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ON SUM-FREE SUBSETS OF THE TORUS GROUP

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Dedicated with best wishes
to Jean-Marc Deshouillers
on the occasion of his 60th birthday

Abstract: Establishing the structure of dense sum-free subsets of the torus group $\mathbb{R}/\mathbb{Z}$, we find an absolute constant $\alpha_0 < 1/3$ such that for any sum-free subset $A \subseteq \mathbb{R}/\mathbb{Z}$ with the inner measure $\mu(A) > \alpha_0$ there exists an integer $q \geqslant 1$ so that

$$A \subseteq \bigcup_{j=0}^{q-1} \left[\frac{j+\mu(A)}{q}, \frac{j+1-\mu(A)}{q} \right].$$

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1. Introduction

The subset A of an additively written semigroup is called *sum-free* if there do not exist $a_1, a_2, a_3 \in A$ with $a_1 + a_2 = a_3$; equivalently, if A is disjoint with its *sumset* $2A := \{a' + a'' : a', a'' \in A\}$. Introduced by Schur in 1916 ("the set of positive integers cannot be partitioned into finitely many sum-free subsets"), sum-free sets become now one of the central objects of study in additive combinatorics. We refer the reader to the papers in the References section and the pending citations for the history and overview of the subject area.

What is the largest possible size of a sum-free subset of a given finite abelian group? Having been studied for several decades, this problem was eventually given a complete solution by Green and Ruzsa in [GR05]. The next natural step is to describe sum-free subsets of the size, close to the largest possible; the ideology here is that *small* sum-free subsets can be sporadic and unstructured, while in order for a *large* subset to be sum-free it has to possess a rigid structure. As the description depends heavily on the algebraic nature of the underlying group, this problem has been considered for a very limited number of groups only. In this connection we mention the papers [DT89] (elementary abelian 2-groups), [L05] (elementary abelian 3-groups), and [DF06, L06, DL] (the group $\mathbb{Z}/p\mathbb{Z}$ of residues

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modulo a prime p). Particularly important in our present context are the last three papers, where the following result is established.

Theorem 1.1 [DF06, L06, DL]**.** *There exists an absolute constant $\alpha_0 < 1/3$ with the following property: if p is a sufficiently large prime and $A \subseteq \mathbb{Z}/p\mathbb{Z}$ is sum-free with $n := |A| > \alpha_0 p$, then there is $d \in \mathbb{Z}/p\mathbb{Z}$ such that $A \subseteq \{nd, (n+1)d, \dots, (p-n)d\}$.*

As shown in [L06], Theorem 1.1 is essentially best possible, except for the value of the constant α_0. The result of [L06] establishes Theorem 1.1 with $\alpha_0 = 0.33$, and this was improved to $\alpha_0 = 0.324$ in [DF06] and further to $\alpha_0 = 0.318$ in [DL].

In this paper we consider sum-free subsets of the torus group $\mathbb{R}/\mathbb{Z}$. This infinite group is endowed with the Haar measure, so that the notion of "large" makes perfect sense. Let μ denote the inner measure, corresponding to the Haar measure on $\mathbb{R}/\mathbb{Z}$, normalized by the condition $\mu(\mathbb{R}/\mathbb{Z}) = 1$; thus if $A \subseteq \mathbb{R}/\mathbb{Z}$ is open, then $\mu(A)$ can be interpreted as the Lebesgue measure of A.

By a result of Raikov [R39], for any $A \subseteq \mathbb{R}/\mathbb{Z}$ we have $\mu(2A) \geqslant \min\{2\mu(A), 1\}$. If A is sum-free, then A is disjoint with $2A$, implying $\mu(A) + \mu(2A) \leqslant 1$ and consequently $\mu(A) \leqslant 1/3$; this bound is sharp as it follows from the observation that the canonical image of the interval $(1/3, 2/3)$ in $\mathbb{R}/\mathbb{Z}$ is sum-free. More generally, for $v, w \in \mathbb{R}$ with $v < w$, denote by $(v, w)_{\mathbb{T}}$ the image of the interval (v, w) under the canonical homomorphism $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. (Similar notation will be used for closed intervals.) For integer $q \geqslant 1$ consider the set

$$A := \bigcup_{j=0}^{q-1} \left(\frac{j+1/3}{q}, \frac{j+2/3}{q} \right)_{\mathbb{T}}.$$

Then $\mu(A) = 1/3$, and we claim that A is sum-free: for, if $a_1, a_2, a_3 \in A$, then $qa_i \in (1/3, 2/3)_{\mathbb{T}}$ for $i \in [1, 3]$, whence $qa_1 + qa_2 \neq qa_3$ and therefore $a_1 + a_2 \neq a_3$.

Our goal in this paper is to establish the structure of sum-free subsets $A \subseteq \mathbb{R}/\mathbb{Z}$ with $\mu(A)$ below $1/3$ and show that this structure is close to that in the example just described.

Main Theorem. *Suppose that $\alpha_0 \in (2/7, 1/3)$ has the property, specified in Theorem 1.1. Then for any sum-free subset $A \subseteq \mathbb{R}/\mathbb{Z}$ with $\alpha := \mu(A) > \alpha_0$ there exists an integer $q \geqslant 1$ such that*

$$A \subseteq \bigcup_{j=0}^{q-1} \left[\frac{j+\alpha}{q}, \frac{j+1-\alpha}{q} \right]_{\mathbb{T}}.$$

For brevity, we denote the union in the statement of the Main Theorem by $\mathcal{C}_q(\alpha)$; observe, that this union is the complement in $\mathbb{R}/\mathbb{Z}$ of the *Bohr neighborhood* $\bigcup_{j=0}^{q-1} \big((j-\alpha)/q, (j+\alpha)/q\big)_{\mathbb{T}}$ (corresponding to the character $x \mapsto \exp(2\pi i q x)$).

As it follows from the discussion above, the assertion of the Main Theorem holds true with $\alpha_0 = 0.318$, and it is worth pointing out that it becomes *wrong* for

$\alpha_0 < 0.2$. To see this, fix $\alpha > 0$ and consider the set $A = (\alpha/2, \alpha)_{\mathbb{T}} \cup (2\alpha, (5/2)\alpha)_{\mathbb{T}}$. Since $2A = (\alpha, 2\alpha)_{\mathbb{T}} \cup ((5/2)\alpha, (7/2)\alpha)_{\mathbb{T}} \cup (4\alpha, 5\alpha)_{\mathbb{T}}$, if $\alpha \leqslant 2/9$ (which is equivalent to $5\alpha \leqslant 1 + \alpha/2$), then A is sum-free. If, moreover, $2/11 < \alpha < 1/5$, then there is no integer $q \geqslant 1$ such that $A \subseteq \mathcal{C}_q(\alpha)$; for, assuming the opposite, we would have $(1-2\alpha)/q \geqslant \alpha/2$, implying $q \leqslant 2/\alpha - 4 < 7$, and the remaining cases $q = 1, \dots, 6$ are not difficult to check "manually". (We notice that if $1/5 \leqslant \alpha \leqslant 2/9$, then A is contained in $\mathcal{C}_3(\alpha)$.)

We also mention, without going into detailed explanations, that the union in the statement of the Main Theorem is best possible of this sort. This follows, for instance, by considering the set $\bigcup_{j=0}^{q-1} \big((j+\alpha)/q, (j+2\alpha)/q\big)_{\mathbb{T}}$: if $q/(3q+1) < \alpha < 1/3$, then this set is sum-free and not contained in any of the sets $\mathcal{C}_s(\alpha + \varepsilon)$ with $s \geqslant 1$ and $\varepsilon > 0$.

2. Proof of the Main Theorem

Given two subsets B and C of an additively written group, we write $B - C := \{b - c \colon b \in B,\, c \in C\}$.

We start with a somewhat technical auxiliary lemma.

Lemma 2.1. *Let l and q be positive integers, and suppose that to every $z \in \mathbb{Z}/q\mathbb{Z}$ there corresponds an integer set $A_z \subseteq [0, l]$. Write $n := q^{-1} \sum_{z \in \mathbb{Z}/q\mathbb{Z}} |A_z|$. If $l < \frac{3}{2} n - 1$, then for any $g \in \mathbb{Z}/q\mathbb{Z}$ the union $\bigcup_{z \in \mathbb{Z}/q\mathbb{Z}} (A_{z+g} - A_z)$ contains all integers from the interval $(-n, n)$.*

Proof. Clearly, it suffices to show that for any integer $h \in [0, n)$ there exists $z \in \mathbb{Z}/q\mathbb{Z}$ such that $h \in A_{z+g} - A_z$; suppose, for the contradiction, that this is wrong.

For every $z \in \mathbb{Z}/q\mathbb{Z}$ the sets A_{z+g} and $A_z + h$ are then disjoint subsets of the interval $[0, l+h]$, whence $|A_{z+g}| + |A_z| \leqslant l + h + 1$; averaging over all z, we obtain

$$2n \leqslant l + h + 1 \tag{1}$$

showing, in particular, that $h > 0$.

For integer u set $Z^{(u)} := \{z \in \mathbb{Z}/q\mathbb{Z} \colon u \in A_z\}$. Since $Z^{(u+h)}$ and $Z^{(u)} + g$ are disjoint subsets of $\mathbb{Z}/q\mathbb{Z}$, we have $|Z^{(u+h)}| + |Z^{(u)}| \leqslant q$, which is equivalent to

$$\sum_{z \in \mathbb{Z}/q\mathbb{Z}} |A_z \cap \{u, u+h\}| \leqslant q. \tag{2}$$

For each $j \in [0, h-1]$ consider the arithmetic progression $P_h(j) := \{j, j+h, \dots, j + \lfloor (l-j)/h \rfloor\}$, so that

$$[0, l] = \bigcup_{j=0}^{h-1} P_h(j). \tag{3}$$

By (2), we have

$$\sum_{z\in\mathbb{Z}/q\mathbb{Z}} |A_z\cap P_h(j)| \leqslant \left\lfloor \frac{|P_h(j)|+1}{2}\right\rfloor q = \left(1+\left\lfloor\frac{l-j}{2h}\right\rfloor\right) q,$$

and taking into account (3) we conclude that

$$n \leqslant h + \sum_{j=0}^{h-1} \left\lfloor \frac{l-j}{2h}\right\rfloor . \tag{4}$$

Along with the assumption $h<n$, the last estimate implies that $l\geqslant 2h$, and we set $j_0 := l-2h$. Observe, that by (1) we have

$$h \geqslant 2n-l-1 > \frac{1}{3}(l+1)$$

and hence $j_0 < h-1$ and furthermore, all the summands $\lfloor (l-j)/2h\rfloor$ in the right-hand side of (4) equal to either 0, or 1. Moreover, the number of those summands, equal to 1, is j_0+1, and using (1) we obtain

$$n \leqslant h+(j_0+1) = l-h+1 \leqslant 2l-2n+2.$$

Therefore $3n\leqslant 2(l+1)$, contradicting the assumptions of the lemma. ∎

Corollary 2.2. *Let q be a positive integer and let v and w be real numbers with $v<w$. Suppose that to every $z\in\mathbb{Z}/q\mathbb{Z}$ there corresponds an open set $A_z\subseteq(v,w)$ of measure α_z, and write $\nu := q^{-1}\sum_{z\in\mathbb{Z}/q\mathbb{Z}}\alpha_z$. If $w-v<\frac{3}{2}\nu$, then for any $g\in\mathbb{Z}/q\mathbb{Z}$ we have*

$$(-\nu,\nu)\subseteq \bigcup_{z\in\mathbb{Z}/q\mathbb{Z}} (A_{z+g}-A_z).$$

Proof. Normalizing the sets A_z, we can assume without loss of generality that $v=0$ and $w=1$, whence $\nu>\frac{2}{3}$. Let p be a prime. For every $z\in\mathbb{Z}/q\mathbb{Z}$ set

$$A_z^{(p)} := \left\{u\in[0,p-1]\colon \left(\tfrac{u}{p},\tfrac{u+1}{p}\right)\subseteq A_z\right\},$$

so that $|A_z^{(p)}|\geqslant \alpha_z p(1+o(1))$ and therefore

$$n^{(p)} := q^{-1}\sum_{z\in\mathbb{Z}/q\mathbb{Z}} |A_z^{(p)}| \geqslant \nu p(1+o(1))$$

as $p\to\infty$. From $\nu>\frac{2}{3}$ it follows now that $p<\frac{3}{2}n^{(p)}$ for all sufficiently large p, and applying Lemma 2.1 with $l=p-1$ we conclude that every integer from the interval $(-n^{(p)}, n^{(p)})$ is contained in $\bigcup_{z\in\mathbb{Z}/q\mathbb{Z}}(A^{(p)}_{z+g} - A^{(p)}_z)$. Since for an integer u the inclusion $u \in A^{(p)}_{z+g} - A^{(p)}_z$ implies that $\left(\frac{u-1}{p}, \frac{u+1}{p}\right) \subseteq A_{z+g} - A_z$, we obtain

$$\bigcup_{z\in\mathbb{Z}/q\mathbb{Z}} (A_{z+g} - A_z) \supseteq \left(\frac{-n^{(p)}}{p}, \frac{n^{(p)}}{p}\right) \supseteq (-\nu + o(1), \nu + o(1)),$$

implying the result. ∎

Clearly, the assertion of Corollary 2.2 remains valid if in the statement of the corollary the sets A_z are open subsets of the *torus group*, and the intervals (v, w) and $(-\nu, \nu)$ are replaced with $(v, w)_{\mathbb{T}}$ and $(-\nu, \nu)_{\mathbb{T}}$, respectively.

We are eventually ready to prove the Main Theorem. Throughout the proof we use the following notation. Let m be a positive integer. For $u \in \mathbb{Z}/m\mathbb{Z}$ by $\frac{u}{m}$ we denote the image of u under the group homomorphism $\mathbb{Z}/m\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, defined by $1 \mapsto \frac{1}{m}$. For $v, w \in \mathbb{R}$ with $v < w$ by $[v, w]_m$ we denote the image of the set $[v, w] \cap \mathbb{Z}$ under the canonical homomorphism $\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$. Finally, for integer $q \geqslant 1$ and a subset A of an abelian group we write $q * A := \{qa\colon a \in A\}$.

Proof of the Main Theorem. As a first step, we prove the Main Theorem under the additional assumption that A is *open*.

Fix a prime p which will be assumed large enough (the little-o-notation below corresponds to $p \to \infty$) and set

$$A^{(p)} := \left\{u \in \mathbb{Z}/p\mathbb{Z}\colon \tfrac{u}{p} \in A\right\}.$$

Since $A \subseteq \mathbb{R}/\mathbb{Z}$ is sum-free, so is $A^{(p)} \subseteq \mathbb{Z}/p\mathbb{Z}$, and since A is open we have

$$n^{(p)} := |A^{(p)}| \geqslant \alpha p(1 + o(1)). \tag{5}$$

By Theorem 1.1 there exists an integer $q^{(p)} \in [1, p/2)$ with

$$q^{(p)} * A^{(p)} \subseteq [n^{(p)}, p - n^{(p)}]_p. \tag{6}$$

Fix $v, w \in \mathbb{R}$ with $v < w$ so that the interval $(v, w)_{\mathbb{T}}$ is entirely contained in A. Corresponding to this interval is a block $I^{(p)}$ of $(w-v+o(1))p$ consecutive elements of $\mathbb{Z}/p\mathbb{Z}$, contained in $A^{(p)}$. Now $q^{(p)} * I^{(p)} \subseteq [n^{(p)}, p-n^{(p)}]_p \subseteq (p/4, 3p/4)_p$ implies that $q^{(p)} < p/(2(|I^{(p)}|-1)) = O_{v,w}(1)$, so that there is an infinite sequence of primes p, sharing the same common value of $q^{(p)}$. In what follows we denote this common value by q and assume that p are so chosen that $q^{(p)} = q$.

Fix arbitrarily $a \in A$ and find $u^{(p)} \in \mathbb{Z}/p\mathbb{Z}$ satisfying

$$a - \frac{u^{(p)}}{p} \in \left[-\frac{1}{2p}, \frac{1}{2p}\right]_{\mathbb{T}}. \tag{7}$$

If p is large enough, then we have $u^{(p)} \in A^{(p)}$, whence

$$q\frac{u^{(p)}}{p} \in (\alpha + o(1), 1 - \alpha + o(1))_{\mathbb{T}}$$

by (6) and (5). Along with (7) this yields $qa \in (\alpha + o(1), 1 - \alpha + o(1))_{\mathbb{T}}$, and consequently $qa \in [\alpha, 1-\alpha]_{\mathbb{T}}$.

We have shown that

$$q * A \subseteq [\alpha, 1-\alpha]_{\mathbb{T}},$$

and this is equivalent to $A \subseteq \mathcal{C}_q(\alpha)$, as wanted.

To complete the proof we drop now the assumption that A is open and turn to the general case. Let I be an arbitrarily fixed open interval, contained in A. Fix $\varepsilon > 0$ and find an open subset $B \subseteq A$ such that $I \subseteq B$ and $\beta := \mu(B) > \max\{\alpha_0, \alpha - \varepsilon\}$. By the above, there is an integer $q \geqslant 1$ with $B \subseteq \mathcal{C}_q(\beta)$, and we observe that then $\mu(I) \leqslant (1-2\beta)/q$, implying that q is bounded by a constant, depending only on I. Accordingly, there is a sequence of values of ε, converging to 0, so that the corresponding values of q are all equal to each other, and we assume below that ε is chosen to be an element of this sequence.

For $z \in \mathbb{Z}/q\mathbb{Z}$ let

$$B_z := \left(B - \frac{z}{q}\right) \cap \left(\frac{\beta}{q}, \frac{1-\beta}{q}\right)_{\mathbb{T}},$$

so that $B = \bigcup_{z\in\mathbb{Z}/q\mathbb{Z}} \left(\frac{z}{q} + B_z\right)$ and consequently,

$$B - B = \bigcup_{g\in\mathbb{Z}/q\mathbb{Z}} \left(\frac{g}{q} + \bigcup_{z\in\mathbb{Z}/p\mathbb{Z}} (B_{z+g} - B_z)\right).$$

As

$$q^{-1} \sum_{z\in\mathbb{Z}/q\mathbb{Z}} \mu(B_z) = \frac{\beta}{q} > \frac{\alpha-\varepsilon}{q} > \frac{2}{3}\,\frac{1-2(\alpha-\varepsilon)}{q} > \frac{2}{3}\,\frac{1-2\beta}{q}$$

for sufficiently small ε (it is here that the assumption $\alpha_0 > 2/7$ is used), we can apply Corollary 2.2 and obtain that

$$B - B \supseteq \bigcup_{g\in\mathbb{Z}/q\mathbb{Z}} \left(\frac{g}{q} + \left(-\frac{\beta}{q}, \frac{\beta}{q}\right)_{\mathbb{T}}\right). \tag{8}$$

Since A is sum-free and $B \subseteq A$, the sets A and $B - B$ are disjoint, and hence (8) gives

$$A \subseteq \bigcup_{g\in\mathbb{Z}/q\mathbb{Z}} \left(\frac{g}{q} + \left[\frac{\beta}{q}, \frac{1-\beta}{q}\right]_{\mathbb{T}}\right).$$

As $\beta > \alpha - \varepsilon$ and ε can be chosen arbitrarily small, this implies that indeed

$$A \subseteq \bigcup_{g \in \mathbb{Z}/q\mathbb{Z}} \left(\frac{g}{q} + \left[\frac{\alpha}{q}, \frac{1-\alpha}{q} \right]_{\mathbb{T}} \right) = \mathcal{C}_q(\alpha),$$

completing the proof. ■

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