Oversemigroups of a valuation semigroup

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Abstract. By a semigroup we mean a submonoid of a torsion-free group in this paper. We investigate some behaviour of prime ideals among semigroups and their oversemigroups. In particular, we consider a oversemigroup \( T \) of a valuation semigroup \( S \) such that the set of prime ideals \( P \) of \( S \) satisfying the property that \( P + T \subseteq T \) equals to \( \text{Spec}(T) \). Also, we study some conditions that a valuation semigroup is completely integrally closed. We give some examples of a valuation semigroup of dimension two and a completely integrally closed semigroup.

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§1. Introduction

Throughout this paper, a semigroup will stand for a commutative cancellation torsion-free additive semigroup, and it is a (non-zero) semigroup with 0.

Let \( S \) be a semigroup. If we set \( G = \{ s - s' \mid s, s' \in S \} \), then \( G \) is a torsion-free abelian group with respect to addition and \( S \) is a subsemigroup of \( G \). \( G \) is called the quotient group of \( S \) and denoted by \( q(S) \).

Any semigroup \( T \) between \( S \) and \( q(S) \) is called an oversemigroup of \( S \).

From now on, assume that \( S \) is a semigroup with quotient group \( G = q(S) \).

In this paper, we study some semigroup versions of [1, §26].

We obtained some results of primary ideals in semigroups in [3].

In §2, we investigate some behaviour of prime ideals among semigroups and their oversemigroups. Also, we prove that the numbers of all quotient semigroups of a semigroup \( S \) equal the numbers of all prime ideals.

A valuation semigroup is seminormal and seminormal semigroups considered in [4] and [5].
In §3, we develop some properties obtained in §2 for valuation semigroups. In particular, we consider an oversegment group \( T \) of a valuation semigroup \( S \) such that the set of prime ideals \( P \) of \( S \) satisfying the property that \( P + T \not\subset T \) equals to \( \text{Spec}(T) \) in Theorem 7.

In §4, using Theorem 7 we will prove that the transform of a principal ideal \( I \) in a valuation semigroup \( S \) containing in \( S_P \) (\( P \in \text{Spec}(S) \)) if and only if \( I \not\subset P \). Also, the ideal transform and conductor inform us about an almost integral element and completely integrally closed semigroups (cf. Theorem 13 and Theorem 14).

Moreover, we give examples semigroups of dimension two and completely integral closures.

Preliminaries

We recall some notations and definitions ([2], [3], [4] and [5]).

Let \( I \) be a subset of \( S \). \( I \) is called an ideal of \( S \) if \( I + S = I \), that is, \( a + s \in I \) for each \( a \in I \) and each \( s \in S \).

An element \( u \) of \( S \) is called a unit if \( u + v = 0 \) for some \( v \in S \). Let \( U(S) \) be the set of units in \( S \). Note that \( U(S) \) is a subgroup of \( G \).

If we put \( M = S - U(S) \), then \( M \) is an ideal of \( S \). Moreover, if \( I \) is an ideal of \( S \) such that \( M \subset I \), then \( M = I \) or \( I = S \). \( M \) is called the maximal ideal of \( S \). A proper ideal \( P \) of \( S \) is called a prime ideal of \( S \) if \( a + b \in P \) with \( a, b \in S \) implies either \( a \in P \) or \( b \in P \).

We note that the maximal ideal of \( S \) is prime and \( \phi \) is a prime ideal of \( S \).

Let \( S \) be a semigroup. Also, let \( \text{Spec}(S) \) be the set of all prime ideals of \( S \).

A finite chain \( P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n \) of \( n + 1 \) prime ideals of \( S \) will be said to have length \( n \). We define the (Krull) dimension of \( S \) in terms of this concept.

There is a non-negative integer \( n \) such that \( S \) contains a chain of prime ideals of length \( n \), but no such chain of length \( n + 1 \). In this case we say that \( S \) has dimension \( n \) or \( S \) is \( n \)-dimensional, and we write \( \dim S = n \).

A nonempty subset \( N \) of a semigroup \( S \) is called an additive system of \( S \) if \( a, b \in N \) implies \( a + b \in N \) and \( 0 \in N \).

Put \( S_N = \{ s - t \mid s, t \in N \} \). Then \( S_N \) is an oversegment of \( S \) and is called the quotient semigroup of \( S \). If \( P \) is a prime ideal of \( S \), then \( T = S - P \) is an additive system of \( S \) and the quotient semigroup \( S_T \) is denoted by \( S_P \).

Also, \( S_\phi = S_S = G = q(S) \) and \( \phi \) is the maximal ideal of \( G \). Each element \( t \in N \) is a unit in \( S_N \).

Let \( P \in \text{Spec}(S) \). We define the height of \( P \), denoted by \( \text{ht}(P) \), to be the dimension of \( S_P \), where \( S_P = \{ s - t \mid s, t \in S, t \not\in P \} \).
For any \( x \in S \), put \( (x) = x + S = \{ x + a \mid a \in S \} \). Then \( (x) \) is an ideal of \( S \) and it is called a \textit{principal ideal} of \( S \). For \( a_1, a_2, \ldots, a_n \in S \), we set \( I = (a_1, a_2, \ldots, a_n) = \bigcup_{i=1}^{n} (a_i) = \bigcup_{i=1}^{n} (a_i + S) \).

\( (a_1, a_2, \ldots, a_n) \) is an ideal of \( S \) and it is called an \textit{ideal generated by} \( a_1, \ldots, a_n \) and \( \{ a_1, a_2, \ldots, a_n \} \) is called a \textit{basis} of \( I \).

We say that \( (a_1, a_2, \ldots, a_n) \) is a \textit{finitely generated ideal} of \( S \). Note that \( (a + b) = (a) + (b) \) for each \( a, b \in S \).

A subset \( F \) of \( q(S) \) is said to be a \textit{fractional ideal} of \( S \) if \( x + F \subset S \) for some \( x \in S \) and \( F + S = F \). If \( S \neq G = q(S) \), then \( G \) is not a fractional ideal of \( S \). Each ideal of \( S \) is a fractional ideal of \( S \) and it is called an \textit{integral ideal}.

For each \( x \in q(S) \), \( (x) = x + S \) is a fractional ideal.

Now, let \( T \) be an oversemigroup of \( S \).

In general, we set \( (I_1 : I_2)_T = \{ x \in T \mid x + I_2 \subset I_1 \} \) for all subsets \( I_1, I_2 \) of \( q(S) \).

If \( I_1, I_2 \) are fractional ideals of \( S \) such that \( I_2 \neq \emptyset \), then \( (I_1 : I_2)_T \) is also a fractional ideal of \( S \) and \( (I_1 : I_2)_S \) is an integral ideal of \( S \). Also, \( (I_1 : \emptyset)_S = G \).

A non-empty fractional ideal \( F \) of \( S \) is said to be \textit{invertible} if there exists a fractional ideal \( F' \) of \( S \) such that \( F + F' = S \).

A principal fractional ideal is a invertible ideal.

Let \( T \) be an oversemigroup of \( S \). An element \( t \in T \) is said to be \textit{integral over} \( S \) if \( nt \in S \) for some positive integer \( n \).

The set of all elements \( t \in T \) that are integral over \( S \) is called the \textit{integral closure} of \( S \) in \( T \).

An oversemigroup \( T \) is said to be \textit{integral over} \( S \) if each element \( t \in T \) is integral over \( S \).

We say that \( \overline{S} \) is the \textit{integral closure} of \( S \) if \( \overline{S} = \{ \alpha \in q(S) \mid \alpha \text{ is integral over } S \} \), \( \overline{S} \) is a semigroup. \( S \) is said to be \textit{integrally closed} if \( \overline{S} = S \). For \( x \in q(S) \), we say that \( x \) is an \textit{almost integral element} over \( S \) if there exists an element \( a \) of \( S \) such that \( a + nx \in S \) for any positive integer \( n \).

The set \( S_0 \) of all elements of \( q(S) \) that are almost integral over \( S \) is called the \textit{complete integral closure} of \( S \) (in \( q(S) \)). If \( S_0 = S \), we say that \( S \) is \textit{completely integrally closed}.

If \( x \) is integral over \( S \), then \( x \) is almost integral over \( S \).

\[ \text{§2. Relations of prime ideals among a semigroup and their oversemigroups} \]

\[ \text{Lemma 1. Let } F \text{ be a fractional ideal of } S. \text{ Then } F \text{ is a principal fractional ideal if and only if } F \text{ is an invertible ideal of } S. \]
Proof. $(\implies)$ Let $F = x + S$ for some element $x \in q(S) = G$. Put $F' = (-x) + S$. Then $F + F' = x + S + (-x) + S = S$.

$(\impliedby)$ Since $F$ is invertible, there exists a fractional ideal $F'$ of $S$ such that $F + F' = S$. As $0 \in S$, write $0 = a + b$ for some $a \in F$ and $b \in F'$. For each $x \in F$, we have that $x = x + 0 = a + (x + b) \in a + S$. Therefore $F \subseteq a + S$. Conversely, since $a + S \subseteq F + S = F$, we have that $F = a + S$. \hfill \Box

Proposition 2. Let $S$ be a semigroup and let $P$ be a prime ideal of $S$. Then the following assertions hold.

1. If $P'$ is a prime ideal of $S$ such that $P' \subseteq P$, then we have that $P' + S_P \subseteq \text{Spec}(S_P)$ and $(P' + S_P) \cap S = P'$.

2. Conversely, if $Q'$ is a prime ideal of $S_P$ and put $P' = Q' \cap S$, then $P'$ is a prime ideal of $S$ such that $P' \subseteq P$ and $P' + S_P = Q'$.

3. Let $\varphi_T : \text{Spec}(T) \rightarrow \text{Spec}(S)$ be the restriction mapping, that is, $\varphi_T(Q') = Q' \cap S$ for each oversemigroup $T$ of $S$. Then we have that $S_{Q' \cap S} \subseteq T_{Q'}$. Moreover, if $T = S_P$ where $P \in \text{Spec}(S)$, then $\varphi_{S_P}$ (in short, $\varphi_N$) is injective.

4. If $T$ is a quotient semigroup of $S$, then there exists a prime ideal $P$ of $S$ such that $T = S_P$. Therefore the numbers of all quotient semigroups of $S$ equal to the numbers of all prime ideals of $S$. Also, we have that $\varphi_{S_N}$ (in short, $\varphi_{N}$) is injective.

Proof. (1) Let $\alpha = s - n \in S_N$ for some $s \in S$ and some $n \in N$. Then $\alpha \in P + S_N$ if and only if $s \in P$. Assume that $z_1, z_2 \in S_P$ such that $z_1 + z_2 \in P' + S_P$ and $z_1 \notin P' + S_P$.

Then there exists an element $n$ of $S - P$ such that $z_1 = a - n$, $z_2 = b - n$ ($a, b \in S$ and $a \notin P'$).

Then $z_1 + z_2 = (a + b) - 2n \in P' + S_P$. Since $P'$ is prime, we have that $b \in P'$. This implies that $z_2 \in P' + S_P$. Consequently, we have that $P' + S_P$ is a prime ideal of $S_P$.

Next, we will prove that $(P' + S_P) \cap S = P'$. Since $(P' + S_P) \cap S \subseteq P'$ is trivial, we will prove the reverse inclusion. Take any $z \in (P' + S_P) \cap S$. Then $S \ni z = a - b$ ($a \in P'$, $b \in S - P$) and so $b + z = a \in P'$. Thus $z \in P'$. Consequently, we have that $(P' + S_P) \cap S = P'$.

(2) Assume that $a, b \in S$ such that $a + b \in P' = Q' \cap S \subseteq Q'$ and $a \notin P'$. Then $a \notin Q'$. Since $Q'$ is a prime ideal of $S_P$, we see that $b \in Q'$. Thus $b \in P'$. Therefore $P' \in \text{Spec}(S)$.

Next, we will prove that $P' \subseteq P$. Take any $z \in P'$. Since $P' \subseteq Q'$, we have that $z \in S$ and $z \in Q'$. Assume that $z \notin P$. Then $-z = 0 - z \in S_P$. Therefore $z$ is a unit of $S_P$. This contradicts that $z \in Q'$. Consequently, we have that $z \in P$, and so we have that $P' \subseteq P$.
Now, we will prove that \( P' + S_P = Q' \). Since \( P' \subseteq Q' \), we have that \( P' + S_P \subseteq Q' + S_P = Q' \). Conversely, take any \( z \in Q' \) \( (z = a - t, \ a \in S, \ t \in S - P) \). Also, \( a \in Q' \cap S = P' \). Therefore \( z = a - t \in P' + S_P \). Consequently, \( P' + S_P = Q' \).

(3) It is clear that \( S_{Q' \cap S} \subset T_{Q'} \). Let \( Q, \ Q' \in \text{Spec}(T) \). If \( \varphi_P(Q) = \varphi_P(Q') \), then \( Q \cap S = Q' \cap S = \phi \). By (2), we have that \( P' \subset P \) and \( Q = P' + S_P = Q' \). Hence \( \varphi_P \) is injective.

(4) Let \( N \) be an additive system of \( S \) such that \( T = S_N \). Let \( M' \) be the maximal ideal of \( T \). Put \( P = M' \cap S \). Then \( M' \cap N = \phi \) and \( S_P \subset T_{M'} = T = S_N \) by (3). To prove the opposite inclusion, take \( \alpha = s - n \in S_N \) where \( s \in S \) and \( n \in N \). Since \( n \not\in M' \cap S = P \), we obtain that \( \alpha = s - n \in S_P \). Hence \( S_P = S_N = T \). The remaining parts are trivial.

**Corollary 3.** Let \( S \) be a semigroup. Then the following assertions hold.

1. If \( P \in \text{Spec}(S) \) then \( P + S_P \) is the maximal ideal of \( S_P \).
2. \( \dim S_P \leq \dim S \).
3. If \( P, \ Q \in \text{Spec}(S) \) such that \( P \neq Q \), then \( S_P \neq S_Q \).

**Proof.** (1) Put \( M_1 = P + S_P \). Let \( M_2 \) be the maximal ideal of \( S_P \). Then \( M_1 \subset M_2 \). By Proposition 2 (2), there exists a prime ideal \( Q \) of \( S \) such that \( Q = M_2 \cap S \subset P \) and \( Q + S_P = M_2 \). Therefore \( M_2 = (M_2 \cap S) + S_P \subset P + S_P = M_1 \). Thus \( M_1 = M_2 \). Therefore \( P + S_P \) is the maximal ideal of \( S_P \).

(2) Assume that \( Q_1 \subset Q_2 \) where \( Q_1, \ Q_2 \in \text{Spec}(S_P) \). Put \( P_i = Q_i \cap S \) where \( i = 1, 2 \). By Proposition 2 (2), we have that \( P_1 \subset P \) and \( P_1 + S_P = Q_1 \). Suppose that \( P_1 = P_2 \). Then \( Q_1 = P_1 + S_P = P_2 + S_P = Q_2 \). This is a contradiction. Therefore \( P_1 \subset P_2 \). Thus \( \dim S_P \leq \dim S \).

(3) Assume that \( P \neq Q \). It is enough to assume that there exists some element \( t \in P - Q \). Since \( -t = 0 - t \in S_Q \) and \( t \in P \subset S \subset S_Q \), we have that \( t \) is a unit of \( S_Q \) and \( t \in P + S_P \). If \( S_Q = S_P \) then \( P + S_P \) is the maximal ideal of \( S_Q \), so this is a contradiction. Therefore \( S_P \neq S_Q \).

§3. Valuation semigroups

**Definition 1**([2], [3], [4] and [5]). Let \( S \) be a semigroup with quotient group \( G \). We say that \( S \) is a valuation semigroup if \( \alpha \in S \) or \( -\alpha \in S \) for each \( \alpha \in G \).

**Lemma 4.** Let \( S \) be a valuation semigroup with quotient group \( G \). Then the following statements hold.

1. If \( T \) is an oversemigroup of \( S \), then \( T \) is also a valuation semigroup.
2. The set of all ideals in \( S \) is a totally ordered set under inclusion relation.
3. If \( I \) is a finitely generated ideal of \( S \), then \( I \) is principal.
4. If $T$ is an oversemigroup of $S$, then there exists a prime ideal $P$ such that $T = S_P$.
5. $S$ is an integrally closed semigroup.

Proof. (1) Take $x \in G$. If $x \notin T$, then $x \notin S$. Since $S$ is a valuation semigroup, we have that $-x \in S$. Hence $-x \in T$. Thus $T$ is a valuation semigroup.

(2) Let $I$ and $J$ be two ideals of $S$ such that $I \not\subseteq J$. Then there exists an element $b$ of $I - J$. For any element $a$ of $J$, put $x = a - b$. Then $x \in G$. So $x \in S$ or $-x \in S$. If $x \in S$ then $a = b + x \in I$. If $-x \in S$, then $b = a + (-x) \notin J$, a contradiction. Therefore $x \in S$, and $J \subseteq I$.

(3) It is trivial from (2).

(4) Let $M$ be the maximal ideal of $T$. Put $T' = S_{M \cap S}$. Then $T'$ is a semigroup. Let $b \in S - M \cap S$. Since $b$ is a unit of $T$, we have that $T' \subseteq T$. By (1), $T'$ is a valuation semigroup. Assume that $T' \not\subseteq T$.

Then there exists an element $a \in T - T'$. Since $T'$ is a valuation semigroup, we have that $-a \in T' \subseteq T$. And so $-a \in T$. Thus $a$ is a unit of $T$. Since $a \notin T'$, $-a$ is not a unit of $T'$. Since $(M \cap S) + T'$ is the maximal ideal of $T'$ by Corollary 3, we have that $-a \in (M \cap S) + T' \subseteq M + T = M$. Since $a$ is a unit of $T$, this is a contradiction. Therefore $T = T'$.

(5) Take $x \in G$. Assume that $a$ is integral over $S$. There exists a positive integer $n$ such that $na \in S$. It is enough to assume that $a \notin S$. Since $S$ is a valuation semigroup, we have that $-a \in S$. Therefore $a = na + (n - 1)(-a) \in S$. Consequently, $S$ is integrally closed. \hfill \Box

Proposition 5. Assume that $S$ is a valuation semigroup. Let $T$ be an oversemigroup of $S$. Then if $J$ is a proper ideal of $T$, then $J$ is a proper ideal of $S$. In particular, $\text{Spec}(T) \subseteq \text{Spec}(S)$.

Proof. Assume that $J \not\subseteq S$. Then there exists some element $z \in J - S$. Since $S$ is a valuation semigroup, we have that $-z \in S \subseteq T$. Hence $z$ is a unit of $T$. This is a contradiction to $J \not\subseteq T$. Therefore $J \subseteq S$. Since $J \not\supseteq 0$, we have that $J \subseteq S$. \hfill \Box

Corollary 6. If $S$ is a valuation semigroup and $T$ is an oversemigroup of $S$, then $T = S_M$ for the maximal ideal $M$ of $T$.

Theorem 7. Assume that $T$ is an oversemigroup of the valuation semigroup $S$, and that $\Omega$ is the set of prime ideals $P$ of $S$ such that $P + T \subseteq T$. Then

1. If $Q$ is a prime ideal of $T$, then $S_Q = T_Q$.
2. If $P$ is a prime ideal of $S$, then $P$ is in $\Omega$ if and only if $S_P \supseteq T$.
3. $\text{Spec}(T) = \{P + T \mid P \in \Omega\} = \Omega$. 

Proof. (1) Since $T_Q \supset S$ and $Q + T_Q$ is the maximal ideal of $T_Q$, we have that $(Q + T_Q) \cap S = Q + S = Q$. By Corollary 6, we have that $T_Q = S_{(Q + T_Q) \cap S} = S_Q$.

(2) ($\Longleftarrow$) If $S_P \supset T$, then $P + T \subset P + S_P \subset S_P$. Since $P + S_P$ does not contain units of $S_P$, $P + T$ does not contain units of $T$. Thus we have that $P + T \subset T$. Hence $P \in \Omega$.

($\Longrightarrow$) If $P \in \Omega$, then $P + T \subset T$. Then there is the maximal ideal $M$ of $T$ containing $P + T$. We have $P \subset M \cap S = M$ so that $S_P \supset S_M$. But by (1), $S_M = T_M = T$. Hence $S_P \supset T$.

(3) Take $Q \in \text{Spec}(T)$. Then we have that $T \supseteq Q = Q + T$. Since $S$ is a valuation semigroup, $Q \in \text{Spec}(S)$ by Proposition 5. Since $S_Q \supset T$, we have that $Q \in \Omega$ by (2). Conversely, if $P \in \Omega$, then $P + S_P$ is the maximal ideal of $S_P$. By $S_P \supset T$, we have that $(P + S_P) \cap T = P + T \in \text{Spec}(T)$. Therefore, the half part of (3) holds.

Next, let $Q \in \text{Spec}(T)$. By Proposition 5, we have that $Q \in \text{Spec}(S)$. Also, since $Q + T = Q \subset T$, it holds that $Q \in \Omega$. Hence $\text{Spec}(T) \subset \Omega$. Conversely, take $P \in \Omega$. Since $P + T \subset S_P$, we have that $(P + S_P) \cap T = P + T \in \text{Spec}(T)$. By Proposition 5, we have that $P + T \in \text{Spec}(S)$. Hence $P + T = (P + T) \cap S = P + S = P$. Therefore $\text{Spec}(T) = \Omega$.

\[
\square
\]

Theorem 8 (cf. [2]). Let $S$ be an integrally closed semigroup. The following conditions are equivalent.

(1) $S$ is a valuation semigroup.

(2) Each oversemigroup of $S$ is a localization $S_P$ for some $P \in \text{Spec}(S)$.

(3) Each oversemigroup of $S$ is integrally closed.

Proof. (1) $\Longrightarrow$ (2). It is proved by Lemma 4 (4).

(2) $\Longrightarrow$ (3). By (2), We have that $T = S_P$ for some $P \in \text{Spec}(S)$. Take any $\alpha = a - t \in G$ ($a, t \in S$) such that $\alpha$ is integral over $T$. So there exists some positive integer $n$ such that $n\alpha \in T$. Also $n\alpha = s - t'$ ($s, t' \in S, t' \notin P$). Hence $n\alpha = s - (n - 1)t' - t' + (n - 1)t'$. Therefore $n(\alpha + t') = s + (n - 1)t' \in S$. Since $S$ is integrally closed, we have that $\alpha + t' \in S$. So $\alpha \in S_P = T$, and so $T$ is integrally closed.

(3) $\Longrightarrow$ (1). Let $M$ be the maximal ideal of $S$. Assume that $\xi \in q(S)$ and $\xi \notin S$, where $q(S)$ denotes the quotient group of $S$. Since $S[2\xi] = \{s + 2n\xi \mid s \in S$ and $n$ is a non-negative integer $\}$ is integrally closed, $\xi$ is in $S[2\xi]$, say $\xi = s + 2n\xi$ ($s \in S$ and $n$ is a positive integer). It follows that $(2n - 1)(-\xi) = s \in S$. Since $2n - 1 > 0$, we have that $-\xi$ is integral over $S$. Since $S$ is integrally closed, we have that $-\xi \in S$. Therefore, $S$ is a valuation semigroup. \[
\square
\]
Conjecture. Let $S$ be an integrally closed semigroup. The following conditions are equivalent.

(1) $S$ is a valuation semigroup.

(4) The prime ideals of each oversemigroup of $S$ are extensions of prime ideals of $S$.

(5) For each prime ideal $P$ of $S$ and each oversemigroup $T$ of $S$, there is at most one prime of $T$ lying over $P$.

Remark 1. In the above Conjecture, we know that (1) $\implies$ (4) $\implies$ (5).

Also, (5) and the following (5') are equivalent.

(5') For each oversemigroup $T$ of $S$, $\varphi_T$ is injective where $\varphi_T : \text{Spec}(T) \to \text{Spec}(S)$ is the restriction mapping, that is, $\varphi_T(Q') = Q' \cap S$ for each $Q' \in \text{Spec}(T)$.

§4. The ideal transform and conductor in semigroups

Theorem 7 provides a tool for attacking the area of almost integrity for a valuation semigroup. We introduce the notion of the transform of an ideal, defined as follows.

Definition 2. If $I$ is an ideal of the semigroup $S$ with quotient group $G$, the transform of $I$ is defined to be $T = \bigcup_{n=1}^{\infty} B_n$, where $B_n = (S : nI)_G = \{x \in G \mid x + nI \subseteq S\}$.

It follows that $B_1 = (S : I)_G = I^{-1}$ if $I$ is principal.

The transform $T$ of $I$ is an oversemigroup of $S$. If the ideal $I$ is principal, then $B_n = n(B_1) = nI^{-1}$ for each $n \geq 1$. If $I$ is principal and $T$ is the transform of $I$, then $I + T = T$.

Lemma 9. Let $I$ be a principal ideal of the semigroup $S$ and let $T$ be the transform of $I$. If $P$ is a prime ideal of $S$, then $I \subseteq P$ if and only if $P + T = T$.

Proof. ($\Rightarrow$) If $I \subseteq P$, then $T = I + T \subseteq P + T$ by the proceeding results, so that $P + T = T$.

($\Leftarrow$) If $P + T = T$, we let $0 = p + t$, where $p$ is in $P$ and $t$ is in $T$.

By definition of $T$, there is a positive integer $m$ such that $t + mI \subseteq S$. Thus, for $x \in mI$, $x = x + 0 = p + t + x \in P + S = P$. It follows that $mI \subseteq P$ and $I \subseteq P$ since $P$ is prime. \(\square\)

Proposition 10. Let $S$ be a valuation semigroup, $I = (a) = a + S$ a proper principal ideal of $S$ and let $T$ be the transform of $I$. If $P$ is a prime ideal of
$S$, then $T \subseteq S_P$ if and only if $I \not\subseteq P$. In particular, if the maximal ideal $M$ of $T$ then $T = S_M$. Also, we have that $M \in \text{Spec}(S)$ and $M \not\subseteq I$.

**Proof.** Since $I$ is principal, $P + T \subseteq T$ if and only if $P \not\subseteq I$ by Lemma 9.

And by Theorem 7, $P \in \Omega = \{P \in \text{Spec}(S) \mid P + T \subseteq T\}$ if and only if $S_P \supseteq T$. Also we have that $T = S_{M \cap S} = S_M$ by Corollary 6. So $M \in \text{Spec}(S)$ and $M \not\subseteq I$. □

**Definition 3.** Let $T$ be a semigroup. If the semigroup $S$ is a subsemigroup of $T$, the conductor of $S$ in $T$ is defined to be $(S : T)_S = \{x \in S \mid x + T \subseteq S\}$.

The conductor of $S$ in $T$ is characterized as the largest ideal of $S$ which is also an ideal of $T$.

If $(S : T)_S \neq \emptyset$, then $S$ and $T$ have the same total quotient group.

**Definition 4.** By a numerical semigroup $S$, we mean an additive subsemigroup of the semigroup $\mathbb{Z}_0$ of all non-negative integers such that $\mathbb{Z}_0 - S$ is finite.

Non-negative integer $g$ is called the conductor number of $S$ if $g - 1$ is the greatest integer which does not belong to the semigroup $S$.

**Example.** (1) Let $T = \mathbb{Z}[X] = \{a + nX \mid a \in \mathbb{Z}, n \in \mathbb{Z}_0\}$ where $\mathbb{Z}$ denotes the set of all integers and $\mathbb{Z}_0$ denotes the set of all non-negative integers.

Also let $S = \mathbb{Z}_0[X]$ be a subsemigroup of $T$. Then $(S : T)_S = \emptyset$.

(2) Let $T = \{0, 2, 3, 4, 5, \ldots\}$ be the numerical semigroup generated by 2 and 3, and let $S = \{0, 3, 5, 6, 8, 9, 10, \ldots\}$ be the numerical semigroup generated by 3 and 5. Then $(S : T)_S = \{6, 8, 9, 10, \ldots\}$.

Let $T$ be a numerical semigroup and $S$ be a numerical subsemigroup of $T$. It is true that $(S : T)_S \supseteq \{g, g + 1, g + 2, \ldots\}$ where $g$ is the conductor number of $S$.

But it is not necessarily true that $(S : T) = \{g - 2, g, g + 1, g + 2, \ldots\}$. For example, let $T = \mathbb{Z}_0 \supseteq S = \{0, 10, 20, 21, 22, \ldots\}$. Then 20 is the conductor number of $S$ and $(S : T)_S = \{20, 21, 22, \ldots\}$.

The next two lemmas relate the concepts of the conductor, the transform and almost integrity.

**Lemma 11.** If $T$ is an oversemigroup of the semigroup $S$ such that the conductor $(S : T)_S$ of $S$ in $T$ is a non-empty set, then $S$ and $T$ have the same complete integral closure.

**Proof.** Suppose that $x \in q(S) = q(T)$ is almost integral over $T$. We choose an element $y$ of $T$ such that $y + nx \in T$ for each positive integer $n$. If $d$ is an
element of the conductor \((S : T)_S\), then \(d + y \in S\) and \((d + y) + nx \in S\) for each positive integer \(n\). Hence \(x\) is almost integral over \(S\).

**Lemma 12.** Let \(I\) be an ideal of the semigroup \(S\) and let \(T\) be the transform of \(I\). Then \(\cap_{n=1}^\infty nI\) is contained in \((S : T)_S\).

**Proof.** It is enough to assume that \(\cap_{n=1}^\infty nI \neq \phi\). Let \(y \in \cap_{n=1}^\infty nI\). Then \(y + T \subset S\) by the definition of \(T\). Thus \(y \in (S : T)_S\).

If \(S\) is a valuation semigroup with quotient group \(G\) and if \(\xi \in G\), we denote by \(B_\xi\) the ideal \((-\xi) \cap S = \{d \in S \mid d + \xi \in S\}\). Note that \(B_\xi\) is a principal ideal of \(S\).

Precisely speaking, if \(-\xi \in S\), then we have that \(B_\xi = (-\xi) = -\xi + S\) and if \(-\xi \notin S\) then we have that \(B_\xi = S\).

**Theorem 13.** Let \(S\) be a valuation semigroup with quotient group \(G\). For \(\xi \in G\), put \(B_\xi = (-\xi) \cap S\). Then the following statements hold.

1. The transform \(T\) of \(B_\xi\) is \(S[\xi]\).
2. \(\xi\) is almost integral over \(S\) if and only if \(\cap_{n=1}^\infty nB_\xi \neq \phi\).

**Proof.** (1) By Proposition 10, \(T = S_M\), where \(M\) is the maximal ideal of \(T\) and so \(M \in \text{Spec}(S)\) and \(B_\xi \notin M\) by Proposition 5 and Lemma 9. We will prove that \(M \not\supset B_\xi\) if and only if \(\xi \in S_M = T\). Assume that \(M \not\supset B_\xi\). Then there exists some element \(a \in B_\xi - M\). Then \(a + \xi \in S\), and so \(a + \xi = s \in S\). Thus \(\xi = s - a \in S_M = T\). Conversely, assume that \(\xi = s - t \in T\) \((s, t \in S, t \notin M)\). Then \(t + \xi = s \in S\), and so \(t \in B_\xi\). Therefore \(t \in B_\xi - M\). Consequently, we have that \(S[\xi] \subset T\).

Proving the reverse inclusion relation, we note that if \(-\xi \in S\) then \(B_\xi = (-\xi) = \xi + S\). For \(x \in T\), we have that \(x + nB_\xi \subset S\), that is, \(x + nS \subset n\xi + S\). Thus \(x = n\xi + s \in S[\xi]\) for some \(s \in S\). So \(T \subset S[\xi]\).

Next, assume that \(-\xi \notin S\). Since \(\xi \in S\), we have that \(B_\xi = S\). For any \(x \in T\), \(x + nB_\xi \subset S\) because \(T\) is the transform of \(B_\xi\). Since \(B_\xi = S\), we have that \(x \in S\), and so \(T \subset S[\xi]\). Therefore \(T = S[\xi]\).

(2) Assume that \(\cap_{n=1}^\infty nB_\xi \neq \phi\). By Lemma 12, we have \((S : T)_S \neq \phi\). Since \(\xi \in T\) by (1), we have that \(0 + n\xi \in T\) for each positive integer \(n\). Thus \(\xi\) is almost integral over \(T\). By Lemma 11, \(\xi\) is an almost integral over \(S\). To prove the opposite implication, assume that \(\xi\) is almost integral over \(S\). Then there exists an element \(a\) of \(S\) such that \(a + n\xi \in S\) for any positive integer \(n\). Since \((S : nB_\xi)_G = nB_\xi^{-1} = (nB_\xi)^{-1}\), for each positive integer \(n\), we obtain that \(a + nB_\xi^{-1} \subset a + T \subset S\). Thus \(a \in nB_\xi + S \subset nB_\xi\). Therefore \(a\) is an element of \(\cap_{n=1}^\infty nB_\xi\).
**Theorem 14.** (1) If $S$ is a completely integrally closed semigroup and if $I$ is a proper principal ideal of $S$, then $\bigcap_{n=1}^{\infty} nI = \phi$.

(2) If $S$ is a valuation semigroup, and if $\bigcap_{n=1}^{\infty} nJ = \phi$ for each proper principal ideal $J$ of $S$, then $S$ is completely integrally closed.

**Proof.** (1) Let $T$ be the transform of $I$. Since $S \subset B_1 \subset B_2 \subset \cdots \subset T$, we shall prove that $T \neq S$. Assume that $T = S$. Since $I$ is a principal ideal, we have that $S = B_1 = I^{-1} = (-a)$. Hence $a$ is a unit. Therefore $I = S$, which is a contradiction to properties of $I$. Since $S$ is completely integrally closed, $T$ is not almost integral over $S$. By Lemma 11, $(S : T)_S = \phi$. Therefore $\bigcap_{n=1}^{\infty} nI = \phi$.

(2) Assume that $\xi \in G - S$ is almost integral. Then $-\xi \in S$ is a non-unit element of $S$. Then $J = B_\xi = (-\xi)$ is a proper ideal of $S$. By Theorem 13 (2), we see that $\bigcap_{n=1}^{\infty} nJ \neq \phi$. This is a contradiction. Hence $S$ is completely integrally closed. □

**Corollary 15.** The valuation semigroup $S$ is completely integrally closed if and only if $\bigcap_{n=1}^{\infty} nI = \phi$ for each proper principal ideal $I$ of $S$.

**Proof.** ($\Longleftarrow$) It is trivial from Theorem 14 (2).

($\Longrightarrow$) It is trivial from Theorem 14 (1). □

**Theorem 16.** Let $S$ be a valuation semigroup with the quotient group $q(S) = G$. Assume that $S \neq G$. Then $S$ is completely integrally closed if and only if $\dim S = 1$.

**Proof.** ($\Longrightarrow$) Let $P, M \in \text{Spec}(S)$. Assume that $P \subsetneq M$. Then there exists some element $x \in M - P$. Since $S$ is completely integrally closed, we have that $\bigcap_{n=1}^{\infty} n(x) = \phi$ by Theorem 14 (1). On the other hand, since $S$ is a valuation semigroup and $x \not\in P$, we obtain that $P \subset n(x)$ for all $n > 0$. Thus $P \subset \bigcap_{n=1}^{\infty} n(x) = \phi$. Therefore $P = \phi$. Consequently, we have that $\dim S = 1$.

($\Longleftarrow$) Take $x \in G - S$. Put $y = -x$. If $\bigcap_{n=1}^{\infty} n(y) \neq \phi$, then this ideal is a prime ideal of $S$. For, put $Q = \bigcap_{n=1}^{\infty} n(y)$. Then $a, b \in S - Q$. There exist positive integers $k$ and $r$ such that $a \not\in k(y), b \not\in r(y)$. Thus $k(y) \subset (a) = a + S$ and $r(y) \subset (b) = b + S$.

Therefore $k(y) + (b + S) \subset (a + b) + S$.

By $r(y) \subset (b) = b + S$, we have that $(k + r)(y) \subset (a + b) + S$. Consequently, $a + b \not\in Q$. Thus $Q$ is a prime ideal of $S$. Since $\dim S = 1$, we deduce that $Q$ is the maximal ideal of $S$. But $Q = (y) = (2y) = (3y) = \cdots$. This implies that $y$ is a unit of $S$. This is a contradiction. Therefore $Q = \bigcap_{n=1}^{\infty} n(y) = \phi$. 
If each \( d \in S \), there exists a positive integer \( n \) such that \((d) \not\subset (ny)\). Thus \( d-ny = d+nx \not\in S \). Since \( d \) is any element of \( S \), \( x \) is not almost integral over \( S \). Therefore \( S \) is a completely integrally closed. \( \square \)

**Remark 2.** Put \( \mathbb{Z}_n = \{ a \in \mathbb{Z} \mid a \geq n \} \) and put \( \mathbb{Z}_1 X = \{ mX \mid m \in \mathbb{Z}_1 \} \). Let \( S = \mathbb{Z}_0 \cup (\mathbb{Z} + \mathbb{Z}_1 X) \). Also put \( P = \mathbb{Z} + \mathbb{Z}_1 X \). Kentaro Hatano proved that \( S \) is a valuation semigroup and \( \text{Spec}(S) = \{ \phi, \ P, \ (1) \} \), that is, \( \dim S = 2 \). It is clear that the ideal \((1)\) of \( S \) is the maximal ideal of \( S \). By Theorem 16, \( S \) is not completely integrally closed. The oversemigroups of \( S \) are three oversemigroups the following: \( S = S_{(1)} \), \( S_P = S_{S-P} = S_{Z_0} = \mathbb{Z} \cup (\mathbb{Z} + \mathbb{Z}_1 X) = \mathbb{Z}[X] \) and \( G \).

Also, for example, put \( \xi = 2 - X \), then \( B_\xi = (X-2) \) and the transform of \( B_\xi \) is the quotient group \( G = S[2-X] \) by Theorem 13.

**Remark 3.** Let \( S = (\mathbb{Z}_1 + \mathbb{Z}_1 X) \cup \{ 0 \} \). It is clear that the maximal ideal of \( S \) is \( M = \mathbb{Z}_1 + \mathbb{Z}_1 X \). If \( P \neq \emptyset \in \text{Spec}(S) \), then \( P \subset M \), precisely speaking \( M = P \).

In fact, let \( P( \neq \emptyset ) \) be a prime ideal of \( S \). There exists an element \( a + bX \in P \) (\( a \geq 1 \), \( b \geq 1 \), say \( a \leq b \)). Since \( P + S = P \), we have that \( P \ni c + dX \) for \( c, \ d \geq b + 1 \). For any \( \alpha + \beta X \in M \), we obtain that \((b+1)(\alpha + \beta X) \in P \). Hence \( \alpha + \beta X \in P \). Thus \( P = M \).

Therefore \( \dim S = 1 \). Some valuation oversemigroups are \( S_1 = \mathbb{Z} + \mathbb{Z}_0 X \), \( S_2 = \mathbb{Z}_0 + \mathbb{Z} X \), \( S_3 = \mathbb{Z}_0 \cup (\mathbb{Z} + \mathbb{Z}_1 X) \) and \( G = q(S) = \mathbb{Z} + \mathbb{Z} X \).

Also, \( S \) is integrally closed and \( S \subset S_3 \). We have that \( \dim S_3 = 2 \) by Remark 2. Let \( S_0 \) be the complete integral closure of \( S \). Then \( S_0 = \mathbb{Z}_0[X] \) by definition, \( \text{Spec}(S_0) = \{ \phi, \ (1), \ (X), \ (1) \cup (X) \} \) and \( \dim S_0 = 2 \). Moreover, \( \varphi_{S_0} : \text{Spec}(S_0) \longrightarrow \text{Spec}(S) \) is not injective because \( \varphi_{S_0}(1) = \varphi_{S_0}(X) = \varphi_{S_0}((1) \cup (X)) = M \).

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