SUT Journal of Mathematics
Vol. 35, No. 2 (1999), 239–245

TOPOLOGICAL PROPERTIES OF COMPOSITION OPERATORS ON SOME FRÉCHET ALGEBRA

Jie Xiao

(Received March 26, 1999)

Abstract. For $\beta > 0$ let F_β denote the Fréchet algebra of all holomorphic functions f on the unit disk Δ for which $\lim_{r\to 1}(1-r)^\beta \log^+[\max_{|z|\le r} |f(z)|] = 0$. Given a holomorphic self-map ϕ of Δ define the composition operator C_ϕ on F_β by: $C_\phi f = f \circ \phi$, $f \in F_\beta$. This note shows that C_ϕ exists always as a continuous operator. Furthermore, this note points out that boundedness and compactness of C_ϕ are not only the same, but also equivalent to $\phi^n \exp[rn^{\beta/(1+\beta)}] \to 0$ in F_β for some $r > 0$.

AMS 1991 *Mathematics Subject Classification.* 30D55, 46E15, 47B38.

Key words and phrases. Fréchet algebra, composition, continuity, boundedness, compactness.

1. Introduction and Theorem

Throughout this note, denote by Δ the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$, and by dm one means the Lebesgue area measure on Δ. Let H be the class of all holomorphic functions on Δ and H^∞ its subclass consisting of all $f \in H$ with $\|f\|_\infty = \sup_{z\in\Delta} |f(z)| < \infty$. It is well-known that H is a Fréchet space with respect to uniform convergence on compact subsets of Δ.

1.1. The Fréchet Algebra. For $\beta > 0$ let F_β be the class of all $f \in H$ for which

$$\lim_{r\to 1}(1-r)^\beta \log^+ M(r,f) = 0,$$

where $M(r,f) = \max_{|z|=r} |f(z)|$ is the maximal modulus of f on the circle $\{|z| = r\}$. In [12] M. Stoll introduced this class and proved that if $f \in H$ with $f(z) = \sum_{n=0}^\infty a_n z^n$, then $f \in F_\beta$ if and only if for all $c > 0$,

$$\|f\|_{c,F_\beta} = \int_0^1 M(r,f) \exp[-c(1-r)^{-\beta}]dr < \infty$$

This project was supported by the Alexander von Humboldt Foundation, Germany.

equivalently

$$|||f|||_{c,F_\beta} = \sum_{n=0}^{\infty} |a_n| \exp[-cn^{\beta/(1+\beta)}] < \infty.$$

Moreover, he showed that F_β is a countably normed Fréchet algebra with respect to the topology given by the seminorms $|||\cdot|||_{c,F_\beta}$.

Indeed, F_β is a natural generalization of F^+, where F^+ (cf. [13]) is the Fréchet envelope of the classical Smirnov class N^+ of all $f \in H$ satisfying $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ a.e. on $[0, 2\pi)$ and

$$\lim_{r\to 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta = \int_0^{2\pi} \log^+ |f(e^{i\theta})| d\theta < \infty.$$

Recently, the article [4] made a great progress and found that for $p \geq 1$ and $\alpha \geq -1$, $F_{(\alpha+2)/p}$ is the Fréchet envelope of the (p, α)-Nevanlinna class N^p_α, where

$$N^p_\alpha = \left\{ f \in H : \int_\Delta [\log^+ |f(z)|]^p (1-|z|^2)^\alpha dm(z) < \infty \right\}, \quad p \geq 1, \quad \alpha > -1;$$

and as to $p > 1$, N^p_{-1} is defined by the Hardy-Orlicz algebra

$$(Log^+ H)^p = \left\{ f \in H : \sup_{r\in(0,1)} \int_0^{2\pi} [\log^+ |f(re^{i\theta})|]^p d\theta < \infty \right\}.$$

In this sense, N^+ is viewed as $\lim_{\alpha\to -1} N^1_\alpha$ or $\lim_{p\to 1} (Log^+ H)^p$. Note that N^+, N^p_α $(p \geq 1, \alpha > -1)$ and $(Log^+ H)^p$ $(p > 1)$ are complete metrizable topological vector spaces. It is clear that $(Log^+ H)^p \subset N^+ \subset N^1_\alpha$ for $p > 1$ and $\alpha > -1$. The inclusions are proper. For a detailed discussion, refer to [4], [8], [12], [13] and [14].

Every holomorphic self-map ϕ of Δ induces a linear composition operator C_ϕ on H via

$$C_\phi f = f \circ \phi, \quad f \in H.$$

This note will pay attention to the (basic) topological properties of C_ϕ on F_β. The reason why we are interested in this topic is because the past three decades have witnessed a flowering of research on composition operators, (cf. [2], [3] and [11]), and in particular, a research about C_ϕ acting on N^+, N^p_α and F^+ has been carried out (cf. [1], [5], [6], [7] and [9]). As a further contribution, this note characterizes continuity, boundedness and compactness of C_ϕ sending F_β to itself in terms of the *function-theoretic properties* of ϕ. Here is our main result.

1.2. Theorem. *Let $\phi : \Delta \to \Delta$ be holomorphic and let $\beta \in (0, \infty)$. Then C_ϕ exists always as a continuous operator on F_β. Moreover, the following are equivalent:*

(i) *C_ϕ exists as a bounded operator on F_β.*

(ii) *C_ϕ exists as a compact operator on F_β.*

(iii) *There is an $r > 0$ such that $\phi^n \exp[rn^{\beta/(1+\beta)}] \to 0$ in F_β.*

The proof of Theorem 1.2 is arranged in the second section where some relevant definitions are set. Moreover, we apply the approach of dealing with C_ϕ to get, as a corollary of the theorem, that continuity, boundedness and compactness of C_ϕ sending F_β to a weighted Bergman space on Δ are equivalent.

1.3. Remark. In the case $\beta = 1$, both the continuity and the equivalence $(ii) \Leftrightarrow (iii)$ were obtained by Roberts and Stoll in [9]. Although somewhat similar to their argument for F^+, our method provides a uniform treatment for C_ϕ on F_β. On the other hand, our result shows that as to C_ϕ living on F_β, continuity is the same as neither boundedness nor compactness.

1.4. Acknowledgment. The author thanks H. Jarchow and K.J. Wirths for interesting discussions during the preparation of this note. In addition, the author is grateful to the referee for helpful comments on previous versions of the note.

2. Proof and Corollary

Given two topological vector spaces X and Y. A linear operator $T : X \to Y$ is called continuous (on X when $X = Y$) if for any neighborhood V of zero (in Y) there is a neighborhood U of zero (in X) such that $TU \subset V$. Further, a linear operator $T : X \to Y$ is said to be bounded resp. compact (on X when $X = Y$) if it takes some neighborhood of zero of X into a bounded resp. relatively compact set of Y. It is clear that any compact operator must be bounded. Moreover, every bounded operator is continuous, and conversely for any Banach space or even any linear topological space possessing a bounded neighborhood of zero. Notice that F_β does not enjoy the last feature, that is to say, F_β is not locally bounded (see also [8]).

With the help of these definitions, we can give the following statement.

2.1. Proof of Theorem 1.2. First of all, we verify that C_ϕ maps F_β into itself, namely, $C_\phi F_\beta \subset F_\beta$ set-theoretically. For $r \in (0, 1)$ and $w \in \Delta$ let $\Delta(w, r) = \{z \in \Delta : |w - z| < r|1 - \bar{w}z|\}$ be the pseudo-hyperbolic disk with the (pseudo-hyperbolic) center w and the (pseudo-hyperbolic) radius r. In fact, $\Delta(w, r)$ is a Euclidean disk on Δ with the (Euclidean) center $\zeta = w(1-r^2)(1-r^2|w|^2)^{-1}$ and the (Euclidean) radius $\rho = r(1-|w|^2)(1-r^2|w|^2)^{-1}$.

By Schwarz's lemma, we see $\phi(\Delta(0,r)) \subset \Delta(w,R(r))$, hereafter $w=\phi(0)$ and $R(r)=(|w|+r)(1+|w|r)^{-1}$. It is obvious that $\lim_{r\to 1} R(r)=1$. If $f\in F_\beta$ then $M(r,C_\phi f)\le M(R(r),f)$ and hence

$$\begin{aligned}\lim_{r\to 1}(1-r)^\beta \log^+ M(r,C_\phi f) &\le \lim_{r\to 1}(1-r)^\beta \log^+ M(R(r),f)\\ &\le \lim_{r\to 1}\frac{[1-R(r)]^\beta \log^+ M(R(r),f)}{[(1-|w|)(1+|w|r)^{-1}]^\beta}\\ &=0,\end{aligned}$$

which implies $C_\phi f\in F_\beta$.

Concerning the continuity of C_ϕ, we can proceed as stated below. Let $c>0$ be arbitrary. Then by the inequality $(1-|w|)(1+|w|)^{-1}\le R'(r)$,

$$\begin{aligned}&\|C_\phi f\|_{c,F_\beta}\\ &=\lim_{t\to 1}\int_0^t \exp[-c(1-r)^{-\beta}]M(r,C_\phi f)dr\\ &\le \lim_{t\to 1}\int_0^t \exp[-c(1-r)^{-\beta}]M(R(r),f)dr\\ &\le (1+|w|)(1-|w|)^{-1}\lim_{t\to 1}\int_0^t \exp[-c_1(1-R(r))^{-\beta}]M(R(r),f)R'(r)dr\\ &\le (1+|w|)(1-|w|)^{-1}\lim_{t\to 1}\int_0^{R(t)} \exp[-c_1(1-s)^{-\beta}]M(s,f)ds\\ &=(1+|w|)(1-|w|)^{-1}\|f\|_{c_1,F_\beta},\end{aligned}$$

where $c_1=c[(1-|w|)(1+|w|)^{-1}]^\beta$. Hence $C_\phi: F_\beta\to F_\beta$ is continuous.

Next, let us prove the equivalence announced in Theorem 1.2. It is sufficient to show (i)⇒(iii)⇒(ii). Assume first that (i) holds. Then there is a $c>0$ such that if $E=\{f\in F_\beta : \|f\|_{c,F_\beta}<1\}$ then $C_\phi E$ is bounded set of F_β. For $n=0,1,2,\ldots$, choose $f_n(z)=z^n\exp[(c/2)n^{\beta/(1+\beta)}]$ which belong to E. Accordingly, $C_\phi f_n=\phi^n\exp[(c/2)n^{\beta/(1+\beta)}]$ lie in $C_\phi E$. From the boundedness of $C_\phi E$ it follows that $a_nC_\phi f_n\to 0$ in F_β as $a_n\to 0$. Upon selecting $r\in(0,c/2)$ and $a_n=\exp[-(c/2-r)n^{\beta/(1+\beta)}]$, we reach (iii).

Suppose secondly that (iii) is true. Since the topology of F_β is determined by the norms $\|\cdot\|_{c,F_\beta}$, in order to verify (ii), it is enough to demonstrate that there is a neighborhood U of 0 in F_β such that $C_\phi U$ is totally bounded with respect to each $\|\cdot\|_{c,F_\beta}$. Pick $r_0\in(0,r)$ and $U=\{f\in F_\beta : \|f\|_{r_0,F_\beta}<1\}$. If $f(z)=\sum_{n=0}^\infty a_nz^n$ is in U, then $|a_n|<\exp[r_0n^{\beta/(1+\beta)}]$. Let $\epsilon>0$ and $c>0$ be given. For $\phi^n\exp[rn^{\beta/(1+\beta)}]$ tends to 0 in F_β (as $n\to\infty$) and $\sum_{n=0}^\infty \exp[(r_0-$

$r)n^{\beta/(1+\beta)}]$ is convergent, there exists an integer $I > 0$ such that as $n > I$, one has $\|\phi^n\|_{c,F_\beta} \exp[rn^{\beta/(1+\beta)}] < 1$ and $\sum_{n=I+1}^{\infty} \exp[(r_0 - r)n^{\beta/(1+\beta)}] < \epsilon$. Thus

$$\|C_\phi f - \sum_{n=0}^{I} a_n \phi^n \|_{c,F_\beta} \leq \sum_{n=I+1}^{\infty} \exp[(r_0 - r)n^{\beta/(1+\beta)}] < \epsilon.$$

In other words, $C_\phi U$ is totally bounded relative to $\| \cdot \|_{c,F_\beta}$, and hence (ii) yields.

The previous idea of studying C_ϕ can be employed to work out the problem of characterizing ϕ such that C_ϕ maps one Fréchet algebra to another such algebra.

2.2. Remark. Let $\phi : \Delta \to \Delta$ be holomorphic and let $\beta, \gamma \in (0, \infty)$. Then $C_\phi : F_\beta \to F_\gamma$ exists as a continuous operator iff $\phi^n \exp[b_n n^{\beta/(1+\beta)}] \to 0$ in F_γ for any $b_n \geq 0$ with $b_n \to 0$. Moreover, boundedness as well as compactness of $C_\phi : F_\beta \to F_\gamma$ holds iff $\phi^n \exp[rn^{\beta/(1+\beta)}] \to 0$ in F_γ for some $r > 0$.

The argument for Theorem 1.2 tells us that only the 'continuity'-part needs checking. Let $C_\phi : F_\beta \to F_\gamma$ be continuous. For any $b_n > 0$ with $b_n \to 0$, consider $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $a_n = \exp[b_n(n^{\beta/(1+\beta)})]$. From Theorem 2.2 of [12] it follows that $f \in F_\beta$. Then for every $c > 0$, $\|f\|_{c,F_\beta} < \infty$ and hence $\|a_n z^n\|_{c,F_\beta} \to 0$. Since C_ϕ is continuous, $\phi^n \exp[b_n n^{\beta/(1+\beta)}] = C_\phi(a_n z^n)$ converges to 0 in F_γ. Conversely, let $\phi^n \exp[b_n n^{\beta/(1+\beta)}] \to 0$ in F_γ for any sequence $b_n : b_n \geq 0,\ b_n \to 0$. If $f \in F_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then by Theorem 2.2 of [12] once again, there is a sequence d_n: $d_n > 0$, $d_n \to 0$ to insure $|a_n| \leq \exp[d_n n^{\beta/(1+\beta)})]$. By letting $b_n = d_n + n^{-\beta/(2+2\beta)}$, one gets an integer $I > 0$ such that as $n > I$, $\|\phi^n\|_{c,F_\gamma} \leq \exp[-b_n n^{\beta/(1+\beta)})]$ for any $c > 0$. Consequently,

$$\|C_\phi f\|_{c,F_\gamma} \leq \sum_{n=0}^{I} \|\phi^n\|_{c,F_\gamma} \exp[d_n n^{\beta/(1+\beta)}] + \sum_{n=I+1}^{\infty} \exp[-n^{\beta/(2+2\beta)}],$$

which implies $C_\phi f \in F_\gamma$. Using Theorem 3.2 in [12] and the uniform boundedness principle [10, p.45], we conclude that $C_\phi : F_\beta \to F_\gamma$ is continuous.

Recall that A^p_α, for $\alpha > -1$ and $p > 0$, is the weighted Bergman space of all $f \in H$ with

$$\|f\|_{p,\alpha} = \left[\int_\Delta |f(z)|^p (1 - |z|^2)^\alpha dm(z)\right]^{1/p} < \infty.$$

The limit case A^p_{-1}, $p > 0$, is given by the classical Hardy space H^p of all $f \in H$ obeying

$$\|f\|_{p,-1} = \left[\sup_{r\in(0,1)} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right]^{1/p} < \infty.$$

2.3. Corollary. *Let $\phi : \Delta \to \Delta$ be holomorphic and let $\beta, p \in (0,\infty)$ and $\alpha \in [-1,\infty)$. Then continuity, boundedness and compactness of $C_\phi : F_\beta \to A^p_\alpha$ are equivalent to $M(\phi, r) = \limsup_{n\to\infty} \|\phi^n\|_{p,\alpha} \exp[rn^{\beta/(1+\beta)}] < \infty$ for some $r > 0$.*

The 'if'-part is a by-product of the proof of Theorem 1.2. And yet, the 'only if'-part will be done once we substantiate $M(\phi, r) < \infty$ for some $r > 0$ under the hypothesis that $C_\phi : F_\beta \to A^p_\alpha$ is continuous. Now assume that there is no such an $r > 0$ to guarantee $M(\phi, r) < \infty$. Then for every $k = 1, 2, 3, ...,$ take n_k such that $n_k > n_{k-1}$ and $\|\phi^{n_k}\|_{p,\alpha} \exp[k^{-1} {n_k}^{\beta/(1+\beta)}] \geq 1$. However, consulting the proof of Remark 2.2, picking $b_m = k^{-1}$ or 0 as $m = n_k$ or others, and applying the continuity of C_ϕ, one get $\|\phi^m\|_{p,\alpha} \exp[b_m m^{\beta/(1+\beta)}] \to 0$, which contradicts the condition on n_k.

2.4. Remark. In the case $\beta = -\alpha = 1$, Corollary 2.3 (except boundedness) is due to Roberts and Stoll [9]. Observe that continuity, boundedness and compactness of $C_\phi : F_\beta \to A^p_\alpha$ coincide. Why? The cause, we think, is that A^p_α is a quasi-Banach space (more precisely, Banach space when $p \geq 1$). This actually reflects a general phenomenon that any continuous operator mapping F_β into a quasi-Banach space Y must be compact and hence bounded. Besides, from Corollary 2.3 it turns out that existence of $C_\phi : F_\beta \to A^p_\alpha$ is independent of p. Nevertheless, this independence does not declare existence of $C_\phi : F_\beta \to H^\infty$. A simple calculation deduces that $(1-z)^{-1} \in F_\beta$ and so that $C_\phi F_\beta \subset H^\infty \Leftrightarrow \|f\|_\infty < 1$. Therefore, there is a holomorphic self-map ϕ of Δ such that $C_\phi F_\beta \subset A^p_\alpha$ succeeds, but $C_\phi F_\beta \subset H^\infty$ fails.

References

[1] J.S. Choa, H.O. Kim and J.H. Shapiro, *Compact composition operators on the Smirnov class*, Proc. Amer. Math. Soc. (to appear).

[2] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, 1995.

[3] F. Jafari et al., editors, *Studies on Composition Operators*, Contemp. Math. **213** (1998), Amer. Math. Soc.

[4] H. Jarchow, V. Montesinos, K.J. Wirths and J. Xiao, *Duality for some large spaces of analytic functions*, Submitted.

[5] H. Jarchow and J. Xiao, *Composition operators between Nevanlinna classes and Bergman spaces with weights*, Submitted.

[6] M. Masri, *Composition operators on the Nevanlinna and Smirnov classes*, Thesis, Univ. of North Carolina at Chapel Hill, 1985.
[7] Y. Nakamura and N. Yanagihara, *Composition operators on* N^+, TRU Math. **14** (1978), 9–16.
[8] M. Nawrocki, *The Fréchet envelopes of vector valued Smirnov classes*, Studia Math. **44** (1989), 163–177.
[9] J.W. Roberts and M. Stoll, *Composition operators on* F^+, Studia Math. **57** (1976), 217–228.
[10] W. Rudin, *Functional Analysis*, McGraw-Hill, Inc., 1973.
[11] J.H. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, 1993.
[12] M. Stoll, *Mean growth and Taylor coefficients of some topologival algebra of analytic functions*, Ann. Polon. Math. **35** (1977), 139–158.
[13] N. Yanagihara, *The containing Fréchet space for the class* N^+, Duke Math. J. **40** (1973), 93-103.
[14] N. Yanagihara, *The second dual space for the space* N^+, Proc. Japan Acad. **49** (1973), 33–36.

Jie Xiao
Department of Mathematics, Peking University
Beijing 100871, China

E-mail: jxiao@sxx0.math.pku.edu.cn:

and
Institute of Analysis, TU-Braunschweig
PK 14, D-38106 Braunschweig, Germany

E-mail: xiao@badbit.math2.nat.tu-bs.de