

WAVELET CHARACTERIZATION OF BESOV-MORREY AND TRIEBEL-LIZORKIN-MORREY SPACES

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Abstract: In the present paper we characterize the Besov-Morrey spaces and the Triebel-Lizorkin-Morrey spaces in terms of wavelet. As an application we investigate some embedding properties and diversity of the function spaces.

Keywords: function spaces, wavelet.

1. Introduction

The goal of the present paper is to characterize the Besov-Morrey spaces and the Triebel-Lizorkin-Morrey spaces in terms of wavelet and to establish the diversity of these spaces along with various examples of functions. As we will see, these function spaces cover a wide range of function spaces and it is very difficult to verify that there is no trivial coincidence of the function spaces.

The Besov-Morrey space emerged originally in [6]. H. Kozono and M. Yamazaki investigated time-local solutions of the Navier-Stokes equations with the initial data in the Besov-Morrey space. Later the Besov-Morrey space was investigated by A. Mazzucato. A. Mazzucato investigated the atomic decomposition and the molecular decomposition [7],[8]. In [6],[7],[8] we have developed a theory of the function space \mathcal{N}_{pqr}^s with $1 \leq q \leq p < \infty$, $1 \leq r \leq \infty$ and $s \in \mathbb{R}$. L. Tang and J. Xu defined the function spaces \mathcal{N}_{pqr}^s and \mathcal{E}_{pqr}^s with $0 < q \leq p < \infty$, $0 < r \leq \infty$ and $s \in \mathbb{R}$. The author and H. Tanaka developed a theory of decompositions in \mathcal{N}_{pqr}^s and \mathcal{E}_{pqr}^s with $0 < q \leq p < \infty$, $0 < r \leq \infty$ and $s \in \mathbb{R}$ in [15].

Before we go into the definitions of the Besov-Morrey spaces and the Triebel-Lizorkin-Morrey spaces, let us recall the definitions of the Besov spaces and the Triebel-Lizorkin spaces, which are prototypes of the Besov-Morrey spaces and the Triebel-Lizorkin-Morrey spaces respectively. To describe these function spaces, we fix some notations.

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For definiteness we adopt the following definition as the Fourier transform and its inverse:

$$\mathcal{F}f(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x)e^{-ix \cdot \xi} dx, \quad \mathcal{F}^{-1}f(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\xi)e^{ix \cdot \xi} d\xi$$

for $f \in L^1$.

Let $\{f_j\}_{j \in \mathbb{N}_0}$ be a sequence of functions. Then define

$$\|f_j : l_q(L_p)\| := \left(\sum_{j \in \mathbb{N}_0} \|f_j : L_p\|^q \right)^{\frac{1}{q}},$$

$$\|f_j : L_p(l_q)\| := \left\| \left(\sum_{j \in \mathbb{N}_0} |f_j|^q \right)^{\frac{1}{q}} : L_p \right\|$$

for $0 < p, q \leq \infty$. Here a natural modification is made if p and/or $q = \infty$.

Denote by $\mathbf{1}_E$ the indicator function of a set E . $B(r)$ means the open ball centered at the origin of radius $r > 0$. Next we fix a sequence of smooth functions $\{\varphi_j\}_{j \in \mathbb{N}_0} \subset \mathcal{S}$ so that

$$\mathbf{1}_{B(2)} \leq \varphi_0 \leq \mathbf{1}_{B(4)}, \quad \mathbf{1}_{B(4) \setminus B(2)} \leq \varphi_1 \leq \mathbf{1}_{B(8) \setminus B(1)}, \quad \varphi_j = \varphi_1(2^{-j+1} \cdot)$$

for $j \in \mathbb{N}$. Given $f \in \mathcal{S}'$ and $\tau \in \mathcal{S}$, we define $\tau(D)f := \mathcal{F}^{-1}(\tau \cdot \mathcal{F}f)$.

Under these notations, we define the Besov norm and the Triebel-Lizorkin norm. Let $f \in \mathcal{S}'$. We define

$$\|f : B_{pq}^s\| := \|2^{js} \varphi_j(D)f : l_q(L_p)\|, \quad 0 < p \leq \infty, \quad 0 < q \leq \infty, \quad s \in \mathbb{R}$$

$$\|f : F_{pq}^s\| := \|2^{js} \varphi_j(D)f : L_p(l_q)\|, \quad 0 < p < \infty, \quad 0 < q \leq \infty, \quad s \in \mathbb{R}.$$

The exhaustive treatment of these function spaces can be found in the textbooks [19, 20, 21, 22, 23]. Different admissible choices of φ_0 and φ_1 will yield equivalent quasi-norms. To unify the formulation in the sequel, as was done in [22], for example, we use A_{pq}^s to denote either B_{pq}^s or F_{pq}^s with $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. If A_{pq}^s means F_{pq}^s , then the case when $p = \infty$ is excluded automatically.

Let us describe the function spaces \mathcal{N}_{pqr}^s and \mathcal{E}_{pqr}^s briefly. Suppose that the parameters p, q, r, s satisfy $0 < q \leq p < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$. Define the Morrey norm of a measurable function f by

$$\|f : \mathcal{M}_q^p\| := \sup_{x \in \mathbb{R}^n, r > 0} r^{\frac{n}{p} - \frac{n}{q}} \left(\int_{B(x,r)} |f(y)|^q dy \right)^{\frac{1}{q}}. \quad (1)$$

p seems to reflect the global integrability while q describes the local integrability.

The Besov-Morrey spaces and the Triebel-Lizorkin-Morrey spaces are obtained by replacing the L_p norm with the Morrey norm $\|\cdot\| : \mathcal{M}_q^p$ given by (1). Given a sequence of measurable functions $\{f_j\}_{j \in \mathbb{N}_0}$, we define

$$\|f_j : l_r(\mathcal{M}_q^p)\| := \left(\sum_{j \in \mathbb{N}_0} \|f_j : \mathcal{M}_q^p\|^r \right)^{\frac{1}{r}}$$

$$\|f_j : \mathcal{M}_q^p(l_r)\| := \left\| \left(\sum_{j \in \mathbb{N}_0} |f_j|^r \right)^{\frac{1}{r}} : \mathcal{M}_q^p \right\|$$

for $0 < q \leq p < \infty$, $0 < r \leq \infty$.

Definition 1.1. Let $f \in \mathcal{S}'$. Then define

$$\|f : \mathcal{N}_{pqr}^s\| := \|2^{js} \varphi_j(D)f : l_r(\mathcal{M}_q^p)\|,$$

$$\|f : \mathcal{E}_{pqr}^s\| := \|2^{js} \varphi_j(D)f : \mathcal{M}_q^p(l_r)\|$$

for $0 < q \leq p < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$. Also one defines $\mathcal{N}_{\infty qr}^s = B_{\infty r}^s$ for $0 < q, r \leq \infty$ and $s \in \mathbb{R}$.

Here and below we denote by \mathcal{A}_{pqr}^s either \mathcal{N}_{pqr}^s or \mathcal{E}_{pqr}^s . If \mathcal{A}_{pqr}^s denotes \mathcal{E}_{pqr}^s , we rule out $p = \infty$ automatically. The function space \mathcal{A}_{pqr}^s covers many families of function spaces such as the Hölder-Zygmund space \mathcal{C}^s , the Morrey space \mathcal{M}_q^p , the Sobolev-Morrey space, the Besov space B_{pq}^s and the Triebel-Lizorkin space F_{pq}^s . As for the Sobolev-Morrey space, we refer to [10, 11, 12]. Recall that the Hölder-Zygmund space \mathcal{C}^s , $0 < s < 1$ is a set of all functions normed by $\|f : \mathcal{C}^s\| := \|f : L_\infty\| + \sup_{\substack{x, y \in \mathbb{R}^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^s}$.

Proposition 1.2. Suppose that $0 < q \leq p \leq \infty$, $0 < r \leq \infty$ and $s \in \mathbb{R}$.

1. Let $k \in \mathbb{N}$. Then $f \in \mathcal{A}_{pqr}^s$ if and only if $f \in \mathcal{A}_{pqr}^{s-k}$ and $\partial_j^k f \in \mathcal{A}_{pqr}^{s-k}$ for every $j = 1, 2, \dots, n$. Furthermore, we have the following norm equivalence

$$\|f : \mathcal{A}_{pqr}^s\| \simeq \|f : \mathcal{A}_{pqr}^{s-k}\| + \sum_{j=1}^n \|\partial_j^k f : \mathcal{A}_{pqr}^{s-k}\| \text{ for all } f \in \mathcal{A}_{pqr}^s.$$

2. $\mathcal{A}_{ppr}^s = \mathcal{A}_{pr}^s$.
3. $\mathcal{E}_{pq2}^0 = \mathcal{M}_q^p$, if $1 < q \leq p < \infty$.
4. $\mathcal{N}_{\infty q\infty}^s = B_{\infty\infty}^s = \mathcal{C}^s$, if $s \in (0, 1)$.

Assertions 1,3,4 can be found in [17, Proposition 2.15], [7, Proposition 4.1] and [16] respectively, while the assertion 2 is immediate from the definition.

Proposition 1.3. *Let the parameters $p, q, r, r_1, r_2, s, \varepsilon$ satisfy*

$$0 < q \leq p \leq \infty, \quad 0 < r, r_1, r_2 \leq \infty, \quad s \in \mathbb{R}, \quad \varepsilon > 0.$$

Then we have

1. $\mathcal{N}_{pqr_1}^{s+\varepsilon} \subset \mathcal{E}_{pqr_2}^s$ and $\mathcal{E}_{pqr_1}^{s+\varepsilon} \subset \mathcal{N}_{pqr_2}^s$.
2. $\mathcal{A}_{pqr_1}^s \subset \mathcal{A}_{pqr_2}^s$, if $r_1 \leq r_2$.
3. $\mathcal{N}_{pq \min(q,r)}^s \subset \mathcal{E}_{pqr}^s \subset \mathcal{N}_{pq\infty}^s$.

The following is obtained by the Plancherel-Polya-Nikolskij inequality.

Proposition 1.4. *Let $0 < q \leq p < \infty$, $0 < r \leq \infty$ and $s > \frac{n}{p}$. Then we have $\mathcal{A}_{pqr}^s \subset BUC$, where BUC stands for a set of all uniformly continuous and bounded functions.*

This is a fundamental property of the function spaces. Now let us formulate our main results.

Wavelet characterization. Let $N \in \mathbb{N}$. According to wavelet theory (see [9, 24], for example) there exists a collection of C^N -functions $\{\psi^{(l)} : l = 0, 1, \dots, 2^n - 1\}$ satisfying the following three conditions:

1. $\int_{\mathbb{R}^n} x^\alpha \psi^{(l)}(x) dx = 0$ for all $l = 1, 2, \dots, 2^n - 1$ and all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$.
2. Each $\psi^{(l)}$ ($l = 0, 1, \dots, 2^n - 1$) is N -regular, namely, $\psi^{(l)} \in C^N$ and

$$\sup_{x \in \mathbb{R}^n} \langle x \rangle^{-m} |\partial^\alpha \psi^{(l)}(x)| < \infty$$

for all $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$.

3. Let $j \in \mathbb{N}_0$, $k \in \mathbb{Z}^n$ and $l \in \{0, 1, 2, \dots, 2^n - 1\}$. In the sequel we write $\psi_{j,k}^{(l)}(x) := 2^{\frac{jn}{2}} \psi^{(l)}(2^j x - k)$. The set

$$\left\{ \psi_{0,k}^{(0)}, \psi_{j,k}^{(l)} : l = 1, 2, \dots, 2^n - 1, \quad j \in \mathbb{N}_0, \quad k \in \mathbb{Z}^n \right\}$$

forms an orthonormal basis in L^2 .

In terms of a multiresolution analysis, we often say that the function $\psi^{(0)}$ is a scaling function and that each $\psi^{(l)}$ ($l = 1, 2, \dots, 2^n - 1$) is a wavelet.

Assume that $N \gg 1$ in the present paper. In Theorem 2.5 we will clarify how large N should be.

Notation. 1. Let $\nu \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Then define $Q_{\nu,m} := \prod_{j=1}^n \left[\frac{m_j}{2^\nu}, \frac{m_j+1}{2^\nu} \right]$. Call $Q_{\nu,m}$ a dyadic cube in \mathbb{R}^n for each $\nu \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$.

2. Let $0 < p < \infty$, $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. $\mathbf{1}_{\nu,m}^{(p)}$ denotes the p -normalized indicator given by $\mathbf{1}_{\nu,m}^{(p)} := 2^{\frac{\nu n}{p}} \mathbf{1}_{Q_{\nu,m}}$.

3. Given a doubly indexed sequence $\lambda = \{\lambda_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$, define

$$\begin{aligned} \|\lambda : \mathbf{n}_{pqr}\| &:= \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \mathbf{1}_{\nu,m}^{(p)} : l_r(\mathcal{M}_q^p) \right\| \\ \|\lambda : \mathbf{n}_{\infty qr}\| &:= \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \mathbf{1}_{\nu,m}^{(p)} : l_r(L_\infty) \right\| \\ \|\lambda : \mathbf{e}_{pqr}\| &:= \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \mathbf{1}_{\nu,m}^{(p)} : \mathcal{M}_q^p(l_r) \right\| \end{aligned}$$

for $0 < q \leq p < \infty$, $0 < r \leq \infty$. In order to unify the formulations in what follows, denote by \mathbf{a}_{pqr} either \mathbf{n}_{pqr} or \mathbf{e}_{pqr} . We rule out \mathbf{a}_{pqr} with $p = \infty$ and $\mathbf{a} = \mathbf{e}$.

Theorem 1.5. *Suppose that the parameters p, q, r, s satisfy*

$$0 < q \leq p \leq \infty, 0 < r \leq \infty, s \in \mathbb{R}.$$

Let N be a large integer depending on p, q, r, s .

1. Suppose that we are given an l_p -sequence $\{\lambda_k\}_{k \in \mathbb{Z}^n}$ and $\{\lambda_{j,k}^{(l)}\}_{j \in \mathbb{N}_0, k \in \mathbb{Z}^n} \in \mathbf{a}_{pqr}$ for each $l = 1, 2, \dots, 2^n - 1$. Then

$$f := \sum_{k \in \mathbb{Z}^n} \lambda_k \psi_{0,k}^{(0)} + \sum_{l=1}^{2^n-1} \sum_{j \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}^n} \lambda_{j,k}^{(l)} \psi_{j,k}^{(l)} \quad (2)$$

belongs to \mathcal{A}_{pqr}^s and satisfies

$$\|f : \mathcal{A}_{pqr}^s\| \leq c \left(\|\{\lambda_k\}_{k \in \mathbb{Z}^n} : l_p\| + \sum_{l=1}^{2^n-1} \left\| \left\{ 2^{j(s+\frac{n}{2}-\frac{n}{p})} \lambda_{j,k}^{(l)} \right\}_{\substack{j \in \mathbb{N}_0 \\ k \in \mathbb{Z}^n}} : \mathbf{a}_{pqr} \right\| \right).$$

2. Conversely any $f \in \mathcal{A}_{pqr}^s$ admits the following decomposition.

$$f = \sum_{k \in \mathbb{Z}^n} \langle f, \psi_{0,k}^{(0)} \rangle \psi_{0,k}^{(0)} + \sum_{l=1}^{2^n-1} \sum_{j \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}^n} \langle f, \psi_{j,k}^{(l)} \rangle \psi_{j,k}^{(l)} \quad (3)$$

and the coefficients satisfy

$$\left\| \{\langle f, \psi_{0,k}^{(0)} \rangle\}_{k \in \mathbb{Z}^n} : l_p \right\| \leq c \|f : \mathcal{A}_{pqr}^s\| \quad (4)$$

and

$$\sum_{l=1}^{2^n-1} \left\| \left\{ 2^{j(s+\frac{n}{2}-\frac{n}{p})} \langle f, \psi_{j,k}^{(l)} \rangle \right\}_{(j,k) \in \mathbb{N}_0 \times \mathbb{Z}^n} : \mathbf{a}_{pqr} \right\| \leq c \|f : \mathcal{A}_{pqr}^s\|. \quad (5)$$

Sharpness of embedding. Let us exhibit some examples showing the sharpness of embedding with the help of wavelet.

Proposition 1.6. 1. Suppose that the parameters p, q, r, r_0, s satisfy

$$0 < q < p < \infty, \quad 0 < r, r_0 \leq \infty, \quad s \in \mathbb{R}.$$

If the continuous embedding $\mathcal{E}_{pqr}^s \subset \mathcal{N}_{pqr_0}^s$ is true, then $r_0 = \infty$.

2. Suppose that the parameters p, q, s satisfy $0 < q \leq p < \infty, s \in \mathbb{R}$. Then the inclusion $\mathcal{E}_{pq\infty}^s \subset \mathcal{N}_{pq\infty}^s$ is strict.

Diversity of function spaces. We are now about to consider the following problem. Suppose that the parameters p_i, q_i, r_i, s_i for $i = 0, 1$ satisfy

$$0 < q_i \leq p_i \leq \infty, \quad 0 < r_i \leq \infty, \quad s_i \in \mathbb{R}.$$

We set $\mathcal{N}_0 := \mathcal{N}_{p_0q_0r_0}^{s_0}$, $\mathcal{E}_0 := \mathcal{E}_{p_0q_0r_0}^{s_0}$, $\mathcal{N}_1 := \mathcal{N}_{p_1q_1r_1}^{s_1}$, $\mathcal{E}_1 := \mathcal{E}_{p_1q_1r_1}^{s_1}$ for the sake of simplicity. Then what can we say for the parameters, if either one of $\mathcal{N}_0 = \mathcal{N}_1$, $\mathcal{E}_0 = \mathcal{E}_1$ or $\mathcal{N}_0 = \mathcal{E}_1$ happens? This problem was referred to as diversity of the function spaces in [19]. Let us formulate our result.

Theorem 1.7.. *The following are true.*

1. $\mathcal{N}_0 = \mathcal{N}_1$ if and only if $(p_0, q_0, r_0, s_0) = (p_1, q_1, r_1, s_1)$.
2. $\mathcal{E}_0 = \mathcal{E}_1$ if and only if $(p_0, q_0, r_0, s_0) = (p_1, q_1, r_1, s_1)$.
3. $\mathcal{N}_0 = \mathcal{E}_1$ if and only if $p_0 = p_1 = q_0 = q_1 = r_0 = r_1 < \infty$ and $s_0 = s_1$.

Having set down elementary facts and our main results, let us describe the organization of the present paper. In Section 2 we collect some elementary results needed for analysis of the function spaces. Section 3 is intended as wavelet characterization of these function spaces: We shall prove Theorem 1.5. For details we refer to [2,3,4,5,23]. As an application of Section 3, we shall establish in Section 4 that $\mathcal{A}_{p_0q_0r_0}^{s_0}$ and $\mathcal{A}_{p_1q_1r_1}^{s_1}$ never agree except in a quite trivial case, where $\mathcal{A}_{p_0q_0r_0}^{s_0}$ and $\mathcal{A}_{p_1q_1r_1}^{s_1}$ can be of different scales.

2. Preliminaries

Maximal operators. We begin with the boundedness of the (powered) maximal operator given by

$$M^{(\eta)}f(x) := \sup_{r>0} \left(\frac{1}{r^n} \int_{B(x,r)} |f|^\eta \right)^{\frac{1}{\eta}}, \quad \eta > 0$$

for a locally integrable function f . It is immediate from the definition that

$$\|M^{(\eta)}f : L_\infty\| \leq c \|f : L_\infty\|, \quad \eta > 0.$$

As for the Morrey boundedness of $M^{(\eta)}$, we have the following.

Proposition 2.1. [14, Theorem 2.2], [17, Theorem 1.4] *Suppose that the parameters p, q, r, s satisfy*

$$0 < q \leq p < \infty, \quad 0 < r \leq \infty, \quad 0 < \eta < \min(q, r).$$

Then, there exists a constant c depending only on p, q, r, η such that

$$\|M^{(\eta)} f_j : \mathcal{M}_q^p(l_r)\| \leq c \|f_j : \mathcal{M}_q^p(l_r)\|$$

for all sequences of measurable functions $\{f_j\}_{j \in \mathbb{N}_0}$.

Multiplier result. Next we turn to the multiplier theorem of Michlin type. The following proposition is a key for boundedness of the multiplier. A “frequency support” of $f \in \mathcal{S}'$ means $\text{supp}(\mathcal{F}f)$ by definition.

Proposition 2.2. *Let $\eta > 0$. Then there exists $c > 0$ such that*

$$\sup_{y \in \mathbb{R}^n} \langle y \rangle^{-\frac{n}{\eta}} |f(x - y)| \leq c M^{(\eta)} f(x)$$

for all $f \in \mathcal{S}'$ with frequency support in $B(r)$, where $\langle a \rangle := \sqrt{1 + |a|^2}$ for $a \in \mathbb{R}^n$.

Let $s > 0$. Recall that H_2^s is a function space consisting of $f \in L_2$ with

$$\|f : H_2^s\| := \|\langle \cdot \rangle^s \mathcal{F}f : L_2\| < \infty.$$

We use the following theorem for later considerations.

Proposition 2.3. [17, Theorem 2.7] *Suppose that the parameters p, q, r satisfy*

$$0 < q \leq p < \infty, \quad 0 < r \leq \infty.$$

1. *Let $\sigma > \frac{n}{\min(1, q)} + \frac{n}{2}$. Then there exists $c > 0$ such that*

$$\|H(D)f : \mathcal{M}_q^p\| \leq c \|H(R_\cdot) : H_2^\sigma\| \cdot \|f : \mathcal{M}_q^p\|,$$

for all $H \in \mathcal{S}$ and $f \in \mathcal{S}'$ with frequency support of f in $B(R)$.

2. *Let $\sigma > \frac{n}{2}$. Then there exists $c > 0$ such that*

$$\|H(D)f : \mathcal{M}_q^p\| \leq c \|H(R_\cdot) : H_2^\sigma\| \cdot \|f : \mathcal{M}_q^p\|,$$

for all $H \in \mathcal{S}$ and $f \in \mathcal{S}'$ with frequency support of f in $B(R)$.

3. *Let $p < \infty$ and $\sigma > \frac{n}{\min(1, q, r)} + \frac{n}{2}$. Then we have*

$$\|H_j(D)f_j : \mathcal{M}_q^p(l_r)\| \leq c \left(\sup_{k \in \mathbb{N}_0} \|H_k(R_k \cdot) : H_2^\sigma\| \right) \cdot \|f_j : \mathcal{M}_q^p(l_r)\|,$$

for all $\{H_j\}_{j \in \mathbb{N}_0} \subset \mathcal{S}$ and $\{f_j\}_{j \in \mathbb{N}_0} \subset \mathcal{S}'$ with frequency support of each f_j in $B(R_j)$.

We remark that this proposition yields that the definition of the norm does not depend on the admissible choice of φ_0 and φ_1 .

Atomic decomposition. To describe the atomic decomposition, we introduce some notations based on those in [21, 22]. Now we keep to the ones in [15, Section 4] to formulate the atomic decomposition.

Following [21], let us recall the definition of atoms.

Definition 2.4. Let $0 < p < \infty$ and $s \in \mathbb{R}$. Fix $K \in \mathbb{N}_0$, $L \in \mathbb{Z} \cap [-1, \infty)$ and $\kappa > 1$.

1. Suppose further that $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. A C^K -function a is said to be an atom centered at $Q_{\nu,m}$, if it is supported on $\kappa Q_{\nu,m}$ and satisfies the differential inequality and the moment condition given below:

$$\|\partial^\alpha a : L_\infty\| \leq 2^{-\nu(s-\frac{n}{p})+\nu|\alpha|} \quad \text{for } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq K \quad (6)$$

and

$$\int_{\mathbb{R}^n} x^\beta a(x) dx = 0 \quad \text{for } \beta \in \mathbb{N}_0^n, \nu \in \mathbb{N}_0 \text{ with } |\beta| \leq L, \nu \geq 1. \quad (7)$$

Here condition (7) means no condition, if $L = -1$.

2. Define

$$\mathbf{Atom}_0 := \{a_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} : \text{each } a_{\nu,m} \text{ is an atom centered at } Q_{\nu,m}\}.$$

With this definition in mind, we formulate our atomic decomposition theorem. For the proof we refer to [15, Section 4.1].

We remark that in [15] \mathcal{M}_{pqr}^s is used instead of \mathcal{M}_{pqr}^s in the present paper.

Theorem 2.5. (Atomic decomposition) *Let $\kappa > 1$ be fixed. Suppose that the parameters $K, L \in \mathbb{Z}$ and $p, q, r, s \in \mathbb{R}$ satisfy*

$$0 < q \leq p \leq \infty, 0 < r \leq \infty, K \geq (1 + [s])_+, L \geq \max(-1, [\sigma_q - s])$$

for the \mathcal{N} -scale and

$$0 < q \leq p < \infty, 0 < r \leq \infty, K \geq (1 + [s])_+, L \geq \max(-1, [\sigma_{qr} - s])$$

for the \mathcal{E} -scale.

1. Assume that $\{a_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{Atom}_0$ and $\lambda = \{\lambda_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{a}_{pqr}$. Then the sum $f := \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}$ converges in \mathcal{S}' and belongs to \mathcal{A}_{pqr}^s with the norm estimate

$$\|f : \mathcal{A}_{pqr}^s\| \leq c \|\lambda : \mathbf{a}_{pqr}\|.$$

Here the constant c does not depend on $\{a_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ nor λ .

2. Conversely any $f \in \mathcal{A}_{pqr}^s$ admits the following decomposition:

$$f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}.$$

The sum converges in \mathcal{S}' . We can arrange $\{a_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{Atom}_0$ and that the coefficient $\lambda = \{\lambda_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{a}_{pqr}$ fulfill the norm estimate

$$\|\lambda : \mathbf{a}_{pqr}\| \leq c \|f : \mathcal{A}_{pqr}^s\|.$$

3. Wavelet characterization

Proof. First, (2) is a direct consequence of the atomic decomposition (Theorem 2.5). Let us turn to the converse. If $f \in \mathcal{S}$, then the desired expansion

$$f = \sum_{k \in \mathbb{Z}^n} \langle f, \psi_{0,k}^{(0)} \rangle \psi_{0,k}^{(0)} + \sum_{l=1}^{2^n-1} \sum_{j \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}^n} \langle f, \psi_{j,k}^{(l)} \rangle \psi_{j,k}^{(l)}$$

is trivial at least in $B_{\infty\infty}^{N-1}$. Observe that $\mathcal{A}_{pqr}^s \subset B_{\infty\infty}^{-N+1}$, if N is large. This is a consequence of Plancherel-Polya-Nikolskij inequality. By duality, we can pass to $f \in \mathcal{A}_{pqr}^s$ and obtain at least (3). Inequality (4) can be proved in the same way as (5). Therefore, what remains to be dealt with is (5), which requires some elaboration. Fix $l \in \{1, 2, \dots, 2^n - 1\}$. Choose $\tau, \theta \in \mathcal{S}$ so that

$$\text{supp}(\mathcal{F}\tau) \subset B(4), \quad \text{supp}(\mathcal{F}\theta) \subset B(4) \setminus B(1), \quad \mathcal{F}\tau^2 + \sum_{m=0}^{\infty} \mathcal{F}[\theta_m]^2 \equiv (2\pi)^{-n}$$

and that τ and θ are even and real-valued. Here and below we define θ_m and τ_m by $\mathcal{F}[\theta_m] := \mathcal{F}\theta(2^{-m}\cdot)$ and $\mathcal{F}[\tau_m] := \mathcal{F}\tau(2^{-m}\cdot)$ for $m \in \mathbb{Z}$. Using this equality, we obtain

$$\langle f, \psi_{j,k}^{(l)} \rangle = \langle f, \tau_j * \tau_j * \psi_{j,k}^{(l)} \rangle + \sum_{m=0}^{\infty} \langle f, \theta_{j+m} * \theta_{j+m} * \psi_{j,k}^{(l)} \rangle = \text{I} + \text{II},$$

where

$$\begin{aligned} \text{I} &:= \int_{\mathbb{R}^n} \langle f, \tau_j(\cdot - y + 2^{-j}k) \rangle \tau_j * \psi_{j,0}^{(l)}(y) dy \\ \text{II} &:= \sum_{m=0}^{\infty} \int_{\mathbb{R}^n} \langle f, \theta_{j+m}(\cdot - y + 2^{-j}k) \rangle \theta_{j+m} * \psi_{j,0}^{(l)}(y) dy. \end{aligned}$$

Observe that

$$\begin{aligned} \mathcal{F}[\theta_{j+m} * \psi_{j,0}^{(l)}](\xi) &= (2\pi)^{\frac{n}{2}} \mathcal{F}[\theta_{j+m}](\xi) \mathcal{F}[\psi_{j,0}^{(l)}](\xi) \\ &= (2\pi)^{\frac{n}{2}} 2^{-\frac{jn}{2}} \mathcal{F}\theta(2^{-j-m}\xi) \mathcal{F}\psi^{(l)}(2^{-j}\xi). \end{aligned}$$

By virtue of the fact that θ has vanishing moment of any order, we see that there exist $M > \frac{n}{\min(1,p,q)} - s$ and $L > \frac{n}{\min(1,p,q)} + n + 1$ such that

$$|\partial^\alpha \mathcal{F}[\theta_{j+m} * \psi_{j,0}^{(l)}](\xi)| \leq c 2^{-mM - \frac{jn}{2}} \langle 2^{-j}\xi \rangle^{-(n+1)}.$$

for all α with $|\alpha| \leq n + 1 + L$. As a consequence, it follows that

$$|\theta_{j+m} * \psi_{j,0}^{(l)}(x)| \leq c 2^{-mM + \frac{jn}{2}} \langle 2^j x \rangle^{-L}.$$

Choose $\eta > 0$ so that $\eta < \min(1, q)$ when $\mathcal{A} = \mathcal{N}$ and that $\eta < \min(1, q, r)$ when $\mathcal{A} = \mathcal{E}$. Inserting this estimate and invoking Proposition 2.2, we obtain

$$\begin{aligned} & |\langle f, \theta_{j+m}(\cdot - y + 2^{-j}k) \rangle \theta_{j+m} * \psi_{j,k}^{(l)}(y)| \\ & \leq c 2^{\frac{jn}{2} - mM} \langle 2^j y \rangle^{-L} |\theta_{j+m}(D)f(2^{-j}k - y)| \\ & \leq c 2^{\frac{jn}{2} - mM} \langle 2^j y \rangle^{-L} \langle 2^{j+m}(x + y - 2^{-j}k) \rangle^{\frac{n}{\eta}} M^{(\eta)}[\theta_{j+m}(D)f](x) \end{aligned}$$

for all $x \in Q_{j,k}$. If $x \in Q_{j,k}$, then we have

$$\langle 2^{j+m}(x + y - 2^{-j}k) \rangle^{\frac{n}{\eta}} \leq c 2^{\frac{mn}{\eta}} \langle 2^j(x + y - 2^{-j}k) \rangle^{\frac{n}{\eta}} \leq c 2^{\frac{mn}{\eta}} \langle 2^j y \rangle^{\frac{n}{\eta}}.$$

If we combine the above inequalities, it follows that

$$\begin{aligned} & |\langle f, \theta_{j+m}(\cdot - y + 2^{-j}k) \rangle \theta_{j+m} * \psi_{j,k}^{(l)}(y)| \\ & \leq c 2^{\frac{jn}{2} - m(M - \frac{n}{\eta})} \langle 2^j y \rangle^{-L + \frac{n}{\eta}} M^{(\eta)}[\theta_{j+m}(D)f](x). \end{aligned}$$

Integrating this inequality over \mathbb{R}^n , we obtain

$$|\text{II}| \leq c 2^{-\frac{jn}{2} - m(M - \frac{n}{\eta})} M^{(\eta)}[\theta_{j+m}(D)f](x)$$

for all $x \in Q_{j,k}$. Now using this pointwise estimate and the counterpart of I proved similarly, we obtain

$$\begin{aligned} & \left| \sum_{k \in \mathbb{Z}^n} 2^{j(s + \frac{n}{2} - \frac{n}{p})} \langle f, \psi_{j,k}^{(l)} \rangle \mathbf{1}_{j,k}^{(p)}(x) \right| \\ & \leq c \sum_{m \in \mathbb{N}} 2^{-m(M - \frac{n}{\eta} + s)} M^{(\eta)}[2^{(j+m)s} \theta_{j+m}(D)f](x) + c M^{(\eta)}[2^{js} \tau_j(D)f](x). \end{aligned}$$

With this pointwise estimate and the Fefferman-Stein vector-valued inequality for the Morrey spaces (see Proposition 2.1), we obtain (5). \blacksquare

4. Diversity of the function spaces

In this section we shall show function spaces for different parameters or scales are different except some trivial cases.

In this section we always keep to the following notations.

Let $\lambda = \{\lambda_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{a}_{pqr}$ and define

$$\Phi(\lambda) := \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \psi_{\nu,m}^{(1)} \quad (8)$$

and $\Psi(\lambda) = \{\Psi(\lambda)_\nu\}_{\nu \in \mathbb{N}_0}$, where

$$\Psi(\lambda)_\nu(x) := \sum_{m \in \mathbb{Z}^n} 2^{\nu(s + \frac{n}{2} - \frac{n}{p})} \lambda_{\nu,m} \mathbf{1}_{\nu,m}^{(p)}(x) \quad (9)$$

for $\nu \in \mathbb{N}_0$. We remark that the definition of Ψ depends on the function space \mathcal{A}_{pqr}^s .

Let us summarize what we have obtained in Section 3.

Lemma 4.1. *Suppose that the parameters p, q, r, s satisfy*

$$0 < q \leq p \leq \infty, \quad 0 < r \leq \infty, \quad s \in \mathbb{R}.$$

Then the following norm equivalence holds:

$$\|\Phi(\lambda) : \mathcal{A}_{pqr}^s\| \simeq \left\| \left\{ 2^{\nu(s + \frac{n}{2} - \frac{n}{p})} \lambda_{\nu, m} \right\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} : \mathbf{a}_{pqr} \right\|.$$

Below in the following lemmas let us tacitly assume that the parameters p, q, r, s satisfy

$$0 < q \leq p \leq \infty, \quad 0 < r \leq \infty, \quad s \in \mathbb{R}.$$

The parameters p and s

Lemma 4.2. *Let $j \in \mathbb{N}$ and define $\alpha^j := \{\alpha_{\nu, m}^j\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{a}_{pqr}$ so that*

$$\alpha_{\nu, m}^j = \begin{cases} 1 & \text{if } j = \nu \text{ and } Q_{\nu, m} \subset [0, 2^j]^n, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $\|\alpha^j : \mathbf{n}_{pqr}\| = \|\alpha^j : \mathbf{e}_{pqr}\| = 2^{j(s + \frac{n}{2})}$.

Proof. Indeed, from the definition of the vector-valued norm, we have

$$\begin{aligned} \|\alpha^j : \mathbf{n}_{pqr}\| &= \|\Phi(\alpha^j)_\nu : l_r(\mathcal{M}_q^p)\| = \|\Phi(\alpha^j)_j : \mathcal{M}_q^p\| \\ \|\alpha^j : \mathbf{e}_{pqr}\| &= \|\Phi(\alpha^j)_\nu : \mathcal{M}_q^p(l_r)\| = \|\Phi(\alpha^j)_j : \mathcal{M}_q^p\|. \end{aligned}$$

As a result, it follows that

$$\|\alpha^j : \mathbf{n}_{pqr}\| = \|\alpha^j : \mathbf{e}_{pqr}\| = 2^{j(s + \frac{n}{2} - \frac{n}{p})} \|\mathbf{1}_{[0, 2^j]^n} : \mathcal{M}_q^p\| = 2^{j(s + \frac{n}{2})}.$$

The proof is now complete. ■

With a minor modification of Lemma 4.2, we can prove the following.

Lemma 4.3. *Let $j \in \mathbb{N}$ and define $\beta^j := \{\beta_{\nu, m}^j\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{a}_{pqr}$ so that*

$$\beta_{\nu, m}^j = \begin{cases} 1 & \text{if } j = \nu \text{ and } Q_{\nu, m} \subset [0, 1]^n, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $\|\beta^j : \mathbf{n}_{pqr}\| = \|\beta^j : \mathbf{e}_{pqr}\| = 2^{j(s + \frac{n}{2} - \frac{n}{p})}$.

The parameter r

Lemma 4.4. *Pick a sequence $\mathbf{a} = \{a_\nu\}_{\nu \in \mathbb{N}_0}$ and define $\delta^{\mathbf{a}} := \{\delta_{\nu,m}^{\mathbf{a}}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ by*

$$\Psi(\delta^{\mathbf{a}})_\nu := a_\nu \mathbf{1}_{[0,1]^n}.$$

Then we have $\|\delta^{\mathbf{a}} : \mathbf{n}_{pqr}\| = \|\delta^{\mathbf{a}} : \mathbf{e}_{pqr}\| = \|\mathbf{a} : l_r\|$.

Proof. From the definition of the vector-valued norm, we have

$$\begin{aligned} \|\delta^{\mathbf{a}} : \mathbf{n}_{pqr}\| &= \left(\sum_{\nu=0}^{\infty} \|a_\nu \mathbf{1}_{[0,1]^n} : \mathcal{M}_q^p\|^r \right)^{\frac{1}{r}} = \|\mathbf{a} : l_r\| \\ \|\delta^{\mathbf{a}} : \mathbf{e}_{pqr}\| &= \left\| \left(\sum_{\nu=0}^{\infty} |a_\nu \mathbf{1}_{[0,1]^n}|^r \right)^{\frac{1}{r}} : \mathcal{M}_q^p \right\| = \|\mathbf{a} : l_r\|, \end{aligned}$$

proving the lemma. ■

The parameter q

Next, let us detect the parameter q . We borrow the following result from [1].

Proposition 4.5. [1, Lemma 2] *Let $n = 1$ and $0 < t < p < \infty$. Choose $\beta > 0$ so that*

$$t = \frac{\beta}{\beta + 1} p. \quad (10)$$

Define $A_N := \bigcup_{j=0}^{N-1} [2^N + jN^\beta, 2^N + (j+1)N^\beta]$. Then we have

$$\left\| \sum_{N \in \mathbb{N}} \chi_{A_N} : \mathcal{M}_t^p \right\| < \infty.$$

Proposition 4.6. *Keep to the same notation as Proposition 4.5. If $t \leq q \leq p$, then we have*

$$\|\chi_{A_N} : \mathcal{M}_q^p\| \geq N^{\frac{1}{t} - \frac{1}{q}}$$

Corollary 4.7. *If one defines*

$$B_N := \left(\bigcup_{j=0}^{N-1} [2^N + [jN^\beta], 2^N + [jN^\beta] + 1] \right)^n \subset \mathbb{R}^n,$$

then $\|\chi_{B_N} : \mathcal{M}_q^p\| \geq N^{\frac{n}{t} - \frac{n}{q}}$.

Let us summarize the observation above in the form we use to prove our results.

Lemma 4.8. Let $j \in \mathbb{N}$ and define γ^j so that

$$\Psi(\gamma^j)_\nu = \delta_{\nu 0} \chi_{B_j}.$$

Then $\lim_{j \rightarrow \infty} \|\gamma^j : \mathbf{a}_{pqr}\| < \infty$ if and only if $q \leq t$, where t is given by (10).

The case when $p \neq q$. If q differs from p , then we can observe a singular phenomenon.

Lemma 4.9. Assume in addition $q < p$. Let $j \in \mathbb{N}$ and define $\theta^j \in \mathbf{a}_{pqr}$ so that

$$\Psi(\theta^j)_\nu = \delta_{\nu,j} 2^{-\frac{n\nu}{p}} \mathbf{1}_{[2^{-\nu}, 2^{-\nu+1}]^n}.$$

Also define $\Theta_j = \sum_{k=1}^j \theta_k$. Then we have $\|\Theta^j : \mathbf{n}_{pqr}\| = j^{\frac{1}{r}}$, $\|\Theta^j : \mathbf{e}_{pqr}\| \simeq 1$.

Proof. Let us begin with $\|\Theta^j : \mathbf{n}_{pqr}\|$.

$$\begin{aligned} \|\Theta^j : \mathbf{n}_{pqr}\| &= \left(\sum_{\nu=0}^{\infty} \left\| \sum_{k=0}^j \Psi(\theta^k)_\nu : \mathcal{M}_q^p \right\|^r \right)^{\frac{1}{r}} \\ &= \left(\sum_{\nu=0}^j \|2^{-\frac{n\nu}{p}} \mathbf{1}_{[2^{-\nu}, 2^{-\nu+1}]^n} : \mathcal{M}_q^p\|^r \right)^{\frac{1}{r}} = j^{\frac{1}{r}}. \end{aligned}$$

It remains to show $\sup_{j \in \mathbb{N}} \|\Theta^j : \mathbf{e}_{pqr}\| < \infty$. We have, for a.e. $x \in \mathbb{R}^n$,

$$\begin{aligned} \left(\sum_{\nu=0}^{\infty} \Psi(\Theta^j)_\nu(x)^r \right)^{\frac{1}{r}} &= \left(\sum_{\nu=0}^j 2^{-\frac{n\nu r}{p}} \mathbf{1}_{[2^{-\nu}, 2^{-\nu+1}]^n}(x) \right)^{\frac{1}{r}} \\ &= \sum_{\nu=0}^j 2^{-\frac{n\nu}{p}} \mathbf{1}_{[2^{-\nu}, 2^{-\nu+1}]^n}(x) \\ &\leq c w_p(x) \cdot \mathbf{1}_{B(n)}(x), \end{aligned}$$

where $w_p(x) = |x|^{-\frac{n}{p}}$. Since we are assuming $q < p$, we have $\|w_p : \mathcal{M}_q^p\| < \infty$. From this we conclude $\sup_{j \in \mathbb{N}} \|\Theta^j : \mathbf{e}_{pqr}\| < \infty$. ■

Lemma 4.10. We rearrange $\{Q_{0,m}\}_{m \in \mathbb{N}_0^n}$ and relabel them as $R_1, R_2, \dots, R_N, \dots$ so that

$$\bigcup_{l=1}^{j^n} R_l = [0, j]^n$$

for every $j \in \mathbb{N}$. We define τ^j so that

$$\Phi(\tau^j)_\nu = \sum_{k=1}^j \delta_{\nu,k} \mathbf{1}_{R_k}$$

for $j \in \mathbb{N}$. Then we have

$$\left\| \tau^{j^n} : \mathbf{n}_{pq\infty} \right\| = j^{\frac{n}{r}}, \left\| \tau^{j^n} : \mathbf{e}_{pq\infty} \right\| = j^{\frac{n}{p}}.$$

Proof. Indeed, we have

$$\left\| \tau^{j^n} : \mathbf{n}_{pqr} \right\| = \left(\sum_{k=1}^{j^n} \left\| \mathbf{1}_{R_k} : \mathcal{M}_q^p \right\|^r \right)^{\frac{1}{r}} = j^{\frac{n}{r}}, \left\| \tau^{j^n} : \mathbf{e}_{pqr} \right\| = \left\| \mathbf{1}_{[0, j^n]} : \mathcal{M}_q^p \right\| = j^{\frac{n}{p}},$$

proving the lemma. ■

Proofs of Proposition 1.6 and Theorem 1.7. Let us refer back to the proof of Proposition 1.6 and Theorem 1.7.

Proposition 1.6 no. 1 and 2 are proved by Lemmas 4.9 and 4.10 with $r = \infty$ respectively. Let us turn to Theorem 1.7. In view of Lemmas 4.2 and 4.3, once we assume one of the conditions in Theorem 1.7, we see $s_0 = s_1$ and $p_0 = p_1$. Let us assume $s_0 = s_1$ and $p_0 = p_1$ below, which yields that the definitions of Ψ given by (9) for $\mathcal{N}_0, \mathcal{N}_1, \mathcal{E}_0, \mathcal{E}_1$ are identical. Therefore, from Lemmas 4.8 and 4.4 we conclude $q_0 = q_1$ and $r_0 = r_1$. Therefore, it remains to show that $p_0 = q_0 = r_0 < \infty$ assuming $\mathcal{N}'_0 = \mathcal{E}'_1$. Proposition 1.6 no. 2 with $r = \infty$ rules out the possibility when $r_0 = r_1 = \infty$. Therefore, we have $r_0 = r_1 < \infty$. In this case Proposition 1.6 no. 1 in turn yields $p_0 = q_0$. Finally another application of Lemma 4.10 gives us $p_0 = r_1$. Therefore, the proof is complete.

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References

- [1] O. Blasco, A. Ruiz and L. Vega, *Non-interpolation in Morrey-Campanato and block spaces*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999), no. 1, 31–40.
- [2] I. Daubechies, *Orthonormal bases of compactly supported wavelets*, Comm. Pure Appl. Math. **41** (1988), 909–996.
- [3] D. Deng, M. Xu and L. Yan, *Wavelet characterization of weighted Triebel-Lizorkin spaces*, Approx. Theory Appl. **18** (2002), no. 4, 76–92.
- [4] E. Hernández, G. Weiss and D. Yang, *The φ -transform and wavelet characterizations of Herz-type spaces*, Collect. Math. **47** (1996), 285–320.
- [5] M. Izuki, *The characterizations of weighted Sobolev spaces by wavelets and scaling functions*, preprint.
- [6] H. Kozono and M. Yamazaki, *Semilinear heat equations and the Navier-Stokes equations with distributions in new function spaces as initial data*, Comm. Partial Differential Equations **19** (1994) no. 5–6, 959–1014.

- [7] A. Mazzucato, *Decomposition of BesovMorrey spaces, Harmonic analysis at Mount Holyoke* (South Hadley, MA, 2001), 279–294, Contemp. Math., **320**, Amer. Math. Soc., Providence, RI, (2003).
- [8] A. Mazzucato, *Besov-Morrey spaces: Function space theory and applications to non-linear PDE*, Trans. Amer. Math. Soc., **355** (2003), no. 4, 1297–1364.
- [9] Y. Meyer, *Wavelets and Operators*, Cambridge University Press, Cambridge, (1992).
- [10] A. M. Najafov, Some properties of functions from the intersection of Besov-Morrey type spaces with dominant mixed derivatives, Proc. A. Razmadze Math. Inst. **139** (2005), 71–82.
- [11] A. M. Najafov, *Embedding theorems in the Sobolev-Morrey type spaces $S_{p,a,\kappa,r}^l W(G)$ with dominant mixed derivatives*, Siberian Math. J., **47** (2006), no 3, 613–625.
- [12] A. M. Najafov, *On some properties of the functions from Sobolev-Morrey type spaces*, Cent. Eur. J. Math. **3** (2005), no. 3, 496–507.
- [13] V. S. Rychkov, *Littlewood-Paley theory and function spaces with A_p^{loc} weights*, Math. Nachr. **224** (2001), 145–180.
- [14] Y. Sawano and H. Tanaka, *Morrey spaces for non-doubling measures*, Acta Math. Sinica, **21** (2005), no. 6, 1535–1544.
- [15] Y. Sawano and H. Tanaka, *Decompositions of Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces*, Math. Z., **257** (2007) no. 4, 871–905.
- [16] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, (1993).
- [17] L. Tang and J. Xu, *Some properties of Morrey type Besov-Triebel spaces*, Math. Nachr **278** (2005), no 7-8, 904–917.
- [18] H. Triebel, *On Besov-Hardy-Sobolev spaces in domains and regular elliptic boundary value operators*, Comm. Partial. Diff. Equations, **3** 12 (1978), 1083–1164.
- [19] H. Triebel, *Theory of Function Spaces*, Birkhäuser (1983).
- [20] H. Triebel, *Theory of Function Spaces II*, Birkhäuser (1992).
- [21] H. Triebel, *Fractal and Spectra*, Birkhäuser (1997).
- [22] H. Triebel, *The Structure of Functions*, Birkhäuser (2001).
- [23] H. Triebel, *Theory of Function Spaces III*, Birkhäuser (2006).
- [24] P. Wojtaszczyk, *A Mathematical Introduction to Wavelets*, Cambridge University Press, Cambridge, (1997).

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