\Theta\in\Delta_\nu$, $\forall\nu$. We shall prove that the first coordinates x_j^{ν,k_ν} of the basis vectors g_j^{ν,k_ν} tend to $+\infty$ as $\nu\to\infty$. Then the lemma will be proved as $z=\sum_{j=1}^{d+1}\lambda_j g_j^{\nu,k_\nu}$ with nonnegative λ_j (and one of the λ_j must be positive). So $x\geq\min_{1\leq j\leq d+1}x_j^{\nu,k_\nu}$, and this is a contradiction.

To do it we must use the finiteness property of our algorithm. Let $a_1,a_2,...,$ a_{d+1} be vertices of the simplex Δ_ν. Now we fix a vertex of this simplex (say a_{d+1}) and show that for large enough ν' this vertex will not be a vertex of $\Delta_{\nu'}$. By the finiteness of the algorithm, we may assume that for $\nu'\geq\nu$ no additional edge with vertex a_{d+1} appears inside simplex $\Delta_{\nu'}$ during our algorithm. Also, during the algorithm only finitely many new edges may appear in the vertices $a_1,..,a_d$. Since at each step of our algorithm we must choose a partition of each simplex into smaller ones, if a_{d+1} is still a vertex of a simplex Δ' from a partition $\mathrm{Til}_{\nu'}$ with large ν', then the other vertices $a_1'=a_1'(\nu'),...,a_d'=a_d'(\nu')$ of Δ' must lie on edges $[a_{d+1},a_j], j=1,..,d$ and do not coincide with the endpoints $a_1,...,a_d$.

Let $\mathcal{E}^{\nu',k_{\nu'}}=\{g_1^{\nu',k_{\nu'}},...,g_{d+1}^{\nu',k_{\nu'}}\}$ be the corresponding basis. Then $g_{d+1}^{\nu',k_{\nu'}}=g_{d+1}^{\nu,k_\nu}$ and for every $j\in[1,..,d]$ we have $g_j^{\nu',k_{\nu'}}=\mu_j g_{d+1}^{\nu,k_\nu}+g_j^{\nu,k_\nu}$ with *positive* μ_j. It means that $a_j'(\nu')\to a_j, 1\leq j\leq d$ when $\nu'\to\infty$. So for large ν' the point Θ will not lie in the simplex with vertex a_{d+1} from the partition $\mathrm{Til}_{\nu'}$. We see that for large ν' all vertices of the simplex $\Delta_{\nu'}$ will differ from the vertices of the simplex Δ_ν. Now $\min\limits_{1\leq j\leq d+1}g_{j,1}^{\nu',k_{\nu'}}>\min\limits_{1\leq l\leq d+1}g_{j,1}^{\nu,k_\nu}$, and the first coordinates of the basis vectors $g_{j,1}^{\nu,k_\nu}$ tend to $+\infty$ as $\nu\to\infty$. The proof is complete. ■

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For a finite GFB algorithm $\mathfrak{F}$ we consider the Dirichlet series

$$L(\mathfrak{F},\beta)=\sum_{v\in V(\mathfrak{F})}\frac{\deg(v)}{(x(v))^\beta},$$

where $x(v)$ is the first coordinate of the integer vector $v=(x,y_1,\ldots,y_d)$. Since for the fixed value of x the number of integer vectors of the form $(x,y_1,...,y_d)$, $0\leq y_l\leq x$, g.c.d.$(x,y_1,...,y_d)=1$ is bounded by $(x+1)^d-2^d$, the series for $L(\mathfrak{F},\beta)$ converges when $\beta>d+1$.

Let $a = \left(\frac{a_1}{q}, ..., \frac{a_d}{q}\right) \in (\mathbb{Q}^+)^d$ and g.c.d.$(q, a_1, ..., a_d) = 1$. For every $a \in (\mathbb{Q}^+)^d$ we define $q(a) = q$. Recall that if the GFBA is finite, then it is complete, and hence the corresponding vertex $a = (q, a_1, \ldots, a_d)$ occurs as a vertex of our graph $T(\mathfrak{F})$. Let $\deg(a)$ be its degree in this graph. Then for $\beta > d+1$ we have

$$L(\mathfrak{F}, \beta) = \sum_{a \in (\mathbb{Q}^+)^d} \frac{\deg(a)}{(q(a))^\beta} = \sum_{q=1}^{+\infty} \frac{\sum_{l=1}^{M(\mathfrak{F})} l \times G_l(q)}{q^\beta}, \tag{6}$$

where $G_l(q)$ denotes the number of rational points a with common denominator q such that $\deg(a) = l$.

In the case $d = 1$ for the algorithm of taking medians of neighbouring on each step (as it was described in Section 1.1.), two vertices ($0/1$ and $1/1$) of the graph T have degree 1 and all other vertices have degree 2. So for the classical one-dimensional algorithm the Dirichlet series is

$$L(\beta) = 2 \times \frac{\zeta(\beta - 1)}{\zeta(\beta)} = 2 \times \sum_{q=1}^{\infty} \frac{\varphi(q)}{q^\beta}$$

(where $\varphi(\cdot)$ is Euler's totient function), and formulas (2), (3) give good asymptotics for the moments of the partition generated by the algorithm under consideration with the coefficient $L(2\beta)$ in the main term.

The main result of this paper is obtaining nice asymptotic formulas for $\sigma_{n,\beta}(\mathfrak{F})$, $n \to \infty$ when $d = 2$ and $\mathfrak{F}$ is one of two simplest finite GFBA algorithms. In Section 2 we consider algorithm $\mathfrak{A}$. It seems to be new but it is related to the (generalized) Poincaré algorithm (see [13], [6, ch. 21], [14]). Also algorithm $\mathfrak{A}$ is related to the construction from [8]. In Section 3 we consider algorithm $\mathfrak{B}$, which was introduced by Mönkemeyer in [12].

We note that both algorithms use a fixed rule for choosing the partition of Til_n from Til_{n-1}, for all partitions Til_n. The description of algorithm $\mathfrak{A}$ seems to be somewhat more complicated than that of algorithm $\mathfrak{B}$, however the rule for the new partitions in algorithm $\mathfrak{A}$ does not depend on the order of the vectors in the preceeding basis, while the rule for algorithm $\mathfrak{B}$ does.

2. Algorithm $\mathfrak{A}$

2.1. The description of Algorithm $\mathfrak{A}$

We fix the initial partition of the unit square $\{z = (x, y_1, y_2) : x = 1, y_{1,2} \in [0, 1]\}$ into two triangles $\Delta^{0,1}$ with vertices $(1, 0, 0), (1, 1, 0), (1, 0, 1)$ and $\Delta^{0,2}$ with vertices $(1, 1, 0), (1, 0, 1), (1, 1, 1)$. Vertices of both triangles form bases $\mathcal{E}^{0,1}$ and $\mathcal{E}^{0,2}$ of the integer lattice $\mathbb{Z}^3$. Now we suppose that a basis

$$\mathcal{E}^{\nu,j} = \{g_1^{\nu,j}, g_2^{\nu,j}, g_3^{\nu,j}\}$$

(and the corresponding triangle $\Delta^{\nu,j}$ of the partition of the unit square $[0,1]^2$ into triangles) occurs in our algorithm, and we define the rule for constructing the bases for the next step of algorithm. In our algorithm $\mathfrak{A}$ the rule also will be the same for each step of the algorithm and for each basis. Namely, for the basis $\mathcal{E}^{\nu,j}$ which occurs at the ν-th step, we take 6 bases $\mathcal{E}^{\nu+1,6(j-1)+i}$, $i = 1,2,3,4,5,6$ by the following formulas

$$\begin{aligned}
\mathcal{E}^{\nu+1,6(j-1)+1} &= \{g_1^{\nu,j}, g_1^{\nu,j} + g_2^{\nu,j}, g_1^{\nu,j} + g_3^{\nu,j}\},\\
\mathcal{E}^{\nu+1,6(j-1)+2} &= \{g_2^{\nu,j}, g_2^{\nu,j} + g_1^{\nu,j}, g_2^{\nu,j} + g_3^{\nu,j}\},\\
\mathcal{E}^{\nu+1,6(j-1)+3} &= \{g_3^{\nu,j}, g_3^{\nu,j} + g_1^{\nu,j}, g_3^{\nu,j} + g_2^{\nu,j}\},\\
\mathcal{E}^{\nu+1,6(j-1)+4} &= \{g_1^{\nu,j} + g_2^{\nu,j}, g_1^{\nu,j} + g_3^{\nu,j}, g_1^{\nu,j} + g_2^{\nu,j} + g_3^{\nu,j}\},\\
\mathcal{E}^{\nu+1,6(j-1)+5} &= \{g_2^{\nu,j} + g_1^{\nu,j}, g_2^{\nu,j} + g_3^{\nu,j}, g_1^{\nu,j} + g_2^{\nu,j} + g_3^{\nu,j}\},\\
\mathcal{E}^{\nu+1,6(j-1)+6} &= \{g_3^{\nu,j} + g_1^{\nu,j}, g_3^{\nu,j} + g_2^{\nu,j}, g_1^{\nu,j} + g_2^{\nu,j} + g_3^{\nu,j}\}.
\end{aligned}$$

We point out that the construction of the set of bases $\mathcal{E}^{\nu+1,6(j-1)+i}, 1 \le i \le 6$ does not depend on the *order* of vectors in the basis $\mathcal{E}^{\nu,j}$.

It is easy to see that the described rule satisfies the conditions (4) (i), (ii) — each new set of vectors $\mathcal{E}^{\nu+1,6(j-1)+i}$ forms a basis of the integer lattice and the cones $\mathcal{C}(\mathcal{E}^{\nu+1,6(j-1)+i})$, $1 \le i \le 6$ form a regular partition of the cone $\mathcal{C}(\mathcal{E}^{\nu,j})$. Obviously this algorithm is finite, and from Lemma 1.3 it follows that algorithm $\mathfrak{A}$ is complete. Hence, for any $\xi = (p, a_1, a_2) \in \mathbb{Z}^3$, $p \ge 1, 0 \le a_1, a_2 \le p$, g.c.d.$(p, a_1, a_2) = 1$ there exist $m \le p$ and j such that $\xi \in \mathcal{E}^{m,j}$.

We shall show in 2.4 that the multidimensional continued fraction algorithm corresponding to algorithm $\mathfrak{A}$ weakly converges everywhere.

2.2. Algorithm $\mathfrak{A}$ in terms of constructing rational points in the square $[0,1]^2$

For two rational points

$$\begin{aligned}
a &= \left(\frac{a_1}{p}, \frac{a_2}{p}\right) \in [0,1]^2, \qquad \text{g.c.d.}(p, a_1, a_2) = 1,\\
b &= \left(\frac{b_1}{q}, \frac{b_2}{q}\right) \in [0,1]^2, \qquad \text{g.c.d.}(q, b_1, b_2) = 1,
\end{aligned}$$

we define the operation

$$a \oplus b = \left(\frac{a_1}{p}, \frac{a_2}{p}\right) \oplus \left(\frac{b_1}{q}, \frac{b_2}{q}\right) = \left(\frac{a_1 + b_1}{p+q}, \frac{a_2 + b_2}{p+q}\right).$$

We note that if integer vectors

$$(p, a_1, a_2), \qquad (q, b_1, b_2), \qquad (r, c_1, c_2)$$

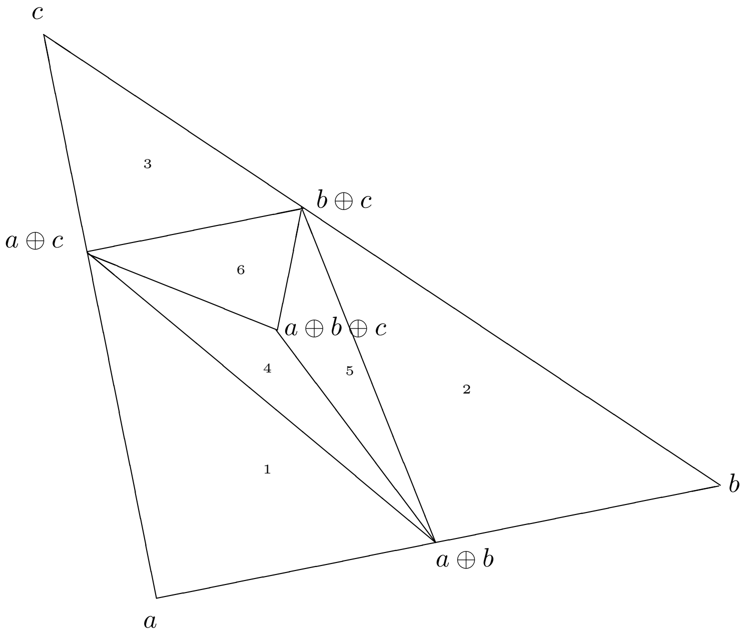

Figure 1.

with corresponding points

$$a = \left(\frac{a_1}{p}, \frac{a_2}{p}\right), \qquad b = \left(\frac{b_1}{q}, \frac{b_2}{q}\right), \qquad c = \left(\frac{c_1}{r}, \frac{c_2}{r}\right),$$

form a basis of integer lattice, then for the derived points $a \oplus b$, and $a \oplus b \oplus c$ the common denominator and both numerators are relatively prime that is $g.c.d.(p + q, a_1 + b_1, a_2 + b_2) = g.c.d.(p + q + r, a_1 + b_1 + c_1, a_2 + b_2 + c_2) = 1$.

Partitions Til_ν may be constructed as follows. The initial partition Til_0 consists of two triangles with vertices $(0,0), (1,0), (0,1)$ and $(0,1), (1,0), (1,1)$. Then a triangle Δ with vertices a, b, c in partition Til_ν is partitioned into six triangles with vertices

$$\begin{gathered} a, a \oplus b, a \oplus c; \\ b, b \oplus a, b \oplus c; \\ c, c \oplus a, c \oplus b; \\ a \oplus b, a \oplus c, a \oplus b \oplus c; \\ b \oplus a, b \oplus c, a \oplus b \oplus c; \\ c \oplus a, c \oplus b, a \oplus b \oplus c. \end{gathered}$$

The corresponding partition of the simplex Δ in this case is shown in Fig 1.

We mention a few simple combinatorial properties of the corresponding partitions Til_ν and graphs T_ν, T and their respective recurrence formula:

1. Til_ν is a partition of the unit square $[0,1]^2$ into $f_\nu = 2 \times 6^\nu$ triangles, with $f_\nu = 6f_{\nu-1}, f_0 = 2$.

2. The number of edges of the graph T_ν is $r_\nu = 2^\nu \times (3^{\nu+1} + 2)$, with $r_\nu = 2r_{\nu-1} + 6f_{\nu-1}, r_0 = 5$.

3. The number of vertices of graph T_ν is $v_\nu = 6^\nu + 2^{\nu+1} + 1$, with $v_\nu = v_{\nu-1} + r_{\nu-1} + f_{\nu-1}, v_0 = 4$. In particular, let $v_\nu^{[d]}$ be the the number of vertices of the graph T_ν with degree d, then $v_\nu^{[2]} = 2$, $v_\nu^{[3]} = (2 \cdot 6^\nu + 8)/5$, with $v_\nu^{[3]} = v_{\nu-1}^{[3]} + f_{\nu-1}, v_0^{[3]} = 2$, $v_\nu^{[5]} = 2^{\nu+2} - 4$, with $v_\nu^{[5]} = v_{\nu-1}^{[5]} + 2^{\nu+1}, v_0^{[5]} = 0$, $v_\nu^{[8]} = (6^{\nu+1} + 14)/10 - 2^{\nu+1}$ with $v_\nu^{[8]} = v_{\nu-1}^{[8]} + (3 \cdot f_{\nu-1} - 2^{\nu+1})/2, v_0^{[8]} = 0$.

4. The degree $\deg(v)$ for any vertex v of the graph T takes values from the set $\{2, 3, 5, 8\}$. Moreover a vertex from the set V_ν has the same degree in the graph T_ν and in the graph T.

5. The areas of the triangles in partition Til_ν vary between (asymptotically) $\frac{8}{(1+\sqrt{2})^{3(\nu+1)}}$, six triangles obtained by always using rule 6, assuming $q(c) \geq q(a), q(b)$ (use the recursion $x_\nu = 2x_{\nu-1} + x_{\nu-2}$ with initial values 1,2 for $q(a)_\nu, q(b)_\nu$ and 1,3 for $q(c)_\nu$), and (precisely) $\frac{1}{2(\nu+1)^2}$, the six triangles with a corner in the original square.

The Dirichlet series $L(\mathfrak{A}, \beta)$ for our algorithm can be written as follows

$$L(\mathfrak{A}, \beta) = \sum_{a \in \mathbb{Q}^2 \cap [0,1]^2} \frac{\deg(a)}{q(a)^\beta} = \sum_{q=1}^{+\infty} \frac{2G_2(q) + 3G_3(q) + 5G_5(q) + 8G_8(q)}{q^\beta},$$

where $G_l(q), l \in \{2, 3, 5, 8\}$ denotes the number of rational points $a \in [0,1]^2$ with $q(a) = q$ and $\deg(a) = l$. Clearly $G_2(q) + G_3(q) + G_5(q) + G_8(q) = \#\{(a_1, a_2) \in \mathbb{Z}^2 : \ 0 \leq a_1, a_2 \leq q, \ \text{g.c.d.}(q, a_1, a_2) = 1\} \leq (q+1)^2$.

$L(\mathfrak{A}, \beta)$ will be used in Section 2.5 to obtain the asymptotic behaviour of $\sigma_{n,\beta}(\mathfrak{A})$ for $n \to \infty$.

2.3. Lemmata about triangles from partition Til_ν

Let a triangle Δ occur in partition Til_n. We define $\Delta^*(\Delta)$ to be the unique triangle from partition Til_{n-1} such that $\Delta \subset \Delta^*(\Delta)$.

Lemma 2.1. *Let $\Delta \in \mathrm{Til}_n$ and a, b, c be the vertices of the triangle $\Delta^*(\Delta)$, with a not a vertex of Δ. Then for all vertices ω of Δ we have $q(\omega) \geq \min\{q(b), q(c)\}$.*

Proof. The lemma is obvious, since the expression for $q(\omega)$ from algorithm $\mathfrak{A}$ (rules 2–6) is the sum of one to three positive summands, and one of them is $q(b)$ or $q(c)$. ∎

For every triangle Δ_n we can consider the unique sequence of nested triangles

$$\Delta_n \subset \Delta_{n-1} \subset \cdots \subset \Delta_1 \subset \Delta_0 \tag{7}$$

where Δ_ν is a triangle from the partition Til_ν and $\Delta_\nu = \Delta^*(\Delta_{\nu+1})$. Sometimes for convenience we shall write $\Delta^{[0]}(\Delta) = \Delta$ and $\Delta^{[k]}(\Delta) = \underbrace{\Delta^*(...(\Delta^*(}_{k \text{ times}}\Delta)))$, so $\Delta_{n-k} = \Delta^{[k]}(\Delta_n)$.

Now, for triangle Δ from partition Til_n we define the value $t(\Delta)$ which is of principal importance for our proof ($t(\Delta)$ is an analogue for the partial quotient of an ordinary one-dimensional continued fraction).

If $\Delta^*(\Delta)$ has no common vertices with Δ then we put $t(\Delta) = 1$. If all triangles $\Delta^{[k]}(\Delta), k = 0, ..., t$ have common vertex a but this vertex a is not a vertex of the triangle $\Delta^{[t+1]}(\Delta)$, we write $t(\Delta) = t$. From the construction of our algorithm we observe that in the case $t(\Delta) \geq 2$ for any $k \in \{1, 2, ..., t\}$ the following holds: If $\Delta^{[k]}(\Delta)$ has vertices a, b, c then $\Delta^{[k-1]}(\Delta)$ has vertices $a, a \oplus b, a \oplus c$.

Lemma 2.2. *Let Δ be a triangle from the partition Til_n, $t(\Delta) = t$, $\Delta = \Delta^{[0]}(\Delta) \subset \cdots \subset \Delta^{[t]}(\Delta)$, and let a be the common vertex for all these triangles. Then $a \in V_{n-t} \setminus V_{n-t-1}$.*

Proof. From the conditions it follows that a is a vertex of $\Delta^{[t]}(\Delta)$ but not a vertex of $\Delta^*(\Delta^{[t]}(\Delta)) = \Delta^{[t+1]}(\Delta)$. From the construction of algorithm $\mathfrak{A}$ one can see that this may happen only when $a \in V_{n-t} \setminus V_{n-t-1}$. ∎

After the definition of $t(\Delta)$ we can construct a subsequence of the sequence (7) in the following way. Put

$$\begin{aligned} \Delta_n \subset \Delta_{n-t_r} \subset \Delta_{n-t_r-t_{r-1}} \subset \cdots \subset \Delta_{n-t_r-t_{r-1}-...-t_2} \\ \subseteq \Delta_1 \subset \Delta_{n-t_r-t_{r-1}-...-t_2-t_1} = \Delta_0, \end{aligned} \tag{8}$$

where t_k are natural numbers,

$$t_1 + ... + t_r = n$$

and

$$t_k = t(\Delta_{n-t_r-...-t_{k+1}}).$$

Hence for each Δ from the partition Til_n we have the correspondence $\Delta \mapsto [t_1, ..., t_r]$, $t_j \in \mathbb{N}$, $t_1 + ... + t_r = n$. We define the sequence $[t_1, ..., t_r]$ as *code* of triangle Δ (different triangles from the partition Til_n may have the same code). We define the *empty* code to correspond to any triangle from the initial partition Til_0.

Lemma 2.3. *Let the triangle $\Delta = \Delta_n$ with vertices a, b, c and code $[t_1, t_2, ..., t_r]$ occur in partition Til_n and $r \geq 2$. Let $\Delta^{[t_r+t_{r-1}]}(\Delta) = \Delta_{n-t_r-t_{r-1}}$ be the triangle with code $[t_1, t_2, ..., t_{r-2}]$ and vertices a', b', c'. Then*

$$\min\{q(a), q(b), q(c)\} \geq 2 \min\{q(a'), q(b'), q(c')\}.$$

Proof. Let $\Delta^{[t_r]}(\Delta)$ have vertices a'', b'', c''. We consider three cases.

1. In the case when $t_r = 1$ and triangles $\Delta, \Delta^{[t_r]}(\Delta)$ have no common vertex we obtain (for the corresponding notation of vertices)

$$a = a'' \oplus b'' \oplus c'', \;\; b = a'' \oplus c'', \;\; c = a'' \oplus b'',$$

and

$$q(a) = q(a'') + q(b'') + q(c''), \;\; q(b) = q(a'') + q(c''), \;\; q(c) = q(a'') + q(b'').$$

Hence

$$\min\{q(a), q(b), q(c)\} \geq 2\min\{q(a''), q(b''), q(c'')\} \geq 2\min\{q(a'), q(b'), q(c')\}.$$

2. In the case when $t_{r-1} = 1$ and triangles $\Delta^{[t_r]}(\Delta), \Delta^{[t_r+t_{r-1}]}(\Delta)$ have no common vertex by the same reasons we obtain

$$\min\{q(a), q(b), q(c)\} \geq \min\{q(a''), q(b''), q(c'')\} \geq 2\min\{q(a'), q(b'), q(c')\}.$$

3. Finally, we must consider the case where the triangles $\Delta, \Delta^{[t_r]}(\Delta)$ have a common vertex, and the triangles $\Delta^{[t_r]}(\Delta), \Delta^{[t_r+t_{r-1}]}(\Delta)$ also have a common vertex, where the first common vertex must not coincide with the second one. Without loss of generality we may assume that $a = a''$ is the common vertex for $\Delta, \Delta^{[t_r]}(\Delta)$ and $b' = b''$ is the common vertex for $\Delta^{[t_r]}(\Delta), \Delta^{[t_r+t_{r-1}]}(\Delta)$. Then

$$b = b'' \underbrace{\oplus a'' \cdots \oplus a''}_{t_r \text{ times}}, \qquad c = c'' \underbrace{\oplus a'' \cdots \oplus a''}_{t_r \text{ times}},$$
$$a'' = a' \underbrace{\oplus b' \cdots \oplus b'}_{t_{r-1} \text{ times}}, \qquad c'' = c' \underbrace{\oplus b' \cdots \oplus b'}_{t_{r-1} \text{ times}}$$

and

$$\begin{aligned} &q(a) = q(a'') = q(a') + t_{r-1}q(b') \geq q(a') + q(b') \geq 2\min\{q(a'), q(b'), q(c')\}, \\ &q(b) = q(b'') + t_r q(a'') \geq 2\min\{q(a'), q(b'), q(c')\}, \\ &q(c) = q(c'') + t_r q(a'') \geq 2\min\{q(a'), q(b'), q(c')\}. \end{aligned}$$

The Lemma is proved. ■

Lemma 2.4. *Let Δ with vertices a, b, c have code $[t_1, ..., t_r]$. Then*

$$\min\{q(a), q(b), q(c)\} \geq 2^{\lfloor r/2 \rfloor}.$$

Proof. Lemma 2.4 follows by induction from Lemma 2.3. ■

We need one more lemma about triangles.

Lemma 2.5. *Let Δ be a triangle from the partition* Til_ν *and a, b, c be vertices of Δ. Let*

$$f = \min\{q(a), q(b), q(c)\}, \qquad F = \max\{q(a), q(b), q(c)\}.$$

Then $F \leq (\nu + 1)f$.

Proof. Induction in ν. The base for $\nu = 0$ is obvious. Let a', b', c' be vertices of $\Delta^*(\Delta)$, w.l.o.g. $q(a') \le q(b') \le q(c')$ and by induction assumption $q(c') \le \nu q(a')$. At the ν-th step of algorithm $\mathfrak{A}$, we obtain the triangle Δ from the triangle $\Delta^*(\Delta)$. There are six possibilities:

1. Δ has vertices $a', a' \oplus b', a' \oplus c'$. Then $f = q(a'), F = q(a') + q(c') \le (\nu+1)q(a')$.
2. Δ has vertices $b', b' \oplus a', b' \oplus c'$. Then $f = q(b'), F = q(b') + q(c') \le q(b') + \nu q(a') \le (\nu+1)q(b')$.
3. Δ has vertices $c', c' \oplus a', c' \oplus b'$. Then $f = q(c'), F = q(b') + q(c') \le q(b') + \nu q(a') \le (\nu+1)q(c')$.
4. Δ has vertices $a' \oplus b' \oplus c', a' \oplus b', a' \oplus c'$. Then $f = q(a') + q(b')$, $F = q(a') + q(b') + q(c') \le (\nu+1)(q(a') + q(b'))$.
5. Δ has vertices $a' \oplus b' \oplus c', a' \oplus b', b' \oplus c'$. Then $f = q(a') + q(b')$, $F = q(a') + q(b') + q(c') \le (\nu+1)(q(a') + q(b'))$.
6. Δ has vertices $a' \oplus b' \oplus c', a' \oplus c', b' \oplus c'$. Then $f = q(a') + q(c')$, $F = q(a') + q(b') + q(c') \le (\nu+1)(q(a') + q(c'))$.

So in every case we have $F \le (\nu+1)f$, and the lemma is proved. ■

2.4. Global weak convergence of Algorithm $\mathfrak{A}$

Theorem 2.1. *The multidimensional continued fraction algorithm corresponding to algorithm $\mathfrak{A}$ weakly converges everywhere.*

Proof. For $\xi \in [0,1]^2$ we look for a sequence of triangles

$$\Delta_0 \supset \Delta_1 \supset \cdots \supset \Delta_\nu \supset \cdots, \qquad \bigcap_\nu \Delta_\nu \ni \xi, \qquad \Delta_\nu \in \mathrm{Til}_\nu.$$

It is sufficient to prove that $\mathrm{diam}\Delta_\nu \to 0$ as $\nu \to \infty$, where $\mathrm{diam}\Omega = \sup_{x,y\in\Omega} |x,y|$, and $|x,y|$ is the distance between points x and y.

Let $\Delta_{\nu-1}$ have vertices

$$a = \left(\frac{a_1}{q(a)}, \frac{a_2}{q(a)}\right), \qquad b = \left(\frac{b_1}{q(b)}, \frac{b_2}{q(b)}\right), \qquad c = \left(\frac{c_1}{q(c)}, \frac{c_2}{q(c)}\right)$$

We consider two cases.

1. Triangles $\Delta_{\nu-1}$ and Δ_ν have a common vertex. Let this common vertex be a. Then Δ_ν has vertices $a' = a, b' = a \oplus b, c' = a \oplus c$. Then we define the quotient

$$s = \frac{|b', a'|}{|b, a|} = \left| \frac{\frac{a_j + b_j}{q(a)+q(b)} - \frac{a_j}{q(a)}}{\frac{b_j}{q(b)} - \frac{a_j}{q(a)}} \right| = \frac{q(b)}{q(a)+q(b)}, \qquad j = 1, 2.$$

By Lemma 2.5 we have $\frac{1}{\nu} \le s \le \frac{\nu-1}{\nu}$, hence $\frac{1}{\nu} \le \frac{|a',b'|}{|a,b|} \le \frac{\nu-1}{\nu}$. By the same reason $\frac{1}{\nu} \le \frac{|a',c'|}{|a,c|} \le \frac{\nu-1}{\nu}$. So obviously

$$|a', b'| \le \left(1 - \frac{1}{\nu}\right)|a, b|, \qquad |a', c'| \le \left(1 - \frac{1}{\nu}\right)|a, c|.$$

As for $|b', c'|$ we easily deduce

$$|b', c'| \leq \left(1 - \frac{1}{\nu}\right) \max\{|a,b|, |a,c|, |b,c|\}$$

(if both angles in vertices b', c' are less than $\pi/2$, we have the bound $|b,c|$; in the other cases $|a,b|$ or $|a,c|$, respectively). Now

$$\begin{aligned}\mathrm{diam}\Delta_\nu &= \max\{|a',b'|, |a',c'|, |b',c'|\} \leq \left(1 - \frac{1}{\nu}\right) \max\{|a,b|, |a,c|, |b,c|\} = \\ &= \left(1 - \frac{1}{\nu}\right) \mathrm{diam}\Delta_{\nu-1}.\end{aligned}$$

2. Triangles $\Delta_{\nu-1}$ and Δ_ν have no common vertex. Then Δ_ν lies inside triangle Δ^+ with vertices $a \oplus b, a \oplus c, b \oplus c$ and

$$\begin{aligned}\mathrm{diam}\Delta_\nu \leq \mathrm{diam}\Delta^+ &= \max\{|a \oplus b, a \oplus c|, |a \oplus b, b \oplus c|, |b \oplus c, a \oplus c|\} \\ &\leq \left(1 - \frac{1}{\nu}\right) \mathrm{diam}\Delta_{\nu-1}\end{aligned}$$

by the same reasons.

In both cases, we have (using $\mathrm{diam}(\Delta_1) = 1$ for all four $\Delta_1 \in \mathrm{Til}_1$)

$$\mathrm{diam}\Delta_\nu \leq \prod_{l=2}^{\nu} \left(1 - \frac{1}{l}\right) = \frac{1}{\nu} \to 0, \;\; \nu \to \infty,$$

and the theorem is proved. ∎

2.5. Asymptotic behaviour of Algorithm $\mathfrak{A}$

In this section, we will prove a formula for the moments of $\sigma_{n,\beta}(\mathfrak{A})$ for algorithm $\mathfrak{A}$ analogous to (2).

Lemma 2.6. *For any $\beta > 1$, we have*

$$\sum_{n=0}^{\infty} \sigma_{n,\beta}(\mathfrak{A}) \leq \frac{16}{3} \zeta(2\beta)\zeta(3\beta - 2).$$

Proof. For triangle Δ we consider the vertex $\alpha(\Delta)$ such that the common denominator $q(\alpha(\Delta))$ is the smallest among all vertices of triangle Δ (it may not be unique and in this case we fix one of the minimal vertices). Then

$$\sigma_{n,\beta}(\mathfrak{A}) = \sum_{\Delta \in \mathrm{Til}_n} (\mathrm{mes}\,\Delta)^\beta = \sum_{m=0}^{n} \sum_{\substack{\Delta \in \mathrm{Til}_n \\ \alpha(\Delta) \in V_m \setminus V_{m-1}}} (\mathrm{mes}\,\Delta)^\beta .$$

Supposing that the following series converges (absolutely) we change the order of summations:

$$\sum_{n=0}^{\infty} \sigma_{n,\beta}(\mathfrak{A}) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{\substack{\Delta \in \mathrm{Til}_n, \\ \alpha(\Delta) \in V_m \setminus V_{m-1}}} (\mathrm{mes}\, \Delta)^{\beta} =$$

$$= \sum_{m=0}^{\infty} \sum_{\alpha \in V_m \setminus V_{m-1}} \sum_{n=m}^{\infty} \sum_{\substack{\Delta \in \mathrm{Til}_n, \\ \alpha(\Delta) = \alpha}} (\mathrm{mes}\, \Delta)^{\beta}.$$

We fix a point $a \in V_m \setminus V_{m-1}$. Among the triangles from partition Til_m there are triangles with vertex a. (The number of these triangles is 2, 3, 5, or 8.) Some of these triangles Δ may admit the property $a = \alpha(\Delta)$ and some may not. If we consider a triangle from Til_m with vertices a, b, c and $a \neq \alpha(\Delta)$ then the triangle Δ' with vertices $a, a \oplus b, a \oplus c$ must appear in the partition Til_{m+1}, and this triangle Δ' has the property $a = \alpha(\Delta')$. Hence the vertex $a \in V_m$ is totally surrounded by triangles

$$\Delta^{(1)}, ..., \Delta^{(r)}, \quad r = \deg(a) \in \{2, 3, 5, 8\}, \tag{9}$$

where the triangle $\Delta^{(i)}$ has vertices $a, b^{(i)}, c^{(i)}$, $a = \alpha(\Delta^{(i)})$, and each of these triangles belongs to partition Til_m or Til_{m+1}. Moreover, every triangle Δ from a partition $\mathrm{Til}_n, n \geq m+1$ with the property $a = \alpha(\Delta)$ may be obtained from one of the triangles in (9) in the following sense. If Δ does not coincide with one of the triangles from (9) and a, b, c are vertices of Δ then

$$b = b^{(i)} \underbrace{\oplus a \cdots \oplus a}_{j \text{ times}}, \qquad c = c^{(i)} \underbrace{\oplus a \cdots \oplus a}_{j \text{ times}},$$

where $j = n - m$ or $j = n - m - 1$. We see that for the rational point $b = b^{(i)} \underbrace{\oplus a \cdots \oplus a}_{j \text{ times}}$ the common denominator is $q(b) = q(b^{(i)}) + jq(a)$ and for the rational point $c = c^{(i)} \underbrace{\oplus a \cdots \oplus a}_{j \text{ times}}$ the common denominator is $q(c) = q(c^{(i)}) + jq(a)$. But for any i we have $\min\{q(a), q(b^{(i)}), q(c^{(i)})\} = q(a)$ and hence $q(b), q(c) \geq (j+1)q(a)$. Recall that by Lemma 1.2, triangle Δ with vertices a, b, c has $\mathrm{mes}\, \Delta = \frac{1}{2q(a)q(b)q(c)}$. Now we see that for fixed $a \in V_m$ we can get the upper bound

$$\sum_{n=m}^{\infty} \sum_{\substack{\Delta \in \mathrm{Til}_n, \\ \alpha(\Delta) = a}} (\mathrm{mes}\, \Delta)^{\beta} \leq \frac{8}{2(q(a))^{3\beta}} \sum_{j=1}^{\infty} \frac{1}{j^{2\beta}} = \frac{4\zeta(2\beta)}{(q(a))^{3\beta}}.$$

We continue our estimate:

$$\sum_{n=0}^{\infty} \sigma_{n,\beta}(\mathfrak{A}) \leq 4\zeta(2\beta) \times \left(\sum_{m=0}^{\infty} \sum_{a \in V_m \setminus V_{m-1}} \frac{1}{(q(a))^{3\beta}} \right).$$

To complete the proof of Lemma 2.6 we must use the estimate

$$\sum_{m=0}^{\infty}\sum_{a\in V_m\setminus V_{m-1}}\frac{1}{(q(a))^{3\beta}}=\sum_{q=1}^{\infty}\sum_{\substack{0\le a,b\le q\\ g.c.d.(q,a,b)=1}}\frac{1}{q^{3\beta}}\le\frac{4}{3}\sum_{q=1}^{\infty}\frac{1}{q^{3\beta-2}}=\frac{4}{3}\zeta(3\beta-2), \tag{10}$$

where the first equality follows from the completeness property of our algorithm, also $\frac{4}{3}=\max_q\frac{(q+1)^2-4}{q^2}$. Observe that all the series under consideration converge (absolutely), ■

In the sequel we shall use not only Lemma 2.6 but also inequality (10) from the proof of Lemma 2.6.

Now we take parameters

$$\gamma=\frac{4(6\beta^2+\beta-1)}{9\log 2\cdot(\beta-1)\beta},\qquad w=(\log n)^{1-\frac{1}{3\beta}}n^{\frac{2\beta+1}{3\beta}} \tag{11}$$

and divide the sum from the definition $\sigma_{n,\beta}(\mathfrak{A})$ into three sums

$$\sigma_{n,\beta}(\mathfrak{A})=\sum_{\Delta\in\mathrm{Til}_n}(\mathrm{mes}\,\Delta)^{\beta}=\Sigma_{(1)}+\Sigma_{(2)}+\Sigma_{(3)},$$

where $\Sigma_{(1)}$ is the sum over all Δ from Til_n for which in the code $[t_1,...,t_r]$ we have

$$r\ge\gamma\log n, \tag{12}$$

$\Sigma_{(2)}$ is the sum over all Δ from Til_n for which in the code $[t_1,...,t_r]$ we have

$$r<\gamma\log n,\qquad 1\le t_r\le n-w, \tag{13}$$

and $\Sigma_{(3)}$ is the sum over all Δ from Til_n for which in the code $[t_1,...,t_r]$ we have

$$r<\gamma\log n,\qquad t_r>n-w. \tag{14}$$

($\Sigma_{(3)}$ will be the dominating term).

Lemma 2.7. $\Sigma_{(1)}\le n^{-(3\log 2\ \gamma(\beta-1))/4}$.

Proof. Obviously,

$$\Sigma_{(1)}\le\max_{\Delta\in\mathrm{Til}_n,\ r\ge\gamma\log n}(\mathrm{mes}\,\Delta)^{\beta-1}\times\sum_{\Delta\in\mathrm{Til}_n}\mathrm{mes}\,\Delta.$$

Let the maximum occur on some triangle Δ with vertices a,b,c. We apply (5) and the inequality

$$\begin{aligned}\max_{\substack{\Delta\in\mathrm{Til}_n,\\ r\ge\gamma\log n}}(\mathrm{mes}\,\Delta)^{\beta-1}&=\frac{1}{(2q(a)q(b)q(c))^{\beta-1}}\le\frac{1}{(2^{1+3\lfloor r/2\rfloor})^{\beta-1}}\\&\le\frac{1}{2^{3\gamma(\beta-1)\log n/4}}=n^{-\frac{3\log 2\ \gamma(\beta-1)}{4}},\end{aligned}$$

which follows from Lemma 1.2 and Lemma 2.4. Lemma 2.7 is proved. ■

Lemma 2.8.

$$\Sigma_{(2)} \leq \frac{2560}{9}(\zeta(3\beta-2))^2\zeta(2\beta)\left(\frac{\gamma\log n}{w}\right)^{3\beta-1}.$$

Proof. Under the conditions (13) we see that $t_1+...+t_{r-1}>w$ and hence there exists $j\leq r-1$ such that $t_j\geq \tau=\left\lceil\frac{w}{\gamma\log n-1}\right\rceil$.

For a triangle Δ with code $[t_1,...,t_r]$ we consider the sequence of triangles (8) and especially the triangle Δ_k from the partition Til_k and the next triangle Δ_{k+t_j} from the partition Til_{k+t_j} with $k=t_1+...+t_{j-1}$. Let a,b,c be the vertices of Δ_k, then $a, b'=b\underbrace{\oplus a\cdots\oplus a}_{t_j\text{ times}}$, $c'=c\underbrace{\oplus a\cdots\oplus a}_{t_j\text{ times}}$ are the vertices of triangle Δ_{k+t_j}. For the corresponding common denominators we have

$$q(b')=t_jq(a)+q(b)\geq t_jq(a),\ q(c')=t_jq(a)+q(c)\geq t_jq(a).$$

As the element t_j is *not the last* element in the code $[t_1,...,t_r]$ in the complete sequence of triangles (7), there exists the triangle Δ_{k+t_j+1}. By Lemma 2.1, for every vertex ω of the triangle Δ_{k+t_j+1} we have

$$q(\omega)\geq t_jq(a). \tag{15}$$

Now we look for the partition Til_n restricted to the triangle Δ_{k+t_j+1}. It is isomorphic to the partition Til_{n-k-t_j-1}. Moreover for any triangle $\Delta\subset\Delta_{k+t_j+1}$ with vertices s,u,v from the partition Til_n and the corresponding triangle Δ' with vertices s',u',v' from the isomorphic partition Til_{n-k-t_j-1}, by (15) we deduce that $q(s)\geq t_jq(a)\cdot q(s'), q(u)\geq t_jq(a)\cdot q(u'), q(v)\geq t_jq(a)\cdot q(v')$, and hence

$$\mathrm{mes}\,\Delta=\frac{1}{2q(s)q(u)q(v)}\leq\frac{1}{2(t_jq(a))^3q(s')q(u')q(v')}=\frac{\mathrm{mes}\,\Delta'}{(t_jq(a))^3}.$$

On the other hand, Lemma 2.2 shows that the vertex a of the triangle Δ_k satisfies $a\in V_k\setminus V_{k-1}$. We take into account that vertex a may be a common vertex for no more than eight triangles from the partition Til_k. Also we must take into account that in partition Til_n there exist just five triangles Δ satisfying the conditions of Lemma 2.1 with the given $\Delta^*(\Delta)$. Hence

$$\Sigma_{(2)}\leq\sum_{\tau\leq t\leq n}\ \sum_{\substack{k\geq 0,h\geq 1:\\ k+h=n-t}}\left(\sum_{a\in V_k\setminus V_{k-1}}\frac{8}{(q(a))^{3\beta}}\right)\times\frac{1}{t^{3\beta}}\times\left(\sum_{\Delta\in\mathrm{Til}_{h-1}}5\,(\mathrm{mes}\,\Delta)^\beta\right)\leq$$

$$\leq\frac{40}{\tau^{3\beta-1}}\times\left(\sum_{k=0}^{\infty}\sum_{a\in V_k\setminus V_{k-1}}\frac{1}{(q(a))^{3\beta}}\right)\times\left(\sum_{h=0}^{\infty}\sum_{\Delta\in\mathrm{Til}_h}(\mathrm{mes}\,\Delta)^\beta\right).$$

But

$$\sum_{h=0}^{\infty}\sum_{\Delta\in\mathrm{Til}_h}(\mathrm{mes}\,\Delta)^\beta=\sum_{h=0}^{\infty}\sigma_{h,\beta}(\mathfrak{A})\leq\frac{16}{3}\zeta(2\beta)\zeta(3\beta-2)$$

by Lemma 2.6 and

$$\sum_{k=0}^{\infty}\sum_{a\in V_k\setminus V_{k-1}}\frac{1}{(q(a))^{3\beta}}\leq\frac{4}{3}\zeta(3\beta-2),$$

applying the upper bound (10). Now the inequality of Lemma 2.8 follows. ■

Lemma 2.9. $\Sigma_{(3)}=\frac{L(\mathfrak{A},3\beta)}{(2n^2)^\beta}+O\left(\Sigma_{(1)}\right)+O\left(\frac{1}{n^{2\beta}w^{3(\beta-1)}}+\frac{w}{n^{2\beta+1}}\right)$

Proof. Obviously

$$\Sigma_{(3)}=\Sigma'_{(3)}+O\left(\Sigma_{(1)}\right),$$

where

$$\Sigma'_{(3)}=\sum_{\substack{\Delta\in\mathrm{Til}_n:\\ \text{code }[t_1,\ldots,t_r]\text{ such that }\ t_r>n-w}}(\mathrm{mes}\,\Delta)^\beta.$$

Let Δ be a triangle from Til_n with code $[t_1,...,t_{r-1},t_r]$ and $t_r>n-w$. Then the triangle $\Delta'=\Delta^{[t_r]}$ belongs to the partition Til_{n-t_r} and its code is $[t_1,...,t_{r-1}]$, $t_1+\ldots+t_{r-1}=n-t_r<w$.

Define a to be the common vertex for $\Delta,\Delta^{[t_r]}$. Then by Lemma 2.2 we have $a\in V_{n-t_r}\setminus V_{n-t_r-1}$.

On the other hand, for any triangle Δ' with code $[t_1,...,t_{r-1}]$ from partition Til_m, $m=n-t_r<w$ with fixed vertex $a\in V_m\setminus V_{m-1}$, there exists only one triangle in Til_n with code $[t_1,...,t_{r-1},t_r]$ and vertex a. Hence

$$\Sigma'_{(3)}=\sum_{m=0}^{w-1}\sum_{a\in V_m\setminus V_{m-1}}\sum_{\substack{\Delta\in\mathrm{Til}_m:\\ a\text{ is a vertex of }\Delta}}(\mathrm{mes}\,\Delta)^\beta,$$

where $\Delta\subset\Delta'$ is the unique triangle with code $[t_1,...,t_{r-1},t_r]$ and common vertex a. Let Δ' from Til_m have vertices a,b,c. Then Δ has vertices a, $b\underbrace{\oplus a\cdots\oplus a}_{t_r\text{ times}}$, $c\underbrace{\oplus a\cdots\oplus a}_{t_r\text{ times}}$, and by Lemma 2.5

$$q(b\underbrace{\oplus a\cdots\oplus a}_{t_r\text{ times}})=t_rq(a)+q(b)\leq(n+1)q(a),$$
$$q(c\underbrace{\oplus a\cdots\oplus a}_{t_r\text{ times}})=t_rq(a)+q(c)\leq(n+1)q(a).$$

Recall that $t_r=n-m>n-w$. Applying Lemma 1.2, we have

$$\sum_{m=0}^{w-1}\sum_{a\in V_m\setminus V_{m-1}}\frac{\deg(a)}{(2(n+1)^2q(a)^3)^\beta}\leq\Sigma'_{(3)}\leq\sum_{m=0}^{w-1}\sum_{a\in V_m\setminus V_{m-1}}\frac{\deg(a)}{(2(n-m)^2q(a)^3)^\beta},$$

and thus

$$\Sigma'_{(3)} = \sum_{m=0}^{w-1} \sum_{a\in V_m\setminus V_{m-1}} \frac{\deg(a)}{(2n^2 q(a)^3)^\beta} \left(1 + O\left(\frac{w}{n}\right)\right).$$

But

$$\sum_{q=1}^{w-1} \frac{\sum_l lG_l(q)}{q^{3\beta}} \le \sum_{a\in V_{w-1}} \frac{\deg(a)}{(q(a))^{3\beta}} \le \sum_{q=1}^{\infty} \frac{\sum_l lG_l(q)}{q^{3\beta}} \tag{16}$$

and

$$\sum_{q=w}^{\infty} \frac{\sum_l lG_l(q)}{q^{3\beta}} \ll \sum_{q=w}^{\infty} \frac{1}{q^{3\beta-2}} \ll_\beta w^{-3(\beta-1)}.$$

Hence

$$\sum_{m=0}^{w-1} \sum_{a\in V_m\setminus V_{m-1}} \frac{\deg(a)}{(q(a))^{3\beta}} = \sum_{a\in V_{w-1}} \frac{\deg(a)}{(q(a))^{3\beta}} = L(\mathfrak{A}, 3\beta) + O\left(w^{-3(\beta-1)}\right). \tag{17}$$

It follows that

$$\Sigma'_{(3)} = \left(\frac{L(\mathfrak{A}, 3\beta)}{(2n^2)^\beta} + O\left(\frac{1}{n^{2\beta} w^{3(\beta-1)}}\right)\right) \left(1 + O\left(\frac{w}{n}\right)\right),$$

and the lemma is proved. ■

Theorem 2.2. *For $\beta > 1$ the following asymptotic formula is valid*

$$\sigma_{n,\beta}(\mathfrak{A}) = \frac{L(\mathfrak{A}, 3\beta)}{(2n^2)^\beta} \left(1 + O\left(\frac{(\log n)^{1-\frac{1}{3\beta}}}{n^{\frac{\beta-1}{3\beta}}}\right)\right).$$

Note that $L(\mathfrak{A}, 3\beta)$ is unbounded as $\beta \to 1+$, but of course $\sigma_{n,1} = 1$ for every n.

Proof. We need to put together the results of Lemmata 2.7, 2.8, 2.9 and take into account the choice of parameters in (11):

$$\begin{aligned}\sigma_{n,\beta}(\mathfrak{A}) &= \frac{L(\mathfrak{A}, 3\beta)}{(2n^2)^\beta} + O\left(\frac{1}{n^{\frac{3\log 2\ \gamma(\beta-1)}{4}}} + \frac{1}{n^{2\beta} w^{3(\beta-1)}} + \frac{w}{n^{2\beta+1}} + \left(\frac{\gamma \log n}{w}\right)^{3\beta-1}\right)\\ &= \frac{L(\mathfrak{A}, 3\beta)}{(2n^2)^\beta} \left(1 + O\left(\frac{(\log n)^{1-\frac{1}{3\beta}}}{n^{\frac{\beta-1}{3\beta}}}\right)\right).\end{aligned}$$

This shows the asymptotic formula. ■

3. Algorithm $\mathfrak{B}$

3.1. The description of Algorithm $\mathfrak{B}$

We fix the initial partition of the unit square $\{z = (x, y_1, y_2) : x = 1, y_{1,2} \in [0,1]\}$ into two triangles $\Delta^{0,1}$ with vertices $(1,0,0), (1,1,0), (1,0,1)$ and $\Delta^{0,2}$ with vertices $(1,1,1), (1,0,1), (1,1,0)$. The vertices of both triangles form bases $\mathcal{E}^{0,1}$ and $\mathcal{E}^{0,2} = \{(1,1,1), (1,0,1), (1,1,0)\}$ (the *order* is of importance here) of the integer lattice $\mathbb{Z}^3$. Now we suppose that a basis

$$\mathcal{E}^{\nu,j} = \{g_1^{\nu,j}, g_2^{\nu,j}, g_3^{\nu,j}\}$$

(and corresponding triangle Δ_j^{ν} of the partition of the unit square $[0,1]^2$ into triangles) occurs in our algorithm and we define the rule for constructing the bases for the next step of algorithm (the rule for dividing cone $\mathcal{C}(\mathcal{E}^{\nu,j})$ of the basis $\mathcal{E}^{\nu,j}$). In our algorithm $\mathfrak{B}$ the rule will be the same for each step of the algorithm and for each basis. Namely, for the basis $\mathcal{E}^{\nu,j}$ which occurs at ν-th step we take 2 bases $\mathcal{E}^{\nu+1,2(j-1)+i}$, $i = 1,2$ by the following formulas

$$\begin{aligned}
\mathcal{E}^{\nu+1,2(j-1)+1} &= \{g_2^{\nu,j} + g_3^{\nu,j}, g_1^{\nu,j}, g_2^{\nu,j}\}, \quad \text{operation “1”}\\
\mathcal{E}^{\nu+1,2(j-1)+2} &= \{g_2^{\nu,j} + g_3^{\nu,j}, g_1^{\nu,j}, g_3^{\nu,j}\}, \quad \text{operation “0”}.
\end{aligned}$$

The construction of the set of bases $\mathcal{E}^{\nu+1,2(j-1)+i}$, $1 \le i \le 2$ depends on the *order* of elements of the basis $\mathcal{E}^{\nu,j}$.

Obviously, this rule satisfies the conditions (4) (i), (ii) — each new set of vectors $\mathcal{E}^{\nu+1,2(j-1)+i}$ is a basis of the integer lattice, and the cones $\mathcal{C}(\mathcal{E}^{\nu+1,2(j-1)+i})$, $1 \le i \le 2$ form a regular partition of the cone $\mathcal{C}(\mathcal{E}^{\nu,j})$. Clearly algorithm $\mathfrak{B}$ is finite and hence complete. The construction described was introduced and studied in [12] (for the general d-dimensional situation). For example, in [12] it was shown that the corresponding multidimensional continued fraction algorithm weakly converges.

3.2. Algorithm $\mathfrak{B}$ in terms of constructing rational points in the square $[0,1]^2$

Note that if integer vectors (p, a_1, a_2), (q, b_1, b_2) can be extended to a basis of the integer lattice $\mathbb{Z}^3$, then the corresponding rational points $a = \left(\frac{a_1}{p}, \frac{a_2}{p}\right)$ and $b = \left(\frac{b_1}{q}, \frac{b_2}{q}\right)$ define a rational point $a \oplus b$, whose common denominator and both numerators are relatively prime.

Partitions Til_ν may be constructed as follows. The initial partition Til_0 consists of two triangles with vertices $(0,0), (1,0), (0,1)$ and $(1,1), (1,0), (0,1)$. Then a triangle Δ with vertices a, b, c in partition Til_ν must be partitioned into two triangles with vertices

$$c \oplus b, a, b, \quad \text{and} \quad c \oplus b, a, c.$$

We must note that the *order* of the enumeration of vertices of triangle Δ is important for the constructing of our partition.

We call the first rule an operation "1", the second one an operation "0". To every triangle $\Delta \in \mathrm{Til}_n$, we then attach a code $\mathbf{c}(\Delta) = \mathbf{c}_1 \dots \mathbf{c}_n$, where $\mathbf{c}_k \in \{0,1\}$ states, which rule was used for the k-th partition. Also, let $|\mathbf{c}|(\Delta) = \sum_{k=1}^{n} \mathbf{c}_k$ be the number of operations "1" to obtain Δ.

Figure 2 shows the first 6 partitions (Til_5 only in the lower left quarter), together with the points (q, a_1, a_2). Walking in the direction of the arrowheads, one obtains a binary tree, infinite in the case of T.

Lemmata 1.2 and 2.5 (consider $q(b)+q(c) \le \nu q(c) + q(c) = (\nu+1)q(c) \le (\nu+1)q(b)$ and $q(b)+q(c) \le (\nu+1)q(c)$ in case of operations "0", and "1", respectively) and inequality (10) remain valid for algorithm $\mathfrak{B}$.

Lemma 3.1. (i) *We have $q(b)+q(c) \ge q(a) \ge q(b) \ge q(c)$ for all triangles.*

(ii) *Let the triangle Δ' be obtained by rule "1" from triangle Δ. Then* $\mathrm{mes}\,\Delta' \le \frac{1}{2}\mathrm{mes}\,\Delta$.

(iii) *Let the triangle Δ' with vertices a', b', c' be obtained by applying rule "0" k times to triangle Δ with vertices a, b, c. Then the vertices of Δ' are* $(a', b', c') = (a \underbrace{\oplus c \cdots \oplus c}_{k/2 \text{ times}}, b \underbrace{\oplus c \cdots \oplus c}_{k/2 \text{ times}}, c)$, *if k is even, and*
$(a', b', c') = (b \underbrace{\oplus c \cdots \oplus c}_{(k+1)/2 \text{ times}}, a \underbrace{\oplus c \cdots \oplus c}_{(k-1)/2 \text{ times}}, c)$, *if k is odd. Also, $q(a'), q(b') \ge \frac{k+1}{2} q(c)$.*

Proof. (i) By induction, obvious for $\nu = 0$. Let (a', b', c') be the new vertices. Then $q(b')+q(c') = q(a)+q(b) \ge q(a') = q(b)+q(c) \ge q(b') = q(a) \ge q(c') = q(b)$ and $q(b')+q(c') = q(a)+q(c) \ge q(a') = q(b)+q(c) \ge q(b') = q(a) \ge q(c') = q(c)$, respectively.

(ii) From (i), $q(b)+q(c) \ge 2 \cdot q(c)$, and using Lemma 1.2,

$$\mathrm{mes}\,\Delta' = (2(q(b)+q(c)) \cdot q(a) \cdot q(b))^{-1} \le (2(q(a) \cdot q(b) \cdot 2q(c))^{-1} = \frac{1}{2}\mathrm{mes}\,\Delta.$$

(iii) Observe the effect of rule "0" on triangle Δ: $(a,b,c) \to (b \oplus c, a, c) \to (a \oplus c, b \oplus c, c) \to (b \oplus c \oplus c, a \oplus c, c) \to (a \oplus c \oplus c, b \oplus c, c) \to \dots$ Iterating the rule k times leads to the stated formula, and then $q(a') = q(a) + \frac{k}{2}q(c), q(b') = q(b) + \frac{k}{2}q(c), q(c') = q(c)$, if k is even, and $q(a') = q(b) + \frac{k+1}{2}q(c), q(b') = q(a) + \frac{k-1}{2}q(c), q(c') = q(c)$, if k is odd, in any case $q(a'), q(b') \ge (\frac{k-1}{2}+1)q(c)$. ∎

It is easy to verify the following properties of partitions Til_ν and graphs T_ν, T by induction.

1. Til_ν is a partition of the unit square $[0,1]^2$ into $f_\nu = 2^{\nu+1}$ triangles.
2. The number of edges of the graph T_ν is equal to

$$r_\nu = \begin{cases} 3 \times 2^{2k} + 2^{k+1}, & \text{if } n = 2k, \\ 6 \times 2^{2k} + 2^{k+1}, & \text{if } n = 2k+1. \end{cases}$$

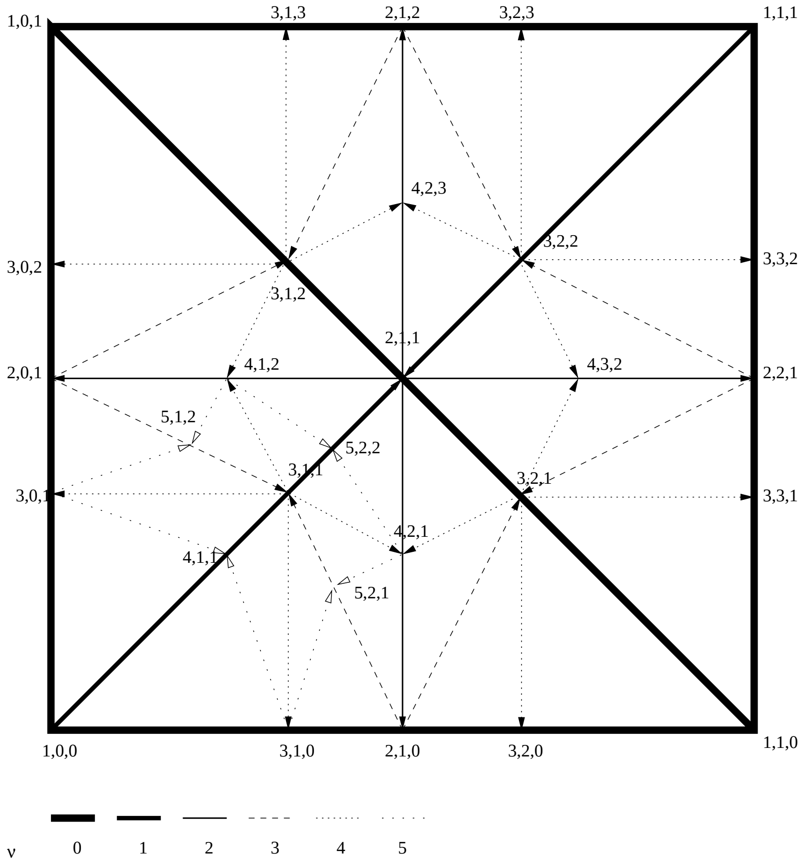

Figure 2.

3. The number of vertices of graph T_ν is equal to

$$v_\nu = \begin{cases} (2^k+1)^2, & \text{if } n = 2k, \\ (2^k+1)^2 + 2^{2k}, & \text{if } n = 2k+1. \end{cases}$$

4. The degree $\deg(v)$ for any vertex v of the graph T takes values from the set $\{3, 5, 8\}$. In each graph T_ν also occur vertices of degree 2 (for $\nu = 0$) or 4 (for $\nu \geq 1$); these vertices lie in $V_\nu \setminus V_{\nu-1}$. The number of vertices from T_ν with the given degree can be easily calculated.

The Dirichlet series $L(\mathfrak{B}, \beta)$ for our algorithm can be written as follows

$$L(\mathfrak{B}, \beta) = \sum_{a \in \mathbb{Q}^2 \cap [0,1]^2} \frac{\deg(a)}{q(a)^\beta} = \sum_{q=1}^{+\infty} \frac{3G_3(q) + 5G_5(q) + 8G_8(q)}{q^\beta},$$

where $G_l(q), l \in \{3, 5, 8\}$ is the number of rational points $a \in [0,1]^2$ with common denominator $q(a) = q$ and $\deg(a) = l$. Obviously $G_3(q) + G_5(q) + G_8(q) =$

$$\#\{(a_1,a_2)\in\mathbb{Z}^2 :\ 0\le a_1,a_2\le q,\ \text{g.c.d.}(q,a_1,a_2)=1\}\le (q+1)^2.$$

Lemma 3.2. *For all $\beta>1$, we have*

$$\sum_{n=0}^{\infty}\sigma_{n,\beta}(\mathfrak{B})\le \frac{32}{3}2^{\beta}\zeta(2\beta)\zeta(3\beta-2).$$

Proof. We follow closely the proof of Lemma 2.6. In algorithm $\mathfrak{B}$, always $\alpha(\Delta)=c$ by Lemma 3.1(i). Hence

$$\sum_{n=0}^{\infty}\sigma_{n,\beta}(\mathfrak{B})=\sum_{m=0}^{\infty}\ \sum_{c\in V_m\setminus V_{m-1}}\ \sum_{n=m}^{\infty}\ \sum_{\substack{\Delta\in\mathrm{Til}_n,\\ \alpha(\Delta)=c}}(\mathrm{mes}\,\Delta)^{\beta}.$$

We fix a point $c\in V_m\setminus V_{m-1}$. Then we have triangles

$$\Delta^{(1)},...,\Delta^{(\deg(c))}\in \mathrm{Til}_m\cup\mathrm{Til}_{m+1}, \tag{18}$$

all including vertex c. Again, every triangle $\Delta\in\mathrm{Til}_n$ with $\alpha(\Delta)=c$ is included in some $\Delta^{(i)}$ with vertices $a',b',c'=c$, and has been obtained by operations "0". With Lemma 1.2 and Lemma 3.1(iii),

$$\sum_{n=m}^{\infty}\ \sum_{\substack{\Delta\in\mathrm{Til}_n,\\ \alpha(\Delta)=c}}(\mathrm{mes}\,\Delta)^{\beta}\le\frac{8}{(2q(c)^3)^{\beta}}\times\sum_{j=1}^{\infty}\frac{2^{2\beta}}{(j+1)^{2\beta}}\le\frac{2^{3+\beta}}{q(c)^{3\beta}}\zeta(2\beta),$$

and thus

$$\sum_{n=0}^{\infty}\sigma_{n,\beta}(\mathfrak{B})\le 2^{3+\beta}\zeta(2\beta)\times\sum_{m=0}^{\infty}\ \sum_{c\in V_m\setminus V_{m-1}}\frac{1}{q(c)^{3\beta}}.$$

Using (10), we obtain the result as in the proof of Lemma 2.6. ∎

We choose parameters

$$\gamma=\frac{4(6\beta^2+\beta-1)}{3\log 2\cdot(\beta-1)\beta},\qquad \text{and}\qquad w=(\log n)^{1-\frac{1}{3\beta}}n^{\frac{2\beta+1}{3\beta}}$$

and again we divide $\sigma_{n,\beta}(\mathfrak{B})$ into three sums, now according to $\mathbf{c}$ and $|\,\mathbf{c}|$,

$$\sigma_{n,\beta}(\mathfrak{B})=\sum_{\Delta\in\mathrm{Til}_n}(\mathrm{mes}\,\Delta)^{\beta}=\Sigma_{(1)}+\Sigma_{(2)}+\Sigma_{(3)},$$

where $\Sigma_{(1)}$ is the sum over all Δ from Til_n with

$$|\,\mathbf{c}|(\Delta)\ge\gamma\log n, \tag{19}$$

$\Sigma_{(2)}$ is the sum over all Δ from Til_n with

$$|\,\mathbf{c}|(\Delta)<\gamma\log n,\qquad \exists k>w:\ \mathbf{c}_k=1, \tag{20}$$

and $\Sigma_{(3)}$ is the sum over all Δ from Til_n with

$$|\,\mathbf{c}|(\Delta)<\gamma\log n,\qquad \mathbf{c}_{w+1}=\cdots=\mathbf{c}_n=0. \tag{21}$$

Lemma 3.3. *For all $\beta > 1$,*

$$\Sigma_{(1)} \leq n^{-\log 2\ \gamma(\beta-1)}.$$

Proof. Obviously,

$$\Sigma_{(1)} \leq \max_{\substack{\Delta\in\mathrm{Til}_n,\\ |\ \mathbf{c}|(\Delta)\geq\gamma\log n}} (\mathrm{mes}\,\Delta)^{\beta-1} \times \sum_{\Delta\in\mathrm{Til}_n} \mathrm{mes}\,\Delta.$$

Let the maximum occur on some triangle Δ with vertices a, b, c. We apply Lemma 3.1(ii), (5) and the inequality

$$\max_{\Delta\in\mathrm{Til}_n,\ |\ \mathbf{c}|(\Delta)\geq\gamma\log n} (\mathrm{mes}\,\Delta)^{\beta-1} \leq \frac{1}{(2^{|\ \mathbf{c}|(\Delta)})^{\beta-1}} \leq \frac{1}{2^{\gamma(\beta-1)\log n}} = n^{-\log 2\ \gamma(\beta-1)}.$$

Lemma 3.3 is proved. ■

Lemma 3.4. *Let some triangle $\Delta = (a, b, c)$ be given, $\Delta' = (a', b', c')$ resulting from Δ via the operation $\delta_0 \in \{0, 1\}$, and $\Delta'' = (a'', b'', c'')$ resulting from Δ' via the three operations $\delta_1, 1, 0$ with $\delta_1 \in \{0, 1\}$. Then c'' was not a vertex of $\Delta, c'' \notin \{a, b, c\}$, but it is a vertex of Δ'.*

Proof. We verify the four cases:

$(a, b, c)\xrightarrow{0}(b\oplus c, a, c)\xrightarrow{0}(a\oplus c, b\oplus c, c)\xrightarrow{1}(b\oplus c\oplus c, a\oplus c, b\oplus c)\xrightarrow{0}(a\oplus b\oplus c\oplus c, b\oplus c\oplus c, b\oplus c)$ with $c'' = b\oplus c = a'$.

$(a, b, c)\xrightarrow{0}(b\oplus c, a, c)\xrightarrow{1}(a\oplus c, b\oplus c, a)\xrightarrow{0}(a\oplus b\oplus c, a\oplus c, b\oplus c)\xrightarrow{0}(a\oplus b\oplus c\oplus c, a\oplus b\oplus c, b\oplus c)$ with $c'' = b\oplus c = a'$.

$(a, b, c)\xrightarrow{1}(b\oplus c, a, b)\xrightarrow{0}(a\oplus b, b\oplus c, b)\xrightarrow{1}(b\oplus b\oplus c, a\oplus b, b\oplus c)\xrightarrow{0}(a\oplus b\oplus b\oplus c, b\oplus b\oplus c, b\oplus c)$ with $c'' = b\oplus c = a'$.

$(a, b, c)\xrightarrow{1}(b\oplus c, a, b)\xrightarrow{1}(a\oplus b, b\oplus c, a)\xrightarrow{1}(a\oplus b\oplus c, a\oplus b, b\oplus c)\xrightarrow{0}(a\oplus b\oplus b\oplus c, a\oplus b\oplus c, b\oplus c)$ with $c'' = b\oplus c = a'$.

Observe that in any case, $c'' = b\oplus c \notin \{a, b, c\}$. ■

Lemma 3.5. $\Sigma_{(2)} \leq \frac{2^8\cdot 59}{9}\cdot 432^\beta(\zeta(3\beta-2))^2\zeta(2\beta)\left(\frac{\gamma\log n}{w}\right)^{3\beta-1}.$

Proof. Condition (20) implies $|\ \mathbf{c}|(\Delta) < \gamma\log n$. As the last operation "1" occurs after the first w partitions, for some $t \geq \tau = \left\lceil\frac{w}{\gamma\log n}\right\rceil - 1$, there exists $k \leq w$ such that (i) $\mathbf{c}_{k+1} = \cdots = \mathbf{c}_{k+t} = 0$ (t consecutive operations "0"), (ii) $\mathbf{c}_{k+t+1} = 1$, (iii) $\mathbf{c}_k = 1$, $\mathbf{c}_{k-1} = \delta \in \{0, 1\}$, or $0 \leq k \leq 1$.

We consider first the case $0 \leq k \leq 1$: This part adds at most $8\cdot(\#V_0 + \#V_1)$ summands of the form $q(c)^{-3\beta} \leq 1$. Since $\#V_0 = 4, \#V_1 = 5$, this amounts to at most $\sum_{k=0,1} \leq 72$.

Let now $k \geq 2$. For the triangle Δ with code $[\ \mathbf{c}_1, ..., \mathbf{c}_n]$ we consider the sequence of triangles (7), especially the triangle $\Delta_{k+1} \in \mathrm{Til}_{k+1}$ and the triangle $\Delta_{k+t} \in \mathrm{Til}_{k+t}$. Let $\Delta_{k+1} = (a, b, c)$ and $\Delta_{k+t} = (a', b', c')$.

By Lemma 3.1(iii), the corresponding common denominators satisfy

$$q(c') = q(c), \quad q(a') \geq \left(\left\lfloor \frac{t}{2} \right\rfloor - 1\right) q(c), \quad q(b') \geq \left(\left\lfloor \frac{t}{2} \right\rfloor - 1\right) q(c).$$

Now, the triangle Δ_{k+t+1} is obtained from Δ_{k+t} by an operation "1"($\mathbf{c}_{k+t+1} = 1$), hence for every vertex $\omega \in \{b' \oplus c', a', b'\}$ of the triangle Δ_{k+t+1}, we have

$$q(\omega) \geq \left(\left\lfloor \frac{t}{2} \right\rfloor - 1\right) q(c) \geq \frac{t}{3} \cdot q(c). \tag{22}$$

Now we consider the partition Til_n restricted to the triangle Δ_{k+t+1}. It is isomorphic to the partition $\mathrm{Til}_{n-k-t-1}$. Moreover, for any triangle $\Delta \subset \Delta_{k+t+1}$ with vertices s, u, v from the partition Til_n and the corresponding triangle Δ' with vertices s', u', v' from the isomorphic partition $\mathrm{Til}_{n-k-t-1}$, by (22), we deduce that

$$\operatorname{mes}\Delta = \frac{1}{2q(s)q(u)q(v)} \leq \frac{1}{2(\frac{t}{3} \cdot q(c))^3 q(s')q(u')q(v')} = \frac{27 \operatorname{mes}\Delta'}{(tq(c))^3}.$$

Lemma 3.4 now states that $c \in V_{k-2} \setminus V_{k-3}$ and we can bound $\Sigma_{(2)}$, distinguishing $k = 0, 1$ from $k \geq 2$ as:

$$\Sigma_{(2)} \leq \sum_{t=\tau}^{n} \left(\sum_{\substack{0 \leq k \leq 1, \\ h=n-t-k}} 8 \cdot \#V_k \cdot \frac{8^\beta}{t^{3\beta}} \times \left(\sum_{\Delta \in \mathrm{Til}_{h-1}} (27 \operatorname{mes}\Delta)^\beta \right) + \right.$$

$$\left. + \sum_{\substack{k \geq 2, h \geq 0: \\ k+h+t=n}} \left(\sum_{c \in V_{k-2} \setminus V_{k-3}} \frac{8}{(q(c))^{3\beta}} \right) \times \frac{8^\beta}{t^{3\beta}} \times \left(\sum_{\Delta \in \mathrm{Til}_{h-1}} (27 \operatorname{mes}\Delta)^\beta \right) \right) \leq$$

$$\leq \frac{216^\beta \cdot 8}{\tau^{3\beta-1}} \times \left(9 + \sum_{k=2}^{\infty} \left(\sum_{c \in V_{k-2} \setminus V_{k-3}} \frac{1}{(q(c))^{3\beta}} \right) \right) \times \sum_{h=0}^{\infty} \left(\sum_{\Delta \in \mathrm{Til}_h} (\operatorname{mes}\Delta)^\beta \right).$$

But

$$\sum_{h=0}^{\infty} \left(\sum_{\Delta \in \mathrm{Til}_h} (\operatorname{mes}\Delta)^\beta \right) = \sum_{h=0}^{\infty} \sigma_{h,\beta}(\mathfrak{B}) \leq \frac{32}{3} 2^\beta \zeta(2\beta)\zeta(3\beta - 2)$$

by Lemma 3.2, and

$$9 + \sum_{k=2}^{\infty} \sum_{c \in V_{k-2} \setminus V_{k-3}} \frac{1}{(q(c))^{3\beta}} \leq 9 \; + \; 8 \sum_{q=1}^{\infty} \sum_{\substack{1 \leq a,b, \leq q \\ g.c.d.(a,b,q)=1}} \frac{1}{q^{3\beta}} \leq \left(9 + \frac{32}{3}\right) \zeta(3\beta - 2),$$

applying the upper bound (10). Now the inequality of Lemma 3.5 follows. ∎

Lemma 3.6. *For all $\beta > 1$,*

$$\Sigma_{(3)} = \frac{L(\mathfrak{B}, 3\beta)}{(n^2/2)^\beta} + O\left(\Sigma_{(1)}\right) + O\left(\frac{1}{n^{2\beta} w^{3(\beta-1)}} + \frac{w}{n^{2\beta+1}}\right)$$

Proof. W.l.o.g., let $n-w$ be even (otherwise use $w' = w+1$, covering even more cases). Let

$$\Sigma'_{(3)} = \sum_{\substack{\Delta \in \mathrm{Til}_n: \\ \mathbf{c}(\Delta) \text{ such that } \mathbf{c}_{w+1} = \cdots = \mathbf{c}_n = 0}} (\operatorname{mes} \Delta)^\beta .$$

Apparently,

$$\Sigma_{(3)} = \Sigma'_{(3)} + O(\Sigma_{(1)}).$$

Now, every triangle $\Delta \in \mathrm{Til}_n$ with code $\mathbf{c}(\Delta) = \mathbf{c}_1 \dots \mathbf{c}_w 0^{n-w}$ is a subset of a unique triangle $\Delta' \in \mathrm{Til}_w$ with code $\mathbf{c}(\Delta') = \mathbf{c}_1 \dots \mathbf{c}_w$.

Let (a, b, c) and (a', b', c') be the vertices of Δ and Δ', respectively. Since Δ is obtained by $k = n - w$ (which is even) operations “0”, with Lemma 1.2 and Lemma 3.1(iii), we get

$$\operatorname{mes} \Delta = \frac{1}{2\left(q(a') + \frac{k}{2} q(c')\right)\left(q(b') + \frac{k}{2} q(c')\right) q(c')}$$

and thus

$$\sum_{c \in V_w} \frac{\deg(c)}{\left(\frac{n^2}{2} q(c)^3\right)^\beta} \le \Sigma'_{(3)} \le \sum_{c \in V_w} \frac{\deg(c)}{\left(\frac{(n-w+1)^2}{2} q(c)^3\right)^\beta}.$$

Since

$$\sum_{q=1}^{w-1} \frac{\sum_l l G_l(q)}{q^{3\beta}} \le \sum_{m=0}^{w-1} \sum_{c \in V_m \setminus V_{m-1}} \frac{\deg(c)}{(q(c))^{3\beta}} = \sum_{c \in V_w} \frac{\deg(c)}{(q(c))^{3\beta}} = L(\mathfrak{B}, 3\beta) + O\left(w^{-3(\beta-1)}\right),$$

as in (16), (17), it follows that

$$\Sigma'_{(3)} = \frac{L(\mathfrak{B}, 3\beta)}{(n^2/2)^\beta} \left(1 + O\left(\frac{w}{n}\right)\right)\left(1 + O\left(\frac{1}{n^2 w^{3(\beta-1)}}\right)\right),$$

and the lemma is proved. ■

Theorem 3.1. *For $\beta > 1$ the following asymptotic formula is valid*

$$\sigma_{n,\beta}(\mathfrak{B}) = \frac{2^\beta \cdot L(\mathfrak{B}, 3\beta)}{n^{2\beta}} \left(1 + O\left(\frac{(\log n)^{1-\frac{1}{3\beta}}}{n^{\frac{\beta-1}{3\beta}}}\right)\right).$$

Note that here (as in Theorem 2) $L(\mathfrak{B}, 3\beta)$ is unbounded as $\beta \to 1+$, but of course $\sigma_{n,1} = 1$ for every n.

Proof. Assembling the results of Lemmata 3.3, 3.5, and 3.6,

$$\sigma_{n,\beta}(\mathfrak{B}) =$$
$$= \frac{L(\mathfrak{B},3\beta)}{(n^2/2)^\beta} + O\left(\frac{1}{n^{\log 2\ \gamma(\beta-1)}} + \frac{1}{n^{2\beta}w^{3(\beta-1)}} + \frac{w}{n^{2\beta+1}} + \left(\frac{\gamma \log n}{w}\right)^{3\beta-1}\right) =$$
$$= \frac{L(\mathfrak{B},3\beta)}{(n^2/2)^\beta}\left(1 + O\left(\frac{(\log n)^{1-\frac{1}{3\beta}}}{n^{\frac{\beta-1}{3\beta}}}\right)\right),$$

we obtain the result. ■

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