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KILLING VECTOR FIELDS OF A SPACETIME

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Abstract. We studied the geodesics of a spacetime with the pseudo-Riemannian metric:

$$ds^2 = \frac{1}{x_4 x_4}\left\{\sum_{b,c=1}^{3}\left(\delta_{bc} - \frac{a x_b x_c}{1+ar^2}\right)dx_b dx_c - \frac{1}{1+a x_4 x_4}dx_4 dx_4\right\}$$

on $R^3 \times R_+$, where $r^2 = \sum_{b=1}^{3} x_b x_b$ and $a =$ constant, which are plane quadratic curves (in [12]). In this paper, we shall determine all the Killing vector fields of this spacetime and choose special pairs out of them with interesting properties for the case $a > 0$.

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§0. Introduction

We investigated the pseudo-Riemannian metric on $R_+^n = R^{n-1} \times R_+$ with the canonical coordinates $(x_1, \dots, x_{n-1}, x_n)$:

$$ds^2 = \frac{1}{{x_n}^2}\left(\frac{1}{Q}dr\,dr + r^2 \sum_{\alpha,\beta=1}^{n-2} h_{\alpha\beta}\,du^\alpha du^\beta - P\,dx_n dx_n\right).$$

where Q and P are functions on $R_+^n - \{0\}$, $r^2 = {x_1}^2 + \cdots + {x_{n-1}}^2$ and

$$d\sigma^2 = \sum_{\alpha,\beta=1}^{n-2} h_{\alpha\beta}\,du^\alpha du^\beta$$

is the standard metric on the unit sphere S^{n-2}: $r^2 = 1$ in R^{n-1}, satisfying the Einstein condition in [9], [10] and [11]. Especially for the metric with

$Q = Q(x,t)$ and $P = P(x,t)$, $x = r/x_n$, $t = x_n$, as a system of partial differential equations of order 2 on the components of the metric tensor the Einstein condition is reduced to the partial differential equation on Q as

$$\begin{aligned}
&(2Q-\varphi)x^2\frac{\partial^2 Q}{\partial x^2} - (3Q-2\varphi)xt\frac{\partial^2 Q}{\partial x\partial t} + (Q-\varphi)t^2\frac{\partial^2 Q}{\partial t^2}\\
&+\big((2n-4)Q - n\varphi\big)x\frac{\partial Q}{\partial x} - \big((n-4)Q-(n-2)\varphi\big)t\frac{\partial Q}{\partial t}\\
&-\frac{1}{Q}\left(x\frac{\partial Q}{\partial x} - t\frac{\partial Q}{\partial t}\right)\left(2(Q-\varphi)x\frac{\partial Q}{\partial x} - (Q-2\varphi)t\frac{\partial Q}{\partial t}\right)\\
&+2(n-3)Q(1-Q) = 0,
\end{aligned}$$

and $P = x^2/(Q-\varphi)$, where $\varphi(x)$ is an auxiliary free integral function of x derived from the original Einstein condition (Theorem 1 in [10]) which is correspond to the first integrals for the ordinary differential equations. This function φ becomes $1-x^2$ for the Minkowski metric

$$ds^2 = \frac{1}{{x_n}^2}\left(\sum_{a=1}^{n-1} dx_a dx_a - dx_n dx_n\right).$$

For $n = 4$ and $\varphi = 1-x^2$, we obtain $Q = 1+ax^2$ and $P = 1+at^2$ as the solution of the above partial differential equation ([11]) and the first metric is written as the one in Abstract in the coordinates (x_1, x_2, x_3, x_4). If we change the coordinates as $x_i \to \bar{x}_i = \sqrt{a}\,x_i$, we may consider as $a = 1$, but we do not use this device in order to avoid miscalculations and for the study of the interesting case: $a < 0$, for which the metric becomes Riemannian in some place in the coming work. Since this metric has constant curvature 1 by (1.4), it will be classified as one of de Sitter spacetimes in the theory of general relativity.

§1. Killing vector fields

Now, we call the above metric Ot-metric in this paper which satisfies the Einstein condition and denote it as

$$ds^2 = \sum_{i,j=1}^{4} g_{ij}\, dx_i dx_j, \quad g_{ij} = g_{ji}, \tag{1.1}$$

where

$$g_{bc} = \frac{1}{x_4x_4}\left(\delta_{bc} - \frac{ax_bx_c}{1+ar^2}\right), \quad g_{b4} = 0, \quad g_{44} = -\frac{1}{x_4x_4(1+ax_4x_4)}, \quad b,c = 1,2,3, \tag{1.2}$$

from which $(g^{ij}) = (g_{ij})^{-1}$ is given by

$$g^{bc} = x_4 x_4(\delta^{bc} + a x_b x_c), \quad g^{b4} = 0, \quad g^{44} = -x_4 x_4(1 + a x_4 x_4).$$

We obtain easily the Christoffel symbols $\{{}_j{}^i{}_h\}$ of (1.2):

$$\{{}_j{}^i{}_h\} = \frac{1}{2}\sum_k g^{ik}\left(\frac{\partial g_{jk}}{\partial x_h} + \frac{\partial g_{kh}}{\partial x_j} - \frac{\partial g_{jh}}{\partial x_k}\right)$$

as

(1.3)

$$\begin{aligned}
&\{{}_b{}^e{}_c\} = -a x_e\left(\delta_{bc} - \frac{a x_b x_c}{1 + a r^2}\right), \quad \{{}_b{}^4{}_c\} = -\frac{1 + a x_4 x_4}{x_4}\left(\delta_{bc} - \frac{a x_b x_c}{1 + a r^2}\right),\\
&\{{}_b{}^e{}_4\} = -\frac{1}{x_4}\delta^e_b, \quad \{{}_b{}^4{}_4\} = 0,\\
&\{{}_4{}^e{}_4\} = 0, \quad \{{}_4{}^4{}_4\} = -\frac{1 + 2a x_4 x_4}{x_4(1 + a x_4 x_4)}.
\end{aligned}$$

The components $R_j{}^i{}_{hk}$ of the curvature tensor:

$$R_j{}^i{}_{hk} = \frac{\partial\{{}_j{}^i{}_k\}}{\partial x_h} - \frac{\partial\{{}_j{}^i{}_h\}}{\partial x_k} + \sum_l \{{}_l{}^i{}_h\}\{{}_j{}^l{}_k\} - \sum_l \{{}_l{}^i{}_k\}\{{}_j{}^l{}_h\}$$

are computed by (1.3) as

$$\begin{gathered}
R_a{}^e{}_{bc} = \delta^e_b g_{ac} - \delta^e_c g_{ab}, \quad R_a{}^4{}_{bc} = 0,\\
R_4{}^e{}_{bc} = 0, \quad R_4{}^4{}_{bc} = 0, \quad R_b{}^e{}_{4c} = 0, \quad R_b{}^4{}_{4c} = g_{bc},\\
R_4{}^e{}_{4c} = -g_{44}\delta^e_c, \quad R_4{}^4{}_{4c} = 0,
\end{gathered}$$

which are written simply as

$$R_j{}^i{}_{hk} = \delta^i_h g_{jk} - \delta^i_k g_{jh}. \tag{1.4}$$

We obtain the Ricci curvature $R_{jk} = \sum_l R_j{}^l{}_{lk}$ and the scalar curvature $R = \sum_{j,k} g^{jk} R_{jk}$ as

$$R_{jk} = 3 g_{jk} \quad \text{and} \quad R = 12, \tag{1.5}$$

which shows that the metric (1.1) satisfies the Einstein condition:

$$R_{ij} = \frac{R}{4} g_{ij}.$$

Now, let $V = \sum_i v^i \, \partial/\partial x^i$ be a Killing field which satisfies the condition:

$$v_{i,j} + v_{j,i} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - 2\sum_k \{{}_i{}^k{}_j\} v_k = 0.$$

By means of (1.2) and (1.3) this condition can be written as

$$\frac{\partial v_b}{\partial x_c} + \frac{\partial v_c}{\partial x_b} + 2a\left(\delta_{bc} - \frac{a x_b x_c}{1+ar^2}\right)\sum_e x_e v_e + \frac{2(1+a x_4 x_4)}{x_4}\left(\delta_{bc} - \frac{a x_b x_c}{1+ar^2}\right) v_4 = 0, \tag{1.6}$$

$$\frac{\partial v_b}{\partial x_4} + \frac{\partial v_4}{\partial x_b} + \frac{2}{x_4} v_b = 0, \tag{1.7}$$

and

$$\frac{\partial v_4}{\partial x_4} + \frac{1 + 2a x_4 x_4}{x_4(1 + a x_4 x_4)} v_4 = 0. \tag{1.8}$$

Integrating (1.8), we obtain easily

$$v_4 = \frac{f}{x_4\sqrt{1 + a x_4 x_4}}, \quad f = f(x_1, x_2, x_3). \tag{1.9}$$

Substituting this relation into (1.7) we obtain

$$\frac{\partial v_b}{\partial x_4} + \frac{2}{x_4} v_b + \frac{1}{x_4\sqrt{1 + a x_4 x_4}} \frac{\partial f}{\partial x_b} = 0,$$

from which we obtain

$$\frac{\partial}{\partial x_4}(x_4 x_4 v_b) = -\frac{x_4}{\sqrt{1 + a x_4 x_4}} \frac{\partial f}{\partial x_b}$$

and integrating this relation we obtain

$$x_4 x_4 v_b = -\frac{\sqrt{1 + a x_4 x_4}}{a} \frac{\partial f}{\partial x_b} + f_b, \quad f_b = f_b(x_1, x_2, x_3),$$

i.e.

$$v_b = -\frac{\sqrt{1 + a x_4 x_4}}{a x_4 x_4} \frac{\partial f}{\partial x_b} + \frac{f_b}{x_4 x_4}. \tag{1.10}$$

From (1.10) we obtain

$$\frac{\partial v_b}{\partial x_c} = -\frac{\sqrt{1 + a x_4 x_4}}{a x_4 x_4} \frac{\partial^2 f}{\partial x_b \partial x_c} + \frac{1}{x_4 x_4} \frac{\partial f_b}{\partial x_c}$$

and substituting these relations into (1.6) we obtain the following conditions regarding integral free functions $f(x_1, x_2, x_3)$ and $f_b(x_1, x_2, x_3)$:

$$(1.11)\quad -2\frac{\sqrt{1+ax_4x_4}}{a}\frac{\partial^2 f}{\partial x_b x_c} + \frac{\partial f_b}{\partial x_c} + \frac{\partial f_c}{\partial x_b} + 2\left(\delta_{bc} - \frac{ax_bx_c}{1+ar^2}\right)\left\{\sqrt{1+ax_4x_4}\left(f - \sum_e \frac{\partial f}{\partial x_e}x_e\right) + a\sum_e f_ex_e\right\} = 0, \quad b, c = 1, 2, 3.$$

If we can find f, f_b satisfying (1.11), then we obtain the solution v_i satisfying (1.6)-(1.8). Noticing the independency of variables, (1.11) can be replaced by

$$(1.12)\qquad \frac{\partial^2 f}{\partial x_b x_c} = a\left(\delta_{bc} - \frac{ax_bx_c}{1+ar^2}\right)\left(f - \sum_e \frac{\partial f}{\partial x_e}x_e\right),$$

$$(1.13)\qquad \frac{\partial f_b}{\partial x_c} + \frac{\partial f_c}{\partial x_b} + 2a\left(\delta_{bc} - \frac{ax_bx_c}{1+ar^2}\right)\sum_e f_ex_e = 0.$$

We see that f and f_b, $b = 1, 2, 3$, can be treated separately.

§2. Solutions of the differential equations (1.12) and (1.13)

Supposing $f(x_1, x_2, x_3)$ is analytic on x_1, x_2, x_3, we put

$$(2.1)\qquad f = \sum_{m=0}^{\infty} P_m(x_1, x_2, x_3),$$

where P_m is a homogeneous polynomial of order m in x_1, x_2, x_3. Substituting this expression into (1.12), we obtain

$$(1+ar^2)\sum_{m=2}^{\infty}\frac{\partial^2 P_m}{\partial x_b \partial x_c} = a\big((1+ar^2)\delta_{bc} - ax_bx_c\big)\left(P_0 - \sum_{m=2}^{\infty}(m-1)P_m\right),$$

which we rewrite in considering the arrangement as

$$(2.2)\quad \sum_{m=2}^{\infty}\frac{\partial^2 P_m}{\partial x_b \partial x_c} + ar^2\sum_{m=2}^{\infty}\frac{\partial^2 P_m}{\partial x_b \partial x_c} = a\delta_{bc}\left(P_0 - \sum_{m=2}^{\infty}(m-1)P_m\right) + a^2(r^2\delta_{bc} - x_bx_c)\left(P_0 - \sum_{m=2}^{\infty}(m-1)P_m\right).$$

Using the equalities

$$(2.3)\qquad \frac{\partial^2 r^{2m}}{\partial x_b \partial x_c} = 2mr^{2m-4}\big(r^2\delta_{bc} + 2(m-1)x_bx_c\big), \quad m = 1, 2, 3, \dots ,$$

we obtain P_m in turn up to $m = 10$ as follows:

$$P_2 = \frac{a}{2}P_0r^2, \quad P_3 = 0, \quad P_4 = -\frac{a^2}{8}P_0r^4, \quad P_5 = 0, \quad P_6 = \frac{a^3}{16}P_0r^6,$$
$$P_7 = 0, \quad P_8 = -\frac{5a^4}{128}P_0r^8, \quad P_9 = 0, \quad P_{10} = \frac{7a^5}{256}P_0r^{10}.$$

Through the arguments determinimg these P_m, we see that we can put

$$P_{2m+1} = 0, \quad m = 1, 2, 3, \dots$$

and

$$f = P_1 + \varphi(X), \quad X = r^2.$$

Denoting the derivative of φ with respect to X by "$\,'\,$", we have

$$\frac{\partial f}{\partial x_c} = \frac{\partial P_1}{\partial x_c} + 2\varphi' x_c,$$
$$\frac{\partial^2 f}{\partial x_b \partial x_c} = 2\varphi'\delta_{bc} + 4\varphi'' x_b x_c,$$

and

$$f - \sum_e \frac{\partial f}{\partial x_e} x_e = \varphi - 2\varphi' r^2.$$

Substituting these into (1.12), we obtain

$$(2\varphi' - a\varphi + 2a\varphi' X)\delta_{bc} + \left(4\varphi'' + \frac{a^2}{1 + aX}(\varphi - 2\varphi' X)\right)x_b x_c = 0. \tag{2.4}$$

Contracting this equality with c by multiplying with x_c, we obtain

$$(2\varphi' - a\varphi + 2a\varphi' X)x_b + (4\varphi'' + \frac{a^2}{1 + aX}(\varphi - 2\varphi' X)r^2 x_b = 0,$$

and hence

$$2\varphi' - a\varphi + 2a\varphi' X + X\left(4\varphi'' + \frac{a^2}{1 + aX}(\varphi - 2\varphi' X)\right) = 0.$$

Substituting this expression into (2.4), we obtain

$$\left\{4\varphi'' + \frac{a^2}{1 + aX}(\varphi - 2\varphi' X)\right\}(X\delta_{bc} - x_b x_c) = 0.$$

Hence it must hold

$$4\varphi'' + \frac{a^2}{1+aX}(\varphi - 2\varphi' X) = 0, \tag{2.5}$$

$$2\varphi' - a\varphi + 2a\varphi' X = 0. \tag{2.6}$$

From (2.6) we obtain by integration

$$\varphi = P_0\sqrt{1+aX} = P_0\sqrt{1+ar^2}. \tag{2.7}$$

We can easily see that this φ satisfies (2.5). Thus we see that the general solution of (1.12) is given by

$$f(x_1, x_2, x_3) = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + P_0\sqrt{1+ar^2}, \tag{2.8}$$

where λ_1, λ_2, λ_3 and P_0 are integral constants.

Next, we shall treat (1.13). First, we put

$$f_b = \sum_{m=0}^{\infty} P_{bm}, \tag{2.9}$$

where P_{bm} is a homogeneous polynomial of order m in x_1, x_2, x_3. Substituting this expression into (1.13) and using the notation

$$Q_{m+1} = \sum_e P_{em} x_e \tag{2.10}$$

for simplicity, we obtain

$$\sum_{m=0}^{\infty}\left(\frac{\partial P_{bm}}{\partial x_c} + \frac{\partial P_{cm}}{\partial x_b}\right) + 2a\left(\delta_{bc} - \frac{a x_b x_c}{1+ar^2}\right)\sum_{m=0}^{\infty} Q_{m+1} = 0,$$

which we rewrite as, considering the arrangement,

$$\begin{aligned}\sum_{m=1}^{\infty}\left(\frac{\partial P_{bm}}{\partial x_c} + \frac{\partial P_{cm}}{\partial x_b}\right) + ar^2\sum_{m=1}^{\infty}\left(\frac{\partial P_{bm}}{\partial x_c} + \frac{\partial P_{cm}}{\partial x_b}\right) \\ + 2a\delta_{bc}\sum_{m=1}^{\infty} Q_m + 2a^2(r^2\delta_{bc} - x_b x_c)\sum_{m=1}^{\infty} Q_m = 0.\end{aligned} \tag{2.11}$$

From the terms of O^{th}-order, we obtain

$$\frac{\partial P_{b1}}{\partial x_c} + \frac{\partial P_{c1}}{\partial x_b} = 0. \tag{2.12}$$

From the terms of order 1 we obtain the relation:

$$\frac{\partial P_{b2}}{\partial x_c}+\frac{\partial P_{c2}}{\partial x_b}+2a\delta_{bc}Q_1=0$$

and, multiplying by x_c and contracting with respect to c,

$$P_{b2}+\frac{\partial Q_3}{\partial x_b}+2ax_bQ_1=0$$

and using the same way for b

$$Q_3=-\frac{a}{2}r^2Q_1.$$

Going back to the previous equality we obtain

$$P_{b2}=\frac{a}{2}(r^2P_{b0}-2x_bQ_1).$$

We see easily that the above expression satisfies the first one.

Next, from the terms of order 2 we obtain the relation:

$$\frac{\partial P_{b3}}{\partial x_c}+\frac{\partial P_{c3}}{\partial x_b}+ar^2\left(\frac{\partial P_{b1}}{\partial x_c}+\frac{\partial P_{c1}}{\partial x_b}\right)+2a\delta_{bc}Q_2=0,$$

which becomes

$$\frac{\partial P_{b3}}{\partial x_c}+\frac{\partial P_{c3}}{\partial x_b}=0$$

by means of (2.12) and $Q_2=\sum_b P_{b1}x_b=0$. We obtain easily from these relations

$$P_{b3}=0 \quad \text{and} \quad Q_4=0.$$

We obtain P_{bm} in turn up to $m=8$ by analogous arguments as follows:

$$\begin{gathered} P_{b4}=-\frac{1}{8}a^2r^2(r^2P_{b0}-4x_bQ_1), \quad P_{b5}=0, \\ P_{b6}=\frac{1}{16}a^3r^4(r^2P_{bo}-6x_bQ_1), \quad P_{b7}=0, \\ P_{b8}=-\frac{5}{128}a^4r^6(r^2P_{b0}-8x_bQ_1). \end{gathered}$$

Through the arguments determining these P_m, for an positive integer m we suppose that

$$\begin{aligned} &P_{b3}=P_{b5}=\cdots=P_{b(2m+1)}=0, \\ &P_{b(2n)}=(-1)^{n-1}k_na^nr^{2n-2}(r^2P_{b0}-2nx_bQ_1), \quad n=1,2,3,\dots,m. \end{aligned} \tag{2.13}$$

From the terms of order $2m+2$ of (2.11), we obtain

$$\frac{\partial P_{b(2m+3)}}{\partial x_c} + \frac{\partial P_{c(2m+3)}}{\partial x_b} + ar^2\left(\frac{\partial P_{b(2m+1)}}{\partial x_c} + \frac{\partial P_{c(2m+1)}}{\partial x_b}\right) + 2a\delta_{bc}Q_{2m+2} + 2a^2(r^2\delta_{bc} - x_b x_c)Q_{2m} = 0,$$

which become by (2.13)

$$\frac{\partial P_{b(2m+3)}}{\partial x_c} + \frac{\partial P_{c(2m+3)}}{\partial x_b} = 0.$$

Multiplying this expression by x_c and contracting with respect to c, we obtain

$$(2m+2)P_{b(2m+3)} + \frac{\partial Q_{2m+4}}{\partial x_b} = 0,$$

which implies $Q_{2m+4} = 0$ and $P_{b(2m+3)} = 0$. Next, from the terms of order $2m+1$, we obtain

$$\frac{\partial P_{b(2m+2)}}{\partial x_c} + \frac{\partial P_{c(2m+2)}}{\partial x_b} + ar^2\left(\frac{\partial P_{b(2m)}}{\partial x_c} + \frac{\partial P_{c(2m)}}{\partial x_b}\right) + 2a\delta_{bc}Q_{2m+1} + 2a^2(r^2\delta_{bc} - x_b x_c)Q_{2m-1} = 0.$$

By means of (2.13) we obtain

$$\frac{\partial P_{b(2m)}}{\partial x_c} = (-1)^{m-1}k_m a^m r^{2m-4} 2m\bigl\{r^2(P_{b0}x_c - P_{c0}x_b) - \bigl(r^2\delta_{bc} + 2(m-1)x_b x_c\bigr)Q_1\bigr\}$$

and

$$Q_{2m+1} = (-1)^m(2m-1)k_m a^m r^{2m} Q_1,$$
$$Q_{2m-1} = (-1)^{m-1}(2m-3)k_{m-1}a^{m-1}r^{2m-2}Q_1.$$

Substituting these into the above expression, we obtain

$$\frac{\partial P_{b(2m+2)}}{\partial x_c} + \frac{\partial P_{c(2m+2)}}{\partial x_b} + (-1)^m 2a^{m+1}r^{2m-2}\bigl\{\bigl((4m-1)k_m - (2m-3)k_{m-1}\bigr)r^2\delta_{bc} + \bigl(4m(m-1)k_m + (2m-3)k_{m-1}\bigr)x_b x_c\bigr\}Q_1 = 0.$$

Multiplying this expression by x_c and contracting with respect to c, we obtain

$$(2m+1)P_{b(2m+2)} + \frac{\partial Q_{2m+3}}{\partial x_b} + (-1)^m 2a^{m+1}r^{2m}(4m^2-1)k_m x_b Q_1 = 0,$$

from which we obtain by the same way

$$Q_{2m+3} = (-1)^{m+1}\frac{4m^2-1}{2(m+1)}k_m a^{m+1} r^{2m+2} Q_1$$

and

$$\frac{\partial Q_{2m+3}}{\partial x_b} = (-1)^{m+1}\frac{4m^2-1}{2(m+1)}k_m a^{m+1} r^{2m}\big(r^2 P_{b0} + 2(m+1)x_b Q_1\big).$$

Using these equalities, we obtain finally

$$P_{b(2m+2)} = (-1)^m k_m \frac{2m-1}{2(m+1)} a^{m+1} r^{2m}\big(r^2 P_{b0} - 2(m+1)x_b Q_1\big)$$

and so we can put

$$k_{m+1} = \frac{2m-1}{2(m+1)}k_m.$$

Thus, we have verified that (2.13) holds for all integers $m > 0$ and

$$k_m = \frac{(2m-3)(2m-5)\cdots 1}{2m\cdot 2(m-1)\cdots 4}k_1 = \frac{(2m-3)!!}{2^m m!}$$

for $m > 1$, since $k_1 = \frac{1}{2}$. Thus, we obtain the formulas:

$$P_{b(2m)} = (-1)^{m-1}\frac{(2m-3)!!}{2^m m!}a^m r^{2m-2}(r^2 P_{b0} - 2m x_b Q_1), \qquad m = 2, 3, \dots \tag{2.14}$$

and

$$P_{b2} = \frac{a}{2}(r^2 P_{b0} - 2x_b Q_1), \quad Q_1 = \sum_e P_{e0} x_e.$$

Arranging the results in this section, we have the following theorem.

Theorem 1. *For the spacetime on $R^4_+ = R^3 \times R_+$ with the metric (1.1) and (1.2) with $a > 0$, any Killing field $V = \sum_i v^i \partial/\partial x_i$ is given by the formula:*

$$\begin{aligned}(v_b) &= \frac{1}{x_4 x_4}\bigg\{-\frac{\sqrt{1+ax_4x_4}}{a}\lambda + \sqrt{1+ar^2}\,p - (\mu\times\tilde{x}) \\ &\quad - \frac{1}{\sqrt{1+ar^2}}\big(p_0\sqrt{1+ax_4x_4} + a(p\cdot\tilde{x})\big)\tilde{x}\bigg\}, \\ v_4 &= \frac{1}{x_4\sqrt{1+ax_4x_4}}\big((\lambda\cdot\tilde{x}) + p_0\sqrt{1+ar^2}\,\big), \quad r^2 = (\tilde{x}\cdot\tilde{x}),\end{aligned} \tag{2.15}$$

where $v_i = \sum_{j=1}^{4} g_{ij} v^j$ *and* $\lambda = \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}$, $p = \begin{pmatrix}p_1\\ p_2\\ p_3\end{pmatrix}$, $\mu = \begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\end{pmatrix}$, $\tilde{x} = \begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$, *considered as vectors in* R^3 *with the standard Euclidean metric:* $ds^2 = \sum_b dx_b dx_b$ *and "* $\cdot$ *" and "* $\times$ *" denote the inner product and the outer product of two vectors. Therefore* V *depends on 10 real constants* p_0, λ_b, p_b, μ_b, $b = 1, 2, 3$.

Proof. We have from (2.14)

$$\begin{aligned} f_b &= P_{b0} + P_{b1} + \frac{a}{2}(r^2 P_{b0} - 2x_b Q_1) \\ &\quad + \sum_{m=2}^{\infty} (-1)^{m-1} \frac{(2m-3)!!}{2^m m!} a^m r^{2m-2} (r^2 P_{b0} - 2m x_b Q_1) \\ &= P_{b1} + \left(1 + \frac{a}{2} r^2 + \sum_{m=2}^{\infty} (-1)^{m-1} \frac{(2m-3)!!}{2^m m!} a^m r^{2m}\right) P_{b0} \\ &\quad - \left(a + \sum_{m=2}^{\infty} (-1)^{m-1} \frac{(2m-3)!!}{2^{m-1}(m-1)!} a^m r^{2m-2}\right) x_b Q_1. \end{aligned}$$

Since we have

$$(1+t)^{\frac{1}{2}} = 1 + \frac{1}{2} t + \sum_{m=2}^{\infty} (-1)^{m-1} \frac{(2m-3)!!}{m!\, 2^m} t^m$$

and

$$(1+t)^{-\frac{1}{2}} = 1 + \sum_{m=2}^{\infty} (-1)^{m-1} \frac{(2m-3)!!}{(m-1)!\, 2^{m-1}} t^{m-1},$$

we obtain

$$f_b = P_{b1} + (1 + ar^2)^{\frac{1}{2}} P_{b0} - a(1 + ar^2)^{-\frac{1}{2}} \left(\sum_e P_{e0} x_e\right) x_b.$$

The 3×3-matrix $\left(\frac{\partial P_{b1}}{\partial x_c}\right)$ is skew by (2.12), we denote it as

$$\left(\frac{\partial P_{b1}}{\partial x_c}\right) = \begin{pmatrix} 0 & \mu_3 & -\mu_2 \\ -\mu_3 & 0 & \mu_1 \\ \mu_2 & -\mu_1 & 0 \end{pmatrix},$$

then we have $(P_{b1}) = -(\mu \times \tilde{x})$. Setting $P_{b0} = p_b$, we obtain

$$(f_b) = -(\mu \times \tilde{x}) + \sqrt{1+ar^2}\,p - \frac{a}{\sqrt{1+ar^2}}(p\cdot\tilde{x})\tilde{x}$$

and

$$f = \lambda\cdot\tilde{x} + p_0\sqrt{1+ar^2}, \quad p_0 = P_0.$$

Finally from (1.10) and the above equalities we obtain

$$\begin{aligned}(v_b) = &-\frac{\sqrt{1+ax_4x_4}}{ax_4x_4}\left(\lambda + \frac{ap_0}{\sqrt{1+ar^2}}\tilde{x}\right)\\ &+\frac{1}{x_4x_4}\left\{-(\mu\times\tilde{x}) + \sqrt{1+ar^2}\,p - \frac{a}{\sqrt{1+ar^2}}(p\cdot\tilde{x})\tilde{x}\right\}\\ =&\ \frac{1}{x_4x_4}\left\{-\frac{\sqrt{1+ax_4x_4}}{a}\lambda + \sqrt{1+ar^2}\,p - (\mu\times\tilde{x})\right.\\ &\left.-\frac{1}{\sqrt{1+ar^2}}\left(p_0\sqrt{1+ax_4x_4} + a(p\cdot\tilde{x})\right)\tilde{x}\right\}\end{aligned}$$

and

$$v_4 = \frac{1}{x_4\sqrt{1+ax_4x_4}}\left((\lambda\cdot\tilde{x}) + p_0\sqrt{1+ar^2}\right).$$

Q.E.D.

Now we compute the norm of Killing field V given by (2.15):

$$N(V) = \sum_{i,j=1}^{4} g_{ij}v^iv^j = \sum_{i,j=1}^{4} g^{ij}v_iv_j. \tag{2.16}$$

Using the notations in Theorem 1 and setting $\tilde{v} = (v_b)$, we have

$$N(V) = x_4x_4\sum_{b,c=1}^{3}(\delta^{bc} + ax_bx_c)v_bv_c - x_4x_4(1+ax_4x_4)v_4v_4$$

and so

$$= x_4x_4\left\{(\tilde{v}\cdot\tilde{v}) + a(\tilde{v}\cdot\tilde{x})^2\right\} - \left((\lambda\cdot\tilde{x}) + p_0\sqrt{1+ar^2}\right)^2$$

$$
\begin{aligned}
&= \frac{1}{x_4x_4}\Big\{\frac{1+ax_4x_4}{a^2}(\lambda\cdot\lambda)+(1+ar^2)(p\cdot p)\\
&\quad+\big((\mu\times\tilde{x})\cdot(\mu\times\tilde{x})\big)+\frac{r^2}{1+ar^2}\big(p_0\sqrt{1+ax_4x_4}+a(p\cdot\tilde{x})\big)^2\\
&\quad-\frac{2\sqrt{1+ax_4x_4}\sqrt{1+ar^2}}{a}(\lambda\cdot p)+\frac{2\sqrt{1+ax_4x_4}}{2}\big((\lambda\times\mu)\cdot\tilde{x}\big)\\
&\quad+\frac{2\sqrt{1+ax_4x_4}}{a\sqrt{1+ar^2}}\big(p_0\sqrt{1+ax_4x_4}+a(p\cdot\hat{x})\big)(\lambda\cdot\tilde{x})\\
&\quad-2\sqrt{1+ar^2}\big((p\times\mu)\cdot\tilde{x}\big)-2\big(p_0\sqrt{1+ax_4x_4}+a(p\cdot\tilde{x})\big)(p\cdot\tilde{x})\Big\}\\
&\quad+\frac{a}{x_4x_4}\Big\{-\frac{\sqrt{1+ax_4x_4}}{a}(\lambda\cdot\tilde{x})+\sqrt{1+ar^2}\,(p\cdot\tilde{x})\\
&\quad-\frac{r^2}{\sqrt{1+ar^2}}\big(p_0\sqrt{1+ax_4x_4}+a(p\cdot\tilde{x})\big)\Big\}^2-\big((\lambda\cdot\tilde{x})+p_0\sqrt{1+ar^2}\,\big)^2
\end{aligned}
$$

which is arranged as follows

(2.17)

$$
\begin{aligned}
x_4x_4N(v) &= \frac{1+ax_4x_4}{a^2}(\lambda\cdot\lambda)+(1+ar^2)(p\cdot p)+r^2(\mu\cdot\mu)\\
&\quad-(\mu\cdot\tilde{x})^2+\frac{1}{a}(\lambda\cdot\tilde{x})^2-a(p\cdot\tilde{x})^2+\frac{2p_0\sqrt{1+ar^2}}{a}(\lambda\cdot\tilde{x})\\
&\quad-2p_0\sqrt{1+ax_4x_4}\,(p\cdot\tilde{x})+\frac{2\sqrt{1+ax_4x_4}}{a}\big((\lambda\times\mu)\cdot\tilde{x}\big)\\
&\quad-2\sqrt{1+ar^2}\,\big((p\times\mu)\cdot\tilde{x}\big)-\frac{2\sqrt{1+ar^2}\sqrt{1+ax_4x_4}}{a}(\lambda\cdot p)\\
&\quad+{p_0}^2(r^2-x_4x_4).
\end{aligned}
$$

Example 1. Case $p_0=0$, $p=0$.

$$
\begin{aligned}
\tilde{v}=(v_b)&=\frac{1}{x_4x_4}\Big\{-\frac{\sqrt{1+ax_4x_4}}{a}\lambda-(\mu\times\tilde{x})\Big\},\\
v_4&=\frac{1}{x_4\sqrt{1+ax_4x_4}}(\lambda\cdot\tilde{x}),\\
N(V)&=\frac{1}{x_4x_4}\Big\{\frac{1+ax_4x_4}{a^2}(\lambda\cdot\lambda)+r^2(\mu\cdot\mu)-(\mu\cdot\tilde{x})^2\\
&\qquad+\frac{1}{a}(\lambda\cdot\tilde{x})^2+\frac{2\sqrt{1+ax_4x_4}}{a}\big((\lambda\times\mu)\cdot\tilde{x}\big)\Big\}\\
&=\frac{1}{x_4x_4}\left\{\left|\frac{\sqrt{1+ax_4x_4}}{a}\lambda+(\mu\times\tilde{x})\right|^2+\frac{1}{a}(\lambda\cdot\tilde{x})^2\right\},
\end{aligned}
$$

which implies that $N(V) \geq 0$ and $N(V) = 0$ is equivalent to

$$\mu \times \tilde{x} = -\frac{\sqrt{1 + ax_4x_4}}{a}\lambda$$

and this relation implies $(\lambda{\cdot}\mu) = 0$. Hence, if $(\lambda{\cdot}\mu) \neq 0$, everywhere $N(V) > 0$, and so V is spacelike.

Example 2. Case $p_0 \neq 0$, $\lambda = p = 0$.

$$\begin{aligned}
\tilde{v} = (v_b) &= \frac{1}{x_4x_4}\left\{-(\mu \times \tilde{x}) - \frac{\sqrt{1 + ax_4x_4}}{\sqrt{1 + ar^2}}p_0\tilde{x}\right\},\\
v_4 &= \frac{p_0\sqrt{1 + ar^2}}{x_4\sqrt{1 + ax_4x_4}},\\
N(V) &= |(\mu \times \tilde{x})|^2 + {p_0}^2(r^2 - x_4x_4),
\end{aligned}$$

which shows that if $r > x_4$, V is spacelike.

Example 3. Case $p_0 \neq 0$, $\lambda = p = \mu = 0$.

$$v_b = -\frac{p_0\sqrt{1 + ax_4x_4}}{x_4x_4\sqrt{1 + ar^2}}x_b, \quad v_4 = \frac{p_0\sqrt{1 + ar^2}}{x_4\sqrt{1 + ax_4x_4}}$$

and

$$\begin{aligned}
v^i &= -p_0\sqrt{1 + ar^2}\sqrt{1 + ax_4x_4}\, x_i, \quad i = 1, \ldots, 4,\\
N(V) &= \frac{{p_0}^2}{x_4x_4}(r^2 - x_4x_4).
\end{aligned}$$

In the following sections, we shall investigate Killing fields of the spacetime with the Ot-metric (1.1) and (1.2), mainly noticing the Killing fields of the above examples and special pairs of two ones which construct a Lie algebra of dimension 2.

§3. Special Killing fields

We say a Killing field V given by (2.15) is *static*, if the Pfaff equation

$$\sum_i v_i dx_i = 0$$

is complete, that is, it admits locally a hypersurface satisfying this equation. As is well known, it is necessary and sufficient that the following equality holds

$$\sum_{i=1}^4 v_i dx_i \wedge d\left(\sum_{j=1}^4 v_j dx_j\right) = 0.$$

Now, we denote the Killing field V of Example 3 in §2 with $p_0 = -1$ by ξ that is

$$\xi^i = \sqrt{1+ar^2}\sqrt{1+ax_4x_4}\,x_i. \tag{3.1}$$

Theorem 2. *Killing field ξ is static.*

Proof. We have

$$\xi := \sum_{i=1}^{4} \xi_i dx_i = \frac{\sqrt{1+ax_4x_4}}{x_4x_4\sqrt{1+ar^2}} \sum_{b=1}^{3} x_b dx_b - \frac{\sqrt{1+ar^2}}{x_4\sqrt{1+ax_4x_4}} dx_4,$$

which is expressed only by r and x_4, since $\sum x_b dx_b = r\,dr$. Hence, $d\xi$ can be written as

$$d\xi = \phi(r, x_4)\, dr \wedge dx_4,$$

which implies the equality $\xi \wedge d\xi = 0$. Q.E.D.

Then, we take another Killing field V given by (2.15) and put

$$\theta := \sum_b v_b dx_b + v_4 dx_4.$$

We search for the condition that the system of Pfaff equations:

$$\xi = 0 \quad \text{and} \quad \theta = 0$$

is complete, that is, it admits locally a surface satisfying both equations. As is well known, it is necessary and sufficient that the following equalities hold:

$$\xi \wedge \theta \wedge d\xi = 0 \quad \text{and} \quad \xi \wedge \theta \wedge d\theta = 0.$$

From Theorem 2, the first equality holds. Regarding the second, we shall compute the three-form $\theta \wedge d\theta$. For simplicity we use the notations

$$L := 1 + ar^2, \quad M := 1 + ax_4x_4 \quad \text{and} \quad d_2\tilde{x} := \begin{pmatrix} dx_2 \wedge dx_3 \\ dx_3 \wedge dx_1 \\ dx_1 \wedge dx_2 \end{pmatrix},$$

then θ can be written as

$$\begin{aligned} \theta = \frac{1}{x_4x_4}\Big[&-\frac{\sqrt{M}}{a}(\lambda \cdot d\tilde{x}) + \sqrt{L}\,(p \cdot d\tilde{x}) - \big((\mu \times \tilde{x}) \cdot d\tilde{x}\big) \\ &- \frac{1}{\sqrt{L}}\big(p_0\sqrt{M} + a(p \cdot \tilde{x})\big) r\, dr\Big] + \frac{1}{x_4\sqrt{M}}\big((\lambda \cdot \tilde{x}) + p_0\sqrt{L}\,\big) dx_4. \end{aligned} \tag{3.2}$$

from which we have

$$
\begin{aligned}
d\theta &= \frac{1}{a}\frac{\partial}{\partial x_4}\left(\frac{\sqrt{M}}{x_4x_4}\right)(\lambda\cdot d\tilde{x})\wedge dx_4 + d\frac{\sqrt{L}}{x_4x_4}\wedge(p\cdot d\tilde{x})\\
&\quad + \frac{2}{(x_4)^3}dx_4\wedge\big((\mu\times\tilde{x})\cdot d\tilde{x}\big) - \frac{r}{\sqrt{L}}\,d\frac{p_0\sqrt{M}+a(p\cdot\tilde{x})}{x_4x_4}\wedge dr\\
&\quad + \frac{1}{x_4\sqrt{M}}\left((\lambda\cdot d\tilde{x}) + \frac{ap_0r}{\sqrt{L}}dr\right)\wedge dx_4 - \frac{2}{x_4x_4}(\mu\cdot d_2\tilde{x})\\
&= \left\{\left(-\frac{2\sqrt{M}}{a(x_4)^3} + \frac{1}{x_4\sqrt{M}}\right)(\lambda\cdot d\tilde{x}) + \frac{2\sqrt{L}}{(x_4)^3}(p\cdot d\tilde{x}) - \frac{2}{(x_4)^3}\big((\mu\times\tilde{x})\cdot d\tilde{x}\big)\right.\\
&\quad + \frac{r}{\sqrt{L}}\left(\frac{ap_0}{x_4\sqrt{M}} - \frac{2p_0\sqrt{M}}{(x_4)^3} - \frac{2a(p\cdot\tilde{x})}{(x_4)^3}\right)dr\\
&\quad \left. + \frac{1}{x_4\sqrt{M}}\left((\lambda\cdot d\tilde{x}) + \frac{ap_0r}{\sqrt{L}}dr\right)\right\}\wedge dx_4 + \frac{2ar}{x_4x_4\sqrt{L}}dr\wedge(p\cdot d\tilde{x})\\
&\quad - \frac{2}{x_4x_4}(\mu\cdot d_2\tilde{x}),
\end{aligned}
$$

and since we have

$$
d\big((\mu\times\tilde{x})\cdot d\tilde{x}\big) = 2(\mu\cdot d_2\tilde{x})
$$

which is arranged as

$$
\begin{aligned}
(3.3)\quad \frac{x_4x_4}{2}d\theta &= \left\{-\frac{1}{ax_4\sqrt{M}}(\lambda\cdot d\tilde{x}) + \frac{\sqrt{L}}{x_4}(p\cdot d\tilde{x}) - \frac{1}{x_4}\big((\mu\times\tilde{x})\cdot d\tilde{x}\big)\right.\\
&\quad \left. - \frac{r}{x_4\sqrt{L}}\left(\frac{p_0}{\sqrt{M}} + a(p\cdot\tilde{x})\right)dr\right\}\wedge dx_4 + \frac{ar}{\sqrt{L}}dr\wedge(p\cdot d\tilde{x}) - (\mu\cdot d_2\tilde{x}).
\end{aligned}
$$

Then, we obtain from (3.2) and (3.3)

$$
\begin{aligned}
\frac{(x_4)^5}{2}\theta\wedge d\theta &= \left[-\frac{\sqrt{M}}{a}(\lambda\cdot d\tilde{x}) + \sqrt{L}\,(p\cdot d\tilde{x}) - \big((\mu\times\tilde{x})\cdot d\tilde{x}\big)\right.\\
&\quad \left. - \frac{r}{\sqrt{L}}\big((p_0\sqrt{M} + a(p\cdot\tilde{x})\big)dr + \frac{x_4}{\sqrt{M}}\big((\lambda\cdot\tilde{x}) + p_0\sqrt{L}\,\big)dx_4\right]\\
&\quad \wedge\left[\left\{-\frac{1}{a\sqrt{M}}(\lambda\cdot d\tilde{x}) + \sqrt{L}\,(p\cdot d\tilde{x}) - \big((\mu\times\tilde{x})\cdot d\tilde{x}\big)\right.\right.\\
&\quad \left.\left. - \frac{r}{\sqrt{L}}\left(\frac{p_0}{\sqrt{M}} + a(p\cdot\tilde{x})\right)dr)\right\}\wedge dx_4 + \frac{arx_4}{\sqrt{L}}dr\wedge(p\cdot d\tilde{x})\right]\\
&\quad - (x_4)^3\theta\wedge(\mu\cdot d_2\tilde{x}),
\end{aligned}
$$

which is arranged by using the relations:

$$\big((\mu\times\tilde{x})\cdot d\tilde{x}\big)\wedge(\mu\cdot d_2\tilde{x})=0,\quad (\lambda\cdot d\tilde{x})\wedge(\mu\cdot d_2\tilde{x})=(\lambda\cdot\mu)dx_1\wedge dx_2\wedge dx_3,$$
$$(p\cdot d\tilde{x})\wedge(\mu\cdot d_2\tilde{x})=(p\cdot\mu)dx_1\wedge dx_2\wedge dx_3,\quad r\,dr\wedge(\mu\cdot d_2\tilde{x})=$$
$$(\mu\cdot\tilde{x})dx_1\wedge dx_2\wedge dx_3,\quad (\lambda\cdot d\tilde{x})\wedge(\mu\cdot d\tilde{x})=\big((\lambda\times\mu)\cdot d_2\tilde{x}\big)$$

after a little cumbersome computation as follows.

(3.4)
$$\begin{aligned}
\frac{(x_4)^3}{2}\theta\wedge d\theta=&\Big\{-\frac{\sqrt{L}}{\sqrt{M}}(\lambda\cdot d\tilde{x})\wedge(p\cdot d\tilde{x})+\frac{1}{\sqrt{M}}(\lambda\cdot d\tilde{x})\wedge\big((\mu\times\tilde{x})\cdot d\tilde{x}\big)\\
&+\frac{ar}{\sqrt{L}\sqrt{M}}\big((p\cdot\tilde{x})(\lambda\cdot d\tilde{x})-(\lambda\cdot\tilde{x})(p\cdot d\tilde{x})-p_0(\mu\times\tilde{x})\cdot d\tilde{x})\big)\wedge dr\Big\}\wedge dx_4\\
&+\frac{r}{x_4\sqrt{L}}\big\{\sqrt{M}\,(\lambda\cdot d\tilde{x})\wedge(p\cdot d\tilde{x})-a(p\cdot d\tilde{x})\wedge\big((\mu\times\tilde{x})\cdot d\tilde{x}\big)\big\}\wedge dr\\
&+\frac{1}{x_4}\Big\{\frac{\sqrt{M}}{a}(\lambda\cdot\mu)-\sqrt{L}\,(p\cdot\mu)+\frac{1}{\sqrt{L}}\big(p_0\sqrt{M}+a(p\cdot\tilde{x})\big)(\mu\cdot\tilde{x})\Big\}dx_1\wedge dx_2\wedge dx_3\\
&-\frac{1}{\sqrt{M}}\big((\lambda\cdot\tilde{x})+p_0\sqrt{L}\,\big)(\mu\cdot d_2\tilde{x})\wedge dx_4.
\end{aligned}$$

Next, since we have

$$\xi=\frac{\sqrt{M}}{x_4x_4\sqrt{L}}r\,dr-\frac{\sqrt{L}}{x_4\sqrt{M}}dx_4$$

by Theorem 1, we obtain from (3.4) the equality

$$\begin{aligned}
&\frac{1}{2}(x_4)^5\xi\wedge\theta\wedge d\theta\\
&=\frac{\sqrt{M}}{\sqrt{L}}r\,dr\wedge\Big\{-\frac{\sqrt{L}}{\sqrt{M}}(\lambda\cdot d\tilde{x})\wedge(p\cdot d\tilde{x})\\
&\quad+\frac{1}{\sqrt{M}}(\lambda\cdot d\tilde{x})\wedge\big((\mu\times\tilde{x})\cdot d\tilde{x}\big)\Big\}\wedge dx_4-\frac{x_4\sqrt{L}}{\sqrt{M}}dx_4\wedge\frac{r}{x_4\sqrt{L}}\\
&\quad\big\{\sqrt{M}(\lambda\cdot d\tilde{x})\wedge(p\cdot d\tilde{x})-a(p\cdot d\tilde{x})\wedge\big((\mu\times\tilde{x})\cdot d\tilde{x}\big)\big\}\wedge dr\\
&\quad-\frac{\sqrt{M}}{\sqrt{L}}(\tilde{x}\cdot d\tilde{x})\wedge\frac{1}{\sqrt{M}}\big((\lambda\cdot\tilde{x})+p_0\sqrt{L}\,\big)(\mu\cdot d_2\tilde{x})\wedge dx_4\\
&\quad+\frac{\sqrt{L}}{\sqrt{M}}\Big\{\frac{\sqrt{M}}{a}(\lambda\cdot\mu)-\sqrt{L}\,(p\cdot\mu)+\frac{1}{\sqrt{L}}\big(p_0\sqrt{M}+a(p\cdot\tilde{x})\big)(\mu\cdot\tilde{x})\Big\}\\
&\quad dx_1\wedge\cdots\wedge dx_4
\end{aligned}$$

$$= \left\{ \frac{r}{\sqrt{L}}(\lambda \cdot d\tilde{x}) \wedge \big((\mu \times \tilde{x}) \cdot d\tilde{x}\big) - \frac{ar}{\sqrt{M}}(p \cdot d\tilde{x}) \wedge \big((\mu \times \tilde{x}) \cdot d\tilde{x}\big) \right\} \wedge dr \wedge dx_4$$
$$+ \left\{ \frac{\sqrt{L}}{a}(\lambda \cdot \mu) - \frac{L}{\sqrt{M}}(p \cdot \mu) + (\mu \cdot \tilde{x})\left(\frac{a}{\sqrt{M}}(p \cdot \tilde{x}) - \frac{1}{\sqrt{L}}(\lambda \cdot \tilde{x}) \right) \right\}$$
$$dx_1 \wedge \cdots \wedge dx_4,$$

which is reduced to

(3.5)
$$\xi \wedge \theta \wedge d\theta = \frac{2}{(x_4)^5}\left\{ \frac{(\lambda \cdot \mu)}{a\sqrt{1+ar^2}} - \frac{(p \cdot \mu)}{\sqrt{1+ax_4x_4}} \right\} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

From this equality, we obtain the following theorem.

Theorem 3. *For the Killing field θ given by (3.2), Pfaffian equation $\theta = 0$ forms a complete system with $\xi = 0$, if its constants λ, p, μ and p_0 satisfy the following conditions: $p_0 \neq 0$ and*

$$\text{(i) } \mu = 0, \quad \text{or} \quad \text{(ii) } \mu \neq 0 \text{ and } (\lambda \cdot \mu) = (p \cdot \mu) = 0,$$

different from $\lambda = \mu = p = 0$ which gives $\theta = \xi$.

In the following we consider ξ and θ with $p_0 \neq 0$ and $\mu = 0$. Here we denote ξ, θ as contravariant vector fields by $X = \sum X^i\, \partial/\partial x_i$, $Y = \sum Y^i\, \partial/\partial x_i$ respectively. By Example 3 and (2.15) we have

$$X^i = \xi^i = \sqrt{L}\sqrt{M}\, x_i$$

and

$$Y^b = \sum_c g^{bc} v_c = x_4 x_4 \sum_c (\delta^{bc} + a x_b x_c) v_c$$
$$= \sum_c (\delta^{bc} + a x_b x_c) \left\{ -\frac{\sqrt{M}}{a}\lambda_c + \sqrt{L}\, p_c - \frac{1}{\sqrt{L}}\big(p_0\sqrt{M} + a(p \cdot \tilde{x})\big) x_c \right\}$$
$$= -\frac{\sqrt{M}}{a}\lambda_b + \sqrt{L}\, p_b - \frac{1}{\sqrt{L}}\big(p_0\sqrt{M} + a(p \cdot \tilde{x})\big) x_b$$
$$+ \left\{ -\sqrt{M}\,(\lambda \cdot \tilde{x}) + a\sqrt{L}\,(p \cdot \tilde{x}) - \frac{ar^2}{\sqrt{L}}\big(p_0\sqrt{M} + a(p \cdot \tilde{x})\big) \right\} x_b$$
$$= -\frac{\sqrt{M}}{a}\lambda_b + \sqrt{L}\, p_b - \sqrt{M}\,\big(p_0\sqrt{L} + (\lambda \cdot \tilde{x})\big) x_b,$$

$$Y^4 = g^{44} v_4 = -x_4 x_4 M \frac{1}{x_4\sqrt{M}}\big(p_0\sqrt{L} + (\lambda \cdot \tilde{x})\big) = -\sqrt{M}\,\big(p_0\sqrt{L} + (\lambda \cdot \tilde{x})\big) x_4.$$

From these expressions we compute the components of $[X,Y]$:

$$[X,Y]^i = \sum_j \Big(X^j \frac{\partial Y^i}{\partial x_j} - Y^i \frac{\partial X^i}{\partial x_j} \Big).$$

First we have

$$\begin{aligned}
&[X,Y]^b = \\
&\sqrt{L}\sqrt{M} \sum_c x_c \Big\{ \frac{a x_c}{\sqrt{L}} p_b - \sqrt{M} \Big(\frac{a p_0 x_c}{\sqrt{L}} + \lambda_c \Big) x_b - \sqrt{M}\,\big(p_0 \sqrt{L} + (\lambda \cdot \tilde{x})\big) \delta_{bc} \Big\} \\
&- \sum_c \Big\{ -\frac{\sqrt{M}}{a} \lambda_c + \sqrt{L}\, p_c - \sqrt{M} \big(p_0 \sqrt{L} + (\lambda \cdot \tilde{x}) x_c \Big\} \sqrt{M} \Big(\frac{a x_c}{\sqrt{L}} + \sqrt{L}\, \delta_{bc} \Big) \\
&+ \sqrt{L}\sqrt{M}\, x_4 \Big\{ -\frac{x_4}{\sqrt{M}} \lambda_b - \frac{a x_4}{\sqrt{M}} \big(p_0 \sqrt{L} + (\lambda \cdot \tilde{x})\big) x_b \Big\} \\
&+ \sqrt{M}\,\big(p_0 \sqrt{L} + (\lambda \cdot \tilde{x})\big) x_4 \frac{\sqrt{L}}{\sqrt{M}} a x_4 x_b,
\end{aligned}$$

which is arranged as follows:

$$[X,Y]^b = \frac{\sqrt{L}}{a} \lambda_b - \sqrt{M}\,\big(p_b + a(p \cdot \tilde{x}) x_b\big).$$

Next, we have

$$\begin{aligned}
[X,Y]^4 = &-\sqrt{L}\sqrt{M} \sum_c x_c \sqrt{M} \Big(\frac{a p_0 x_c}{\sqrt{L}} + \lambda_c \Big) x_4 \\
&- \sum_c \Big\{ -\frac{\sqrt{M}}{a} \lambda_c + \sqrt{L}\, p_c - \sqrt{M}\,\big(p_0 \sqrt{L} + (\lambda \cdot \tilde{x})\big) x_c \Big\} \sqrt{M}\, \frac{a x_c x_4}{\sqrt{L}} \\
&- \sqrt{L}\,\sqrt{M}\, x_4 \big(p_0 \sqrt{L} + (\lambda \cdot \tilde{x})\big) \Big(\sqrt{M} + \frac{a x_4 x_4}{\sqrt{M}} \Big) \\
&+ \sqrt{M} \big(p_0 \sqrt{L} + (\lambda \cdot \tilde{x})\big) x_4 \sqrt{L} \Big(\sqrt{M} + \frac{a x_4 x_4}{\sqrt{M}} \Big),
\end{aligned}$$

which is arranged as follows:

$$[X,Y]^4 = -a\sqrt{M}\,(p \cdot \tilde{x}) x_4.$$

From these expressions we obtain the relations

$$\text{(3.6)} \quad \begin{aligned}
[X,Y]^b - Y^b - p_0 X^b &= \frac{\sqrt{L} + \sqrt{M}}{a} (\lambda_b - a p_b) + \sqrt{M}\,\big((\lambda - ap) \cdot \tilde{x}\big) x_b, \\
[X,Y]^4 - Y^4 - p_0 X^4 &= \qquad\qquad\qquad\qquad\quad + \sqrt{M}\,\big((\lambda - ap) \cdot \tilde{x}\big) x_4.
\end{aligned}$$

Theorem 4. *For the Killing fields* $X = \sum_i \sqrt{L}\sqrt{M}\, x_i \partial/\partial x_i$ *and* $Y = \sum_i Y^i\, \partial/\partial x_i$ *as*

$$Y^b = \big(\sqrt{M} - \sqrt{L}\,\big)p_b + a\sqrt{M}\,(p\cdot\tilde{x})x_b, \quad Y^4 = a\sqrt{M}\,(p\cdot\tilde{x})x_4$$

we have the equality: $[X, Y] = Y$.

Proof. If we put $\lambda = ap$ in (3.6) and replace $Y + p_0 X$ by $-Y$, then we obtain $[X, Y] = Y$. Since p_0 is constant, $-Y - p_0 X$ is also a Killing field and its components become above. Q.E.D.

From Theorem 4 two vectors X and Y generate a Lie group of motion of dimension 2. We denote this new Killing field Y by η in the following.

§4. Integral submanifolds related with ξ and η

By the definition of ξ and η in §3, we have

$$\xi_b = \frac{\sqrt{M}}{x_4x_4\sqrt{L}}x_b, \quad \xi_4 = -\frac{\sqrt{L}}{x_4\sqrt{M}}, \quad \xi^i = \sqrt{L}\sqrt{M}\,x_i$$

and

$$\eta_b = \frac{1}{x_4x_4}\left\{\big(\sqrt{M} - \sqrt{L}\,\big)p_b + \frac{a}{\sqrt{L}}(p\cdot\tilde{x})x_b\right\}, \quad \eta_4 = -\frac{a}{x_4\sqrt{M}}(p\cdot\tilde{x}),$$
$$\eta^b = \big(\sqrt{M} - \sqrt{L}\,\big)p_b + a\sqrt{M}\,(p\cdot\tilde{x})x_b, \quad \eta^4 = a\sqrt{M}\,(p\cdot\tilde{x})x_4.$$

First, regarding Theorem 2, we integrate the Pfaff equation:

$$\sum_i \xi_i dx_i = \frac{\sqrt{M}}{x_4x_4\sqrt{L}}\sum_b x_b dx_b - \frac{\sqrt{L}}{x_4\sqrt{M}}dx_4 = 0,$$

which is written as

$$\frac{\sqrt{1 + ax_4x_4}}{x_4x_4\sqrt{1 + ar^2}}\, r\, dr - \frac{\sqrt{1 + ar^2}}{x_4\sqrt{1 + ax_4x_4}}dx_4 = 0.$$

From the above equality, we get

$$d\log(1 + ar^2) = d\log(1 + ax_4x_4)$$

and hence

$$1 + ax_4x_4 = c^2(1 + ar^2) \tag{4.1}$$

where $c > 0$ is an integral constant. We denote this hypersurface by Σ_c.

Second, regarding Theorem 3, we integrate the Pfaff equations:

$$\sum_i \xi_i dx_i = 0 \quad \text{and} \quad \sum_i \eta_i dx_i = 0. \tag{4.2}$$

From the first one we have (4.1) and $c^2 r\, dr = x_4 dx_4$. Using this we obtain

$$\begin{aligned}\sum_i \eta_i dx_i &= \frac{1}{x_4x_4}\left\{\big(\sqrt{M}-\sqrt{L}\big)(p\cdot d\tilde{x}) + \frac{a}{\sqrt{L}}(p\cdot\tilde{x})r\,dr\right\} - \frac{a}{x_4\sqrt{M}}(p\cdot\tilde{x})dx_4 \\ &= \frac{1}{x_4x_4}\left[\left\{(c-1)\sqrt{L}\,(p\cdot d\tilde{x}) + \frac{a}{\sqrt{L}}(p\cdot\tilde{x})r\,dr\right\} - \frac{ac}{\sqrt{L}}(p\cdot\tilde{x})r\,dr\right] \\ &= \frac{(c-1)}{x_4x_4}\left[\sqrt{L}\,(p\cdot d\tilde{x}) - \frac{a}{\sqrt{L}}(p\cdot\tilde{x})r\,dr\right] = 0,\end{aligned}$$

from which we obtain $c = 1$ or

$$\sqrt{L}\,(p\cdot d\tilde{x}) - \frac{a}{\sqrt{L}}(p\cdot\tilde{x})r\,dr = 0.$$

Integrating the above equation, we obtain $(p\cdot\tilde{x})^2 = (1+ar^2)\times\text{const}$. Setting

$$\bar{p} = p/\sqrt{(p\cdot p)},$$

we write the above equality as

$$1 + ar^2 = {c_1}^2(\bar{p}\cdot\tilde{x})^2, \tag{4.3}$$

where c_1 is an integral constant such that $c_1 > \sqrt{a}$, since this equality can be written as

$$({c_1}^2 - a)u^2 - av^2 = 1, \quad u = (\bar{p}\cdot\tilde{x}), \quad r^2 = u^2 + v^2,$$

and which is the equation of a rotating hyperbolic surface of order 2 with its center at the origin of R^3 and its axis is the line on p. We denote this surface in R^3 by Π_{c_1}. Hence, the solution of the Pfaff equation (4.2) is the intersection surface:

$$\Gamma(c, c_1) = \Sigma_c \cap (\Pi_{c_1} \times R). \tag{4.4}$$

For any point $\tilde{x} \in \Pi_{c_1}$, $\{\tilde{x}\} \times R \cap \Sigma_c$ is given by

$$c^2{c_1}^2(\bar{p}\cdot\tilde{x})^2 = 1 + ax_4x_4.$$

In order to get the value x_4, it is necessary and sufficient

$$|(\bar{p}\cdot\tilde{x})| > \frac{1}{cc_1}$$

which means that $\tilde{x}$ lies outside of the closed domain between the two planes:

$$(\bar{p} \cdot \tilde{x}) = \pm \frac{1}{cc_1}.$$

Third, regarding Theorem 4, we shall set up the surface generated by the tangent vector fields $X = \sum_i \xi^i \partial/\partial x_i$ and $Y = \sum_i \eta^i \partial/\partial x_i$. We see easily that the integral curves of X are straight half lines starting the origin of $R^3 \times R_+$. Since Y is written as

$$Y = a\sqrt{M}\,(p \cdot \tilde{x})x + \left(\sqrt{M} - \sqrt{L}\,\right)\begin{pmatrix} p \\ 0 \end{pmatrix},$$

we see that the integral surface through x is the upper half plane ${E_x}^2$ through the half straight line joining the point x and the origin and including the vector p.

Theorem 5. *The integral curve of the vector field* $Y = \sum_i \eta^i \partial/\partial x_i$ *is an algebraic plane curve of order 4 on the plane* ${E_y}^2$.

Proof. On the plane ${E_y}^2$ through a fixed point y we denote any point $x \in {E_y}^2$ as

$$x = \lambda_1 p + \lambda_2 y$$

and consider (λ_1, λ_2) as Descartes coordinate on ${E_y}^2$. Then we have

$$\begin{gathered} 1 + ar^2 = 1 + a\big(\lambda_1\lambda_1\,|p|^2 + 2\lambda_1\lambda_2(p \cdot \tilde{y}) + \lambda_2\lambda_2(\tilde{y} \cdot \tilde{y})\big) = L(\lambda_1, \lambda_2), \\ 1 + ax_4x_4 = 1 + a\lambda_2\lambda_2 y_4 y_4 = M(\lambda_2), \\ (p \cdot \tilde{x}) = \lambda_1\,|p|^2 + \lambda_2(p \cdot \tilde{y}) = \frac{1}{2a}\frac{\partial L}{\partial \lambda_1} \end{gathered}$$

and

$$\eta = \left(\sqrt{M} - \sqrt{L}\,\right)p + a\sqrt{M}\,(p \cdot \tilde{x})x = \left(\sqrt{M} - \sqrt{L}\,\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\sqrt{M}}{2}\frac{\partial L}{\partial \lambda_1}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.$$

Hence the differential equation

$$\frac{dx}{dt} = \eta$$

becomes

$$\frac{d\lambda_1}{dt} = \sqrt{M} - \sqrt{L} + \frac{1}{2}\lambda_1\sqrt{M}\frac{\partial L}{\partial \lambda_1}, \quad \frac{d\lambda_2}{dt} = \frac{1}{2}\lambda_2\sqrt{M}\frac{\partial L}{\partial \lambda_1},$$

form which we obtain

$$\begin{aligned}\frac{d}{dt}\big(\sqrt{M}-\sqrt{L}\,\big) &= \frac{1}{2\sqrt{M}}\frac{\partial M}{\partial\lambda_2}\frac{d\lambda_2}{dt} - \frac{1}{2\sqrt{L}}\left(\frac{\partial L}{\partial\lambda_1}\frac{d\lambda_1}{dt} + \frac{\partial L}{\partial\lambda_2}\frac{d\lambda_2}{dt}\right)\\ &= -\frac{1}{2\sqrt{L}}\frac{\partial L}{\partial\lambda_1}\left(\sqrt{M}-\sqrt{L}+\frac{1}{2}\lambda_1\sqrt{M}\frac{\partial L}{\partial\lambda_1}\right)\\ &\quad+\frac{1}{4}\left(\frac{1}{\sqrt{M}}\frac{\partial M}{\partial\lambda_2}-\frac{1}{\sqrt{L}}\frac{\partial L}{\partial\lambda_2}\right)\lambda_2\sqrt{M}\frac{\partial L}{\partial\lambda_1}\\ &= -\frac{1}{4\sqrt{L}}\left(\lambda_1\frac{\partial L}{\partial\lambda_1}+\lambda_2\frac{\partial L}{\partial\lambda_2}\right)\sqrt{M}\frac{\partial L}{\partial\lambda_1}\\ &\quad+\frac{1}{2}(M-1)\frac{\partial L}{\partial\lambda_1}-\frac{1}{2\sqrt{L}}\frac{\partial L}{\partial\lambda_1}\big(\sqrt{M}-\sqrt{L}\,\big)\\ &= \big(\sqrt{M}-\sqrt{L}\big)\frac{1}{2}\sqrt{M}\frac{\partial L}{\partial\lambda_1}\end{aligned}$$

and hence

$$\frac{d}{dt}\log\big(\sqrt{M}-\sqrt{L}\,\big) = \frac{1}{2}\sqrt{M}\frac{\partial L}{\partial\lambda_1} = \frac{d}{dt}\log\lambda_2.$$

Integrating the above equation, we obtain

$$\sqrt{L}-\sqrt{M} = c_2\lambda_2,$$

where c_2 is an integral constant. Then, we have

$$\sqrt{L}+\sqrt{M} = \frac{L-M}{\sqrt{L}-\sqrt{M}} = \frac{L-M}{c_2\lambda_2}$$

and

$$2\sqrt{M} = \frac{L-M}{c_2\lambda_2} - c_2\lambda_2 = \frac{L-M-(c_2\lambda_2)^2}{c_2\lambda_2},$$

from which we obtain

$$4M(c_2\lambda_2)^2 = (L-M)^2 - 2(L-M)(c_2\lambda_2)^2 + (c_2\lambda_2)^4,$$

which is also written as

$$(L-M)^2 - 2(L+M)(c_2\lambda_2)^2 + (c_2\lambda_2)^4 = 0$$

that is

$$\begin{aligned}(4.5)\quad & a^2\big(\lambda_1\lambda_1\,|p|^2 + 2\lambda_1\lambda_2(p\cdot\tilde{y}) + \lambda_2\lambda_2(y\mid y)\big)^2 + (c_2\lambda_2)^4\\ &- 2a\big(\lambda_1\lambda_1\,|p|^2 + 2\lambda_1\lambda_2(p\cdot\tilde{y}) + \lambda_2\lambda_2(y\cdot y)\big)(c_2\lambda_2)^2 - 4(c_2\lambda_2)^2 = 0,\end{aligned}$$

where we used the notations

$$(y \mid y) = \sum_b y_b y_b - y_4 y_4, \quad (y \cdot y) = \sum_b y_b y_b + y_4 y_4.$$

This expression shows the integral curve of the Killing field η is an algebraic plane curve of order 4 on ${E_y}^2$. Q.E.D.

Note. If we put $\lambda_1 = 0$, $\lambda_2 = 1$ in (4.5), we obtain

$$(c_2)^4 - 2\big(a(y \cdot y) + 2\big)(c_2)^2 + a^2(y \mid y)^2 = 0. \tag{4.6}$$

As a quadratic equation of $(c_2)^2$, its discriminant becomes

$$\begin{aligned} 4\big(a(y \cdot y) + 2\big)^2 - 4a^2(y \mid y)^2 &= 4a^2\big\{(y \cdot y)^2 - (y \mid y)^2\big\} + 16a(y \cdot y) + 16 \\ &= 16\big\{a^2(\tilde{y} \cdot \tilde{y})y_4 y_4 + a(y \cdot y) + 1\big\} > 0. \end{aligned}$$

Hence, (4.6) gives two positive roots $(c_2)^2$. We see that there exist four solution curves through the print y.

§5. Another pair of Killing fields

Now, we consider the second case in Theorem 3, the θ given by (3.2) satisfies

$$p_0 \neq 0, \quad \mu \neq 0 \quad \text{and} \quad (\lambda \cdot \mu) = (p \cdot \mu) = 0. \tag{5.1}$$

Then, by (3.5) we have

$$\xi \wedge \theta \wedge d\theta = 0$$

and hence the pair of Pfaff equations:

$$\xi = 0 \quad \text{and} \quad \theta = 0 \tag{5.2}$$

is completely integrable.

We consider the contravariant vector fields X and Y corresponding to ξ and θ, respectively. By (3.1) we have

$$X^i = \sqrt{L}\sqrt{M}\, x_i$$

and by (3.2) we have

$$\begin{aligned} \theta_b &= \frac{1}{x_4 x_4}\left[-\frac{\sqrt{M}}{a}\lambda_b + \sqrt{L}\, p_b - (\mu \times \tilde{x})_b - \frac{1}{\sqrt{L}}\big(p_0\sqrt{M} + a(p \cdot \tilde{x})\big)x_b\right], \\ \theta_4 &= \frac{1}{x_4\sqrt{M}}\big((\lambda \cdot \tilde{x}) + p_0\sqrt{L}\big), \end{aligned}$$

and

$$
\begin{aligned}
Y^b &= \sum_c g^{bc}\theta_c = x_4x_4\sum_c(\delta^{bc}+ax_bx_c)\theta_c \\
&= -\frac{\sqrt{M}}{a}\lambda_b + \sqrt{L}\,p_b - (\mu\times\tilde{x})_b - \frac{1}{\sqrt{L}}\big(p_0\sqrt{M}+a(p\cdot\tilde{x})\big)x_b \\
&\quad + ax_b\Big\{-\frac{\sqrt{M}}{a}(\lambda\cdot\tilde{x}) + \sqrt{L}\,(p\cdot\tilde{x}) - \frac{r^2}{\sqrt{L}}\big(p_0\sqrt{M}+a(p\cdot\tilde{x})\big)\Big\} \\
&= -\frac{\sqrt{M}}{a}\lambda_b + \sqrt{L}\,p_b - (\mu\times\tilde{x})_b - \sqrt{M}\,\big(p_0\sqrt{L}+(\lambda\cdot\tilde{x})\big)x_b, \\
Y^4 &= -\sqrt{M}\,\big(p_0\sqrt{L}+(\lambda\cdot\tilde{x})\big)x_4.
\end{aligned}
$$

If we use the notations

$$
\hat{\lambda} = \begin{pmatrix}\lambda\\0\end{pmatrix}, \quad \hat{p} = \begin{pmatrix}p\\0\end{pmatrix}, \quad (\mu\times\tilde{x})\hat{} = \begin{pmatrix}(\mu\times\tilde{x})\\0\end{pmatrix}
$$

then we have

(5.3)

$$
X = \sqrt{L}\sqrt{M}\,x, \quad Y = -\frac{\sqrt{M}}{a}\hat{\lambda} + \sqrt{L}\,\hat{p} - \sqrt{M}\big(p_0\sqrt{L}+(\lambda\cdot\tilde{x})\big)x - (\mu\times\tilde{x})\hat{}.
$$

We shall compute $[X,Y]$ as follows. Since we have the equalities:

$$
\begin{aligned}
&[x,\hat{\lambda}] = -\hat{\lambda}, \quad (x\cdot\nabla L) = 2ar^2, \quad (x\cdot\nabla M) = 2ax_4x_4, \\
&\big[x,(\mu\times\tilde{x})\hat{}\,\big] = 0,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&[X,Y] \\
&= -\frac{1}{a}\big[\sqrt{L}\sqrt{M}\,x,\sqrt{M}\,\hat{\lambda}\big] + \big[\sqrt{L}\sqrt{M}\,x,\sqrt{L}\,\hat{p}\big] - \big[\sqrt{L}\sqrt{M}\,x,\sqrt{M}\,(\lambda\cdot\tilde{x})x\big] \\
&\quad - \big[\sqrt{L}\sqrt{M}\,x,(\mu\times\tilde{x})\hat{}\,\big] \\
&= \frac{\sqrt{L}\,M}{a}\hat{\lambda} + \frac{\sqrt{M}}{a}\Bigg(\hat{\lambda}\cdot\nabla\big(\sqrt{L}\sqrt{M}\,\big)x - \frac{\sqrt{L}\sqrt{M}}{a}\big(x\cdot\nabla\sqrt{M}\,\big)\hat{\lambda} - L\sqrt{M}\,\hat{p} \\
&\quad - \sqrt{L}\,\Big(\hat{p}\cdot\nabla\big(\sqrt{L}\sqrt{M}\,\big)\Big)x + \sqrt{L}\sqrt{M}\,\big(x\cdot\nabla\sqrt{L}\,\big)\hat{p} \\
&\quad + \Big\{\Big(\sqrt{M}\,(\lambda\cdot\tilde{x})x\cdot\nabla\big(\sqrt{L}\sqrt{M}\,\big)\Big) - \Big(\sqrt{L}\sqrt{M}\,x\cdot\nabla\big(\sqrt{M}\,(\lambda\cdot\tilde{x})\big)\Big)\Big\}x \\
&\quad + \Big((\mu\times\tilde{x})\hat{}\cdot\nabla\big(\sqrt{L}\sqrt{M}\,\big)\Big)x
\end{aligned}
$$

$$
\begin{aligned}
&= \left\{ \frac{\sqrt{L}\,M}{a} - \frac{\sqrt{L}}{2a}(x \cdot \nabla M) \right\} \hat{\lambda} - \left\{ L\sqrt{M} - \frac{\sqrt{M}}{2}(x \cdot \nabla L) \right\} \hat{p} \\
&\quad + \left\{ \frac{\sqrt{M}}{a}\Big(\hat{\lambda} \cdot \nabla\big(\sqrt{L}\sqrt{M}\,\big)\Big) - \sqrt{L}\left(\hat{p} \cdot \nabla\big(\sqrt{L}\sqrt{M}\,\big)\right) \right. \\
&\quad + \sqrt{M}\,(\lambda \cdot \tilde{x})\Big(x \cdot \nabla\big(\sqrt{L}\sqrt{M}\,\big)\Big) - \sqrt{L}\sqrt{M}\left(x \cdot \nabla\big(\sqrt{M}\,(\lambda \cdot \tilde{x})\big)\right) \\
&\quad \left. + \Big((\mu \times \tilde{x})\hat{}\, \cdot \nabla\big(\sqrt{L}\sqrt{M}\,\big)\Big) \right\} x \\
&= \frac{\sqrt{L}}{a}\hat{\lambda} - \sqrt{M}\,\hat{p} \\
&\quad + \left\{ \frac{M}{\sqrt{L}}(\lambda \cdot \tilde{x}) - a\sqrt{M}\,(p \cdot \tilde{x}) + \frac{aMr^2}{\sqrt{L}}(\lambda \cdot \tilde{x}) - \sqrt{L}\,M(\lambda \cdot \tilde{x}) \right\} x \\
&= \frac{\sqrt{L}}{a}\hat{\lambda} - \sqrt{M}\,\hat{p} - a\sqrt{M}\,(p \cdot \tilde{x})x,
\end{aligned}
$$

that is

$$
[X, Y] = \frac{\sqrt{L}}{a}\hat{\lambda} - \sqrt{M}\,\hat{p} - a\sqrt{M}\,(p \cdot \tilde{x})x. \tag{5.4}
$$

This expression shows that even though if we suppose that $\lambda = ap$, X and Y could not generate a Lie algebra, then in fact we have

$$
\begin{aligned}
Y &= \big(\sqrt{L} - \sqrt{M}\,\big)\hat{p} - \sqrt{M}\,\big(p_0\sqrt{L} + a(p \cdot \tilde{x})\big)x - (\mu \times \tilde{x})\hat{}, \\
[X, Y] &= \big(\sqrt{L} - \sqrt{M}\,\big)\hat{p} - a\sqrt{M}\,(p \cdot \tilde{x})x,
\end{aligned}
$$

from which we obtain the identity

$$
[X, Y] - Y - p_0 X = (\mu \times \tilde{x})\hat{} \quad \text{with } \lambda = ap. \tag{5.4*}
$$

Now, we shall solve the Pfaff equations (5.2). We already knew that the solution of $\xi = 0$ is given by

$$
1 + ax_4x_4 = c^2(1 + ar^2) \quad \text{or} \quad M = c^2 L, \tag{5.5}
$$

where $c\ (> 0)$ is an integral constant. Using this relation, the equation

$$
\begin{aligned}
\theta = \sum_b \theta_b dx_b + \theta_4 dx_4 &= \frac{1}{x_4x_4}\left[-\frac{\sqrt{M}}{a}(\lambda \cdot d\tilde{x}) + \sqrt{L}\,(p \cdot d\tilde{x}) \right. \\
&\quad \left. - \frac{1}{\sqrt{L}}\big(p_0\sqrt{M} + a(p \cdot \tilde{x})\big)r\,dr - \big((\mu \times \tilde{x}) \cdot d\tilde{x}\big) \right] \\
&\quad + \frac{1}{x_4\sqrt{M}}\big((\lambda \cdot \tilde{x}) + p_0\sqrt{L}\,\big)dx_4 = 0
\end{aligned}
$$

can be replaced by

$$-\frac{c}{a}\sqrt{L}\,(\lambda\cdot d\tilde{x})+\sqrt{L}\,(p\cdot d\tilde{x})-\frac{1}{\sqrt{L}}\big(p_0c\sqrt{L}+a(p\cdot\tilde{x})\big)r\,dr \\ -\big((\mu\times\tilde{x})\cdot d\tilde{x}\big)+\frac{1}{c\sqrt{L}}\big((\lambda\cdot\tilde{x})+p_0\sqrt{L}\,\big)x_4dx_4=0.$$

We suppose here that λ and p are independent as vectors of R^3, that is $\lambda\times p\neq 0$. Then, by (5.1) we can put

$$\mu=\mu_0(\lambda\times p),\quad \mu_0\neq 0, \tag{5.6}$$

and so we have

$$(\mu\times\tilde{x})=\mu_0\big((\lambda\times p)\times\tilde{x}\big)=\mu_0\big\{(\lambda\cdot\tilde{x})p-(p\cdot\tilde{x})\lambda\big\} \\ \big((\mu\times\tilde{x})\cdot d\tilde{x}\big)=\mu_0\big\{(\lambda\cdot\tilde{x})(p\cdot d\tilde{x})-(p\cdot\tilde{x})(\lambda\cdot d\tilde{x})\big\}$$

Then, the above equation becomes

$$\sqrt{L}\left((p\cdot d\tilde{x})-\frac{c}{a}(\lambda\cdot d\tilde{x})\right)-\frac{1}{\sqrt{L}}\big(p_0c\sqrt{L}+a(p\cdot\tilde{x})\big)r\,dr \\ -\mu_0\big\{(\lambda\cdot\tilde{x})(p\cdot d\tilde{x})-(p\cdot\tilde{x})(\lambda\cdot d\tilde{x})\big\}+\frac{c}{\sqrt{L}}\big((\lambda\cdot\tilde{x})+p_0\sqrt{L}\,\big)r\,dr=0,$$

which is written as

$$\sqrt{L}\left(\frac{(\lambda\cdot d\tilde{x})}{a}-\frac{(p\cdot d\tilde{x})}{c}\right)-\left(\frac{(\lambda\cdot\tilde{x})}{a}-\frac{(p\cdot\tilde{x})}{c}\right)d\sqrt{L} \\ -\frac{\mu_0}{c}(\lambda\cdot\tilde{x})(p\cdot\tilde{x})\left(\frac{(\lambda\cdot d\tilde{x})}{(\lambda\cdot\tilde{x})}-\frac{(p\cdot d\tilde{x})}{(p\cdot\tilde{x})}\right)=0.$$

For simplicity we set

$$\frac{(\lambda\cdot\tilde{x})}{a}=u,\quad \frac{(p\cdot\tilde{x})}{c}=v,$$

then the above equality becomes

$$\omega:=\sqrt{L}\,(du-dv)-(u-v)\,d\sqrt{L}-a\mu_0(v\,du-u\,dv)=0. \tag{5.7}$$

Since the Pfaff equation (5.2) is completely integrable, we take an integral multiplier Φ, that is $\Phi\bar{\omega}$ is exact. Considering Φ as function of u, v and

$z = \sqrt{L}$, then we have

$$\begin{aligned} d(\Phi\omega) &= d\Phi \wedge \omega + \Phi d\omega = \left(\frac{\partial \Phi}{\partial u} du + \frac{\partial \Phi}{\partial v} dv + \frac{\partial \Phi}{\partial z} d\sqrt{L} \right) \wedge \omega + \Phi d\omega \\ &= \left\{ -\big(\sqrt{L} - a\mu_0 u\big) \frac{\partial \Phi}{\partial u} - \big(\sqrt{L} - a\mu_0 v\big) \frac{\partial \Phi}{\partial v} + 2a\mu_0 \Phi \right\} du \wedge dv \\ &\quad + \left\{ -(u-v) \frac{\partial \Phi}{\partial u} - \big(\sqrt{L} - a\mu_0 v\big) \frac{\partial \Phi}{\partial z} - 2\Phi \right\} du \wedge d\sqrt{L} \\ &\quad + \left\{ -(u-v) \frac{\partial \Phi}{\partial v} + \big(\sqrt{L} - a\mu_0 u\big) \frac{\partial \Phi}{\partial z} + 2\Phi \right\} dv \wedge d\sqrt{L}. \end{aligned}$$

Hence, $\Phi(u, v, z)$ must satisfy the following equalities:

$$\begin{aligned} (z - a\mu_0 u) \frac{\partial \Phi}{\partial u} + (z - a\mu_0 v) \frac{\partial \Phi}{\partial v} - 2a\mu_0 \Phi = 0, \\ (u - v) \frac{\partial \Phi}{\partial u} + (z - a\mu_0 v) \frac{\partial \Phi}{\partial z} + 2\Phi = 0, \\ -(u - v) \frac{\partial \Phi}{\partial v} + (z - a\mu_0 u) \frac{\partial \Phi}{\partial z} + 2\Phi = 0. \end{aligned} \tag{5.8}$$

In order to solve (5.8) with respect to Φ, here we take a change of variables u, v and z to

$$u^* = a\mu_0 u - z, \quad v^* = a\mu_0 v - z, \quad z^* = z,$$

then we have

$$\frac{\partial \Phi}{\partial u} = a\mu_0 \frac{\partial \Phi}{\partial u^*}, \quad \frac{\partial \Phi}{\partial v} = a\mu_0 \frac{\partial \Phi}{\partial v^*}, \quad \frac{\partial \Phi}{\partial z} = -\frac{\partial \Phi}{\partial u^*} - \frac{\partial \Phi}{\partial v^*} + \frac{\partial \Phi}{\partial z^*}$$

and

$$u - v = \frac{1}{a\mu_0}(u^* - v^*).$$

(5.8) turns into respectively

$$\begin{aligned} u^* \frac{\partial \Phi}{\partial u^*} + v^* \frac{\partial \Phi}{\partial v^*} + 2\Phi = 0, \\ u^* \frac{\partial \Phi}{\partial u^*} + v^* \frac{\partial \Phi}{\partial v^*} + 2\Phi - v^* \frac{\partial \Phi}{\partial z^*} = 0, \\ u^* \frac{\partial \Phi}{\partial u^*} + v^* \frac{\partial \Phi}{\partial v^*} + 2\Phi - u^* \frac{\partial \Phi}{\partial z^*} = 0, \end{aligned}$$

which are equivalent to

$$\frac{\partial \Phi}{\partial z^*} = 0 \quad \text{and} \quad u^* \frac{\partial \Phi}{\partial u^*} + v^* \frac{\partial \Phi}{\partial v^*} = -2\Phi.$$

Hence, we obtain the general solution of (5.8) given by

$$\begin{aligned}\Phi &= \frac{c_1}{(u^*)^2} + \frac{c_2}{(v^*)^2} + \frac{c_3}{u^* v^*} \\ &= \frac{c_1}{(a\mu_0 u - z)^2} + \frac{c_2}{(a\mu_0 v - z)^2} + \frac{c_3}{(a\mu_0 u - z)(a\mu_0 v - z)},\end{aligned}$$

where c_1, c_2 and c_3 are integral constant. Since we have

$$\begin{aligned}\omega &= z(du - dv) - (u - v)dz - a\mu_0(v\,du - u\,dv) \\ &= (z - a\mu_0 v)du - (z - a\mu_0 u)dv - (u - v)dz \\ &= -v^* \frac{du^* + dz^*}{a\mu_0} + u^* \frac{dv^* + dz^*}{a\mu_0} - \frac{u^* - v^*}{a\mu_0} dz^* = \frac{-v^* du^* + u^* dv^*}{a\mu_0},\end{aligned}$$

which implies the relations

$$\begin{aligned}&\frac{1}{(u^*)^2}\omega = \frac{1}{a\mu_0}\, d\left(\frac{v^*}{u^*}\right), \quad \frac{1}{(v^*)^2}\omega = -\frac{1}{a\mu_0}\, d\left(\frac{u^*}{v^*}\right), \\ &\frac{1}{u^* v^*}\omega = \frac{1}{a\mu_0}\, d \log \frac{v^*}{u^*}.\end{aligned}$$

Thus, we obtain the general solution of (5.7) as follows. Setting $F = F(x_1, x_2, x_3, c)$ by

$$F := \frac{v^*}{u^*} = \frac{a\mu_0 v - z}{a\mu_0 u - z} = \frac{a\mu_0 (p \cdot \tilde{x})/c - \sqrt{1 + ar^2}}{\mu_0(\lambda \cdot \tilde{x}) - \sqrt{1 + ar^2}},$$

(5.7) is equivalent to

$$d(c_1 F - c_2/F + c_3 \log F) = 0.$$

We reach the following conclusion:

Theorem 6. *The solutions of the pair of Pfaff equations*

$$\xi = 0 \quad \textit{and} \quad \theta = 0$$

with $p_0 \neq 0$, $\lambda \times p \neq 0$ and $\mu = \mu_0(\lambda \times p)$, $\mu_0 \neq 0$ are given by

$$\begin{aligned}&1 + a x_4 x_4 = c^2(1 + ar^2), \\ &a\mu_0(p \cdot \tilde{x}) - c\sqrt{1 + ar^2} = c_1\big(\mu_0(\lambda \cdot \tilde{x}) - \sqrt{1 + ar^2}\,\big),\end{aligned} \tag{5.9}$$

where $c > 0$ and c_1 are integral constants.

Finally, we investigate the rest case: $\lambda \times p = 0$ in the previous argument, that is

$$p_0 \neq 0,\ \mu \neq 0 \quad \text{and} \quad (\lambda \cdot \mu) = (p \cdot \mu) = 0,\ (\lambda \times p) = 0. \tag{5.10}$$

Choosing suitable coordinates (x_1, x_2, x_3) in R^3, we may put

$$\lambda = \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix}, \quad p = \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} \quad \text{with } \mu \neq 0.$$

Then, by (3.2) θ is expressed as

$$\theta = \frac{1}{x_4 x_4}\left[-\frac{\sqrt{M}}{a}\lambda\, dx_1 + \sqrt{L}\, p\, dx_1 - \mu(x_3 dx_1 - x_1 dx_3) \right.$$
$$\left. - \frac{1}{\sqrt{L}}(p_0\sqrt{M} + apx_1) r\, dr\right] + \frac{1}{x_4\sqrt{M}}\big(\lambda x_1 + p_0\sqrt{L}\big)dx_4.$$

In order to solve the Pfaff equations (5.2), we can put

$$M = c^2 L, \quad L = 1 + ar^2, \quad M = 1 + ax_4x_4, \quad r^2 = x_1x_1 + x_2x_2 + x_3x_3$$

and so $\theta = 0$ turns into

$$\left(-\frac{c\lambda}{a} + p\right)\sqrt{L}\, dx_1 - \mu(x_3dx_1 - x_1dx_3) - \frac{1}{\sqrt{L}}\big(cp_0\sqrt{L} + apx_1\big) r\, dr$$
$$+ \frac{c}{\sqrt{L}}\big(\lambda x_1 + p_0\sqrt{L}\big) r\, dr = 0,$$

that is

$$-\frac{1}{a}(c\lambda - ap)\sqrt{L}\, dx_1 - \mu(x_3 dx_1 - x_1 dx_3) + \frac{1}{\sqrt{L}}(c\lambda - ap)x_1 r\, dr = 0. \tag{5.11}$$

For simplicity we set

$$A = c\lambda - ap \quad \text{and}$$

$$\omega := -\frac{A}{a}\sqrt{L}\, dx_1 - \mu(x_3dx_1 - x_1dx_3) + \frac{A}{\sqrt{L}}x_1 r\, dr,$$
$$= \left(\frac{A}{\sqrt{L}}{x_1}^2 - \frac{A}{a}\sqrt{L} - \mu x_3\right)dx_1 + \frac{A}{\sqrt{L}}x_1x_2dx_2 + x_1\left(\mu + \frac{A}{\sqrt{L}}x_3\right)dx_3$$

then we obtain

$$d\omega = -\frac{Ar}{\sqrt{L}}dr \wedge dx_1 - 2\mu\, dx_3 \wedge dx_1 + \frac{Ar}{\sqrt{L}}dx_1 \wedge dr$$
$$= \frac{2A}{\sqrt{L}}x_2 dx_1 \wedge dx_2 - 2\left(\mu + \frac{A}{\sqrt{L}}x_3\right)dx_3 \wedge dx_1.$$

Now, let $\Phi(x_1, x_2, x_3)$ be an integral multiplier for $\omega = 0$. Then, we obtain from the above expressions

$$\begin{aligned}
d(\Phi\omega) &= \frac{\partial\Phi}{\partial x_1} dx_1 \wedge \omega + \frac{\partial\Phi}{\partial x_2} dx_2 \wedge \omega + \frac{\partial\Phi}{\partial x_3} dx_3 \wedge \omega + \Phi\, d\omega \\
&= \frac{\partial\Phi}{\partial x_1}\left\{ \frac{A}{\sqrt{L}} x_1 x_2 dx_1 \wedge dx_2 - x_1\left(\mu + \frac{A}{\sqrt{L}} x_3\right) dx_3 \wedge dx_1 \right\} \\
&\quad + \frac{\partial\Phi}{\partial x_2}\left\{ -\left(\frac{A}{\sqrt{L}} {x_1}^2 - \frac{A}{a}\sqrt{L} - \mu x_3\right) dx_1 \wedge dx_2 \right. \\
&\qquad\qquad \left. + x_1\left(\mu + \frac{A}{\sqrt{L}} x_3\right) dx_2 \wedge dx_3 \right\} \\
&\quad + \frac{\partial\Phi}{\partial x_3}\left\{ \left(\frac{A}{\sqrt{L}} {x_1}^2 - \frac{A}{a}\sqrt{L} - \mu x_3\right) dx_3 \wedge dx_1 - \frac{A}{\sqrt{L}} x_1 x_2 dx_2 \wedge dx_3 \right\} \\
&\quad + \Phi\left\{ \frac{2A}{\sqrt{L}} x_2 dx_1 \wedge dx_2 - 2\left(\mu + \frac{A}{\sqrt{L}} x_3\right) dx_3 \wedge dx_1 \right\},
\end{aligned}$$

from which we see that Φ must satisfy the following equalities:

$$\begin{aligned}
& x_1\left(\mu + \frac{A}{\sqrt{L}} x_3\right)\frac{\partial\Phi}{\partial x_2} - \frac{A}{\sqrt{L}} x_1 x_2 \frac{\partial\Phi}{\partial x_3} = 0, \\
& -x_1\left(\mu + \frac{A}{\sqrt{L}} x_3\right)\frac{\partial\Phi}{\partial x_1} + \left(\frac{A}{\sqrt{L}} {x_1}^2 - \frac{A}{a}\sqrt{L} - \mu x_3\right)\frac{\partial\Phi}{\partial x_3} \\
& \qquad\qquad -2\left(\mu + \frac{A}{\sqrt{L}} x_3\right)\Phi = 0, \\
& \frac{A}{\sqrt{L}} x_1 x_2 \frac{\partial\Phi}{\partial x_1} - \left(\frac{A}{\sqrt{L}} {x_1}^2 - \frac{A}{a}\sqrt{L} - \mu x_3\right)\frac{\partial\Phi}{\partial x_2} + \frac{2A}{\sqrt{L}} x_2 \Phi = 0.
\end{aligned} \tag{5.12}$$

Since we have

$$\frac{A}{\sqrt{L}} {x_1}^2 - \frac{A}{a}\sqrt{L} - \mu x_3 = -\frac{A}{\sqrt{L}}\left(\frac{1}{a} + x_2 x_2\right) - x_3\left(\mu + \frac{A}{\sqrt{L}} x_3\right)$$

the above equalities turns into respectively

(5.12')

$$\begin{aligned}
& \left(\mu + \frac{A}{\sqrt{L}} x_3\right)\frac{\partial\Phi}{\partial x_2} - \frac{A}{\sqrt{L}} x_2 \frac{\partial\Phi}{\partial x_3} = 0, \\
& \left(\mu + \frac{A}{\sqrt{L}} x_3\right)\left(x_1 \frac{\partial\Phi}{\partial x_1} + x_3 \frac{\partial\Phi}{\partial x_3} + 2\Phi\right) + \frac{A}{\sqrt{L}}\left(\frac{1}{a} + x_2 x_2\right)\frac{\partial\Phi}{\partial x_3} = 0, \\
& \frac{A}{\sqrt{L}} x_2\left(x_1 \frac{\partial\Phi}{\partial x_1} + x_2 \frac{\partial\Phi}{\partial x_2} + 2\Phi\right) + \left(x_3\left(\mu + \frac{A}{\sqrt{L}} x_3\right) + \frac{A}{a\sqrt{L}}\right)\frac{\partial\Phi}{\partial x_2} = 0.
\end{aligned}$$

Now, using the notations:

$$u = Ax_3 + \mu\sqrt{L}, \quad A = c\lambda - ap, \quad B = a\mu^2 - A^2$$

we change the variables (x_1, x_2, x_3) to the new ones $({x_1}^*, {x_2}^*, u^*)$ by

$${x_1}^* = x_1, \quad {x_2}^* = x_2x_2 + \frac{u^2}{B}, \quad u^* = x_2x_2 - \frac{u^2}{B},$$

where we suppose $B \neq 0$. Then, we have

$$\frac{\partial u}{\partial x_1} = \frac{a\mu}{\sqrt{L}}x_1, \quad \frac{\partial u}{\partial x_2} = \frac{a\mu}{\sqrt{L}}x_2, \quad \frac{\partial u}{\partial x_3} = A + \frac{a\mu}{\sqrt{L}}x_3$$

and

$$\begin{aligned}
\frac{\partial \Phi}{\partial x_1} &= \Phi_{{x_1}^*} + \frac{2u}{B}\frac{a\mu x_1}{\sqrt{L}}\Phi_{{x_2}^*} - \frac{2u}{B}\frac{a\mu x_1}{\sqrt{L}}\Phi_{u^*} = \Phi_{{x_1}^*} + \frac{2a\mu u}{B\sqrt{L}}x_1(\Phi_{{x_2}^*} - \Phi_{u^*}),\\
\frac{\partial \Phi}{\partial x_2} &= 2x_2\left(1 + \frac{a\mu u}{B\sqrt{L}}\right)\Phi_{{x_2}^*} + 2x_2\left(1 - \frac{a\mu u}{B\sqrt{L}}\right)\Phi_{u^*},\\
\frac{\partial \Phi}{\partial x_3} &= \frac{2u}{B}\left(A + \frac{a\mu}{\sqrt{L}}x_3\right)(\Phi_{{x_2}^*} - \Phi_{u^*}),
\end{aligned}$$

where we consider Φ as a function of $({x_1}^*, {x_2}^*, u^*)$ and denote its partial derivatives with respect to the new variables as $\Phi_{{x_1}^*}$, $\Phi_{{x_2}^*}$, Φ_{u^*}. The first equality of (5.12') becomes

$$\begin{aligned}
u\frac{\partial \Phi}{\partial x_2} - Ax_2\frac{\partial \Phi}{\partial x_3}&\\
&= 2ux_2\left\{\left(1 + \frac{a\mu u}{B\sqrt{L}}\right)\Phi_{{x_2}^*} + \left(1 - \frac{a\mu u}{B\sqrt{L}}\right)\Phi_{u^*}\right\}\\
&\quad - \frac{2Ax_2u}{B}\left(A + \frac{a\mu}{\sqrt{L}}x_3\right)(\Phi_{{x_2}^*} - \Phi_{u^*})\\
&= 2ux_2\left[\left\{1 + \frac{a\mu u}{B\sqrt{L}} - \frac{A}{B}\left(A + \frac{a\mu}{\sqrt{L}}x_3\right)\right\}\Phi_{{x_2}^*}\right.\\
&\qquad \left. + \left\{1 - \frac{a\mu u}{B\sqrt{L}} + \frac{A}{B}\left(A + \frac{a\mu}{\sqrt{L}}x_3\right)\right\}\Phi_{u^*}\right]\\
&= 2ux_2\left[\left\{1 - \frac{A^2}{B} + \frac{a\mu}{B\sqrt{L}}(u - Ax_3)\right\}\Phi_{{x_2}^*}\right.\\
&\qquad \left. + \left\{\frac{B + A^2}{B} + \frac{a\mu}{B\sqrt{L}}(Ax_3 - u)\right\}\Phi_{u^*}\right]\\
&= 2ux_2\left[\left\{1 - \frac{A^2}{B} + \frac{a\mu^2}{B}\right\}\Phi_{{x_2}^*} + \left\{\frac{a\mu^2}{B} + \frac{a\mu}{B\sqrt{L}}(-\mu\sqrt{L})\right\}\Phi_{u^*}\right.\\
&= 4ux_2\Phi_{{x_2}^*} = 0,
\end{aligned}$$

which implies

$$\Phi_{x_2{}^*} = \partial\Phi/\partial {x_2}^* = 0. \tag{5.13}$$

Using this equality, we have

$$\frac{\partial\Phi}{\partial x_1} = \Phi_{x_1{}^*} - \frac{2a\mu u x_1}{B\sqrt{L}}\Phi_{u^*}, \quad \frac{\partial\Phi}{\partial x_2} = 2x_2\left(1 - \frac{a\mu u}{B\sqrt{L}}\right)\Phi_{u^*},$$
$$\frac{\partial\Phi}{\partial x_3} = -\frac{2u}{B}\left(A + \frac{a\mu}{\sqrt{L}}x_3\right)\Phi_{u^*}.$$

Hence the second equality of (5.12') becomes

$$\begin{aligned}
&u\left(x_1\frac{\partial\Phi}{\partial x_1} + x_3\frac{\partial\Phi}{\partial x_3} + 2\Phi\right) + A\left(\frac{1}{a} + x_2x_2\right)\frac{\partial\Phi}{\partial x_3}\\
&\quad = ux_1\left(\Phi_{x_1{}^*} - \frac{2a\mu u}{B\sqrt{L}}x_1\Phi_{u^*}\right)\\
&\qquad - \left(ux_3 + A\left(\frac{1}{a} + x_2x_2\right)\right)\frac{2u}{B}\left(A + \frac{a\mu}{\sqrt{L}}x_3\right)\Phi_{u^*} + 2u\Phi\\
&\quad = ux_1\Phi_{x_1{}^*}\\
&\qquad - \left\{\frac{2a\mu u^2}{B\sqrt{L}}{x_1}^2 + \frac{2u}{B\sqrt{L}}\left(ux_3 + A\left(\frac{1}{a} + x_2x_2\right)\right)\left(a\mu x_3 + A\sqrt{L}\,\right)\right\}\Phi_{u^*}\\
&\qquad + 2u\Phi\\
&\quad = 0,
\end{aligned}$$

which is equivalent to

$${x_1}^*\Phi_{x_1{}^*} - 2H_2\Phi_{u^*} + 2\Phi = 0, \tag{5.14}$$

where we set

$$H_2 = \frac{1}{B\sqrt{L}}\left\{a\mu u{x_1}^2 + \left(ux_3 + A\left(\frac{1}{a} + x_2x_2\right)\right)\left(a\mu x_3 + A\sqrt{L}\,\right)\right\}.$$

The third equality of (5.12') becomes

$$\begin{aligned}
&Ax_2\left(x_1\frac{\partial\Phi}{\partial x_1} + x_2\frac{\partial\Phi}{\partial x_2} + 2\Phi\right) + \left(ux_3 + \frac{A}{a}\right)\frac{\partial\Phi}{\partial x_2}\\
&= Ax_1x_2\left(\Phi_{x_1{}^*} - \frac{2a\mu u x_1}{B\sqrt{L}}\Phi_{u^*}\right) + \left(A{x_2}^2 + ux_3 + \frac{A}{a}\right)2x_2\left(1 - \frac{a\mu u}{B\sqrt{L}}\right)\Phi_{u^*}\\
&\quad + 2Ax_2\Phi\\
&= Ax_1x_2\Phi_{x_1{}^*} - 2x_2\left\{\frac{a\mu A u{x_1}^2}{B\sqrt{L}} + \left(A{x_2}^2 + ux_3 + \frac{A}{a}\right)\left(\frac{a\mu u}{B\sqrt{L}} - 1\right)\right\}\Phi_{u^*}\\
&\quad + 2Ax_2\Phi\\
&= 0,
\end{aligned}$$

which is equivalent to

$$x_1{}^*\Phi_{x_1{}^*} - 2H_3\Phi_{u^*} + 2\Phi = 0, \tag{5.15}$$

where we set

$$H_3 = \frac{1}{B\sqrt{L}}\left\{a\mu u x_1{}^2 + \left(x_2{}^2 + \frac{ux_3}{A} + \frac{1}{a}\right)(a\mu u - B\sqrt{L}\,)\right\}.$$

From (5.14) and (5.15) we obtain the equality

$$(H_2 - H_3)\Phi_{u^*} = 0.$$

Since we have

$$\begin{aligned} B\sqrt{L}\,(H_2 - H_3) = &\left(ux_3 + A\left(\frac{1}{a} + x_2x_2\right)\right)(a\mu x_3 + A\sqrt{L}\,) \\ &- \left(x_2{}^2 + \frac{ux_3}{A} + \frac{1}{a}\right)(a\mu u - B\sqrt{L}\,) \end{aligned}$$

and

$$\begin{aligned} &\left(x_2{}^2 + \frac{ux_3}{A} + \frac{1}{a}\right)(a\mu u - B\sqrt{L}\,) \\ &\qquad = \frac{1}{A}\left(ux_3 + A\left(\frac{1}{a} + x_2x_2\right)\right)(a\mu A x_3 + a\mu^2\sqrt{L} - B\sqrt{L}\,) \\ &\qquad = \left(ux_3 + A\left(\frac{1}{a} + x_2x_2\right)\right)(a\mu x_3 + A\sqrt{L}\,), \end{aligned}$$

we obtain $H_2 = H_3$. Therefore (5.14) and (5.15) are identical.

Now we shall express H_2 by $x_1{}^*$, $x_2{}^*$ and u^*. We have

$$\begin{aligned} H_2 &= \frac{1}{B\sqrt{L}}\Big[a\mu u x_1{}^2 + a\mu u x_3{}^2 + \big(A\mu(1 + ax_2{}^2) + Au\sqrt{L}\,\big)x_3 \\ &\quad + \frac{A^2}{a}(1 + ax_2{}^2)\sqrt{L}\Big] \\ &= \frac{1}{B\sqrt{L}}\Big[a\mu u x_1{}^2 + a\mu u x_3{}^2 + \big(\mu(1 + ax_2{}^2) + u\sqrt{L}\,\big)(u - \mu\sqrt{L}\,) \\ &\quad + \frac{A^2}{a}(1 + ax_2{}^2)\sqrt{L}\Big] \\ &= \frac{1}{B\sqrt{L}}\Big[a\mu u x_1{}^2 + a\mu u x_3{}^2 + \mu u(1 + ax_2{}^2) - \mu u(1 + ax_1{}^2 + ax_2{}^2 + ax_3{}^2) \\ &\quad + \left\{u^2 - \mu^2(1 + ax_2{}^2) + \frac{A^2}{a}(1 + ax_2{}^2)\right\}\sqrt{L}\Big] \\ &= \frac{1}{B}\left\{u^2 - \left(\mu^2 - \frac{A^2}{a}\right)(1 + ax_2{}^2)\right\} = \frac{1}{B}u^2 - \frac{1}{a}(1 + ax_2{}^2) = -\frac{1}{a} - u^*. \end{aligned}$$

Thus, the equality (5.14) turns into

$$x_1{}^*\Phi_{x_1{}^*} + 2\left(\frac{1}{a} + u^*\right)\Phi_{u^*} + 2\Phi = 0. \tag{5.14'}$$

If we take $v = \sqrt{u^* + \frac{1}{a}}$ in place of u^*, then we have

$$2\left(\frac{1}{a} + u^*\right)\Phi_{u^*} = 2v^2\Phi_v \frac{1}{2\sqrt{u^* + 1/a}} = v\Phi_v$$

and hence (5.14') can be replaced by

$$x_1{}^*\Phi_{x_1{}^*} + v\Phi_v + 2\Phi = 0, \tag{5.14''}$$

whose solution is given by

$$\Phi = \frac{c_1}{x_1x_1} + \frac{c_2}{vv} + \frac{c_2}{x_1v}, \quad v = \sqrt{\frac{1}{a} + x_2x_2 - \frac{uu}{B}}, \quad u = Ax_3 + \mu\sqrt{L},$$

where c_1, c_2 and c_3 are integral constants.

Finally we shall integrate the Pfaff equation

$$\omega = 0.$$

Since we have

$$\begin{aligned}\frac{1}{x_1x_1}\omega &= -\frac{A}{a}\sqrt{L}\,\frac{dx_1}{x_1{}^2} - \mu\left(\frac{x_3}{x_1{}^2}dx_1 - \frac{1}{x_1}dx_3\right) + \frac{A}{\sqrt{L}}\frac{1}{x_1}r\,dr \\ &= \frac{A}{a\sqrt{L}}\,d\frac{1}{x_1} + \mu\left(x_3d\frac{1}{x_1} + \frac{1}{x_1}dx_3\right) + \frac{A}{a}\frac{1}{x_1}\,d\sqrt{L} = d\left(\frac{A}{a}\frac{\sqrt{L}}{x_1} + \mu\frac{x_3}{x_1}\right),\end{aligned}$$

the solution of the above Pfaff equation is given by

$$\frac{A}{a}\sqrt{L} + \mu x_3 = c_1x_1 \quad \text{or} \quad A^2(1 + ar^2) = a^2(c_1x_1 - \mu x_3)^2,$$

which is given by the original coordinates x_i by

$$\big(c\,|\lambda| - a\,|p|\,\big)^2(1 + ar^2) = \frac{a^2}{|\lambda|^2}\big\{c_1(\lambda\cdot\tilde{x}) - \big((\lambda\times\mu)\cdot\tilde{x}\big)\big\}^2,$$

where λ is supposed $\lambda \neq 0$ and c_1 is an integral constant.

Theorem 7. *The solutions of the pair of Pfaff equations*

$$\xi = 0 \quad \textit{and} \quad \theta = 0$$

with $p_0 \neq 0$, $\mu \neq 0$, $\lambda \neq 0$, $p \neq 0$, $(\lambda\times p) = 0$, $(\lambda\cdot\mu) = (p\cdot\mu) = 0$ *and* $a\,|\mu|^2 \neq \big(c\,|\lambda| - a\,|p|\,\big)^2$, *are given by*

$$\begin{aligned} &1 + ax_4x_4 = c^2(1 + ar^2), \\ &|\lambda|^2\big(c\,|\lambda| - a\,|p|\,\big)^2(1 + ar^2) = a^2\big\{c_1(\lambda\cdot\tilde{x}) - \big((\lambda\times\mu)\cdot\tilde{x}\big)\big\}^2. \end{aligned} \tag{5.16}$$

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