KILLING VECTOR FIELDS OF A SPACETIME

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Abstract. We studied the geodesics of a spacetime with the pseudo-Riemannian metric:

\[ ds^2 = \frac{1}{x_4 x_4} \left\{ \sum_{b,c=1}^{3} \left( \delta_{bc} - \frac{ax_b x_c}{1 + ar^2} \right) dx_b dx_c - \frac{1}{1 + ax_4 x_4} dx_4 dx_4 \right\} \]

on \( \mathbb{R}^3 \times \mathbb{R}_+ \), where \( r^2 = \sum_{b=1}^{3} x_b x_b \) and \( a \) = constant, which are plane quadratic curves (in [12]). In this paper, we shall determine all the Killing vector fields of this spacetime and choose special pairs out of them with interesting properties for the case \( a > 0 \).

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§0. Introduction

We investigated the pseudo-Riemannian metric on \( \mathbb{R}_+^n = \mathbb{R}^{n-1} \times \mathbb{R}_+ \) with the canonical coordinates \( (x_1, \ldots, x_{n-1}, x_n) \):

\[ ds^2 = \frac{1}{x_n^2} \left( \frac{1}{Q} dr \, dr + r^2 \sum_{\alpha, \beta=1}^{n-2} h_{\alpha \beta} du^\alpha du^\beta - P \, dx_n dx_n \right). \]

where \( Q \) and \( P \) are functions on \( \mathbb{R}_+^n - \{0\} \), \( r^2 = x_1^2 + \cdots + x_{n-1}^2 \) and

\[ ds^2 = \sum_{\alpha, \beta=1}^{n-2} h_{\alpha \beta} du^\alpha du^\beta \]

is the standard metric on the unit sphere \( S^{n-2} \); \( r^2 = 1 \) in \( \mathbb{R}^{n-1} \), satisfying the Einstein condition in [9], [10] and [11]. Especially for the metric with
\[ Q = Q(x, t) \text{ and } P = P(x, t), \quad x = r/x_n, \quad t = x_n, \] as a system of partial differential equations of order 2 on the components of the metric tensor the Einstein condition is reduced to the partial differential equation on \( Q \) as

\[
(2Q - \varphi)x^2 \frac{\partial^2 Q}{\partial x^2} - (3Q - 2\varphi)x^t \frac{\partial^2 Q}{\partial x \partial t} + (Q - \varphi)t^2 \frac{\partial^2 Q}{\partial t^2} \\
+ \left((2n - 4)Q - n\varphi\right)x^2 \frac{\partial Q}{\partial x} - \left((n - 4)Q - (n - 2)\varphi\right)t \frac{\partial Q}{\partial t} \\
- \frac{1}{Q} \left(x \frac{\partial Q}{\partial x} - t \frac{\partial Q}{\partial t}\right) \left(2(Q - \varphi)x^2 \frac{\partial Q}{\partial x} - (Q - 2\varphi)t \frac{\partial Q}{\partial t}\right) \\
+ 2(n - 3)Q(1 - \varphi) = 0,
\]

and \( P = x^2/(Q - \varphi) \), where \( \varphi(x) \) is an auxiliary free integral function of \( x \) derived from the original Einstein condition (Theorem 1 in [10]) which is correspond to the first integrals for the ordinary differential equations. This function \( \varphi \) becomes \( 1 - x^2 \) for the Minkowski metric

\[
ds^2 = \frac{1}{x_n^2} \left( \sum_{a=1}^{n-1} dx_a dx_a - dx_n dx_n \right).
\]

For \( n = 4 \) and \( \varphi = 1 - x^2 \), we obtain \( Q = 1 + ax^2 \) and \( P = 1 + a\ell^2 \) as the solution of the above partial differential equation ([11]) and the first metric is written as the one in Abstract in the coordinates \((x_1, x_2, x_3, x_4)\). If we change the coordinates as \( x_i \rightarrow \tilde{x}_i = \sqrt[4]{ax_i} \), we may consider as \( a = 1 \), but we do not use this device in order to avoid miscalculations and for the study of the interesting case: \( a < 0 \), for which the metric becomes Riemannian in some place in the coming work. Since this metric has constant curvature 1 by (1.4), it will be classified as one of de Sitter spacetimes in the theory of general relativity.

§1. Killing vector fields

Now, we call the above metric \( \text{Ot-metric} \) in this paper which satisfies the Einstein condition and denote it as

\[
ds^2 = \sum_{i,j=1}^{4} g_{ij} dx_i dx_j, \quad g_{ij} = g_{ji},
\]

where

\[
g_{bc} = \frac{1}{x_4 x_4} \left( \delta_{bc} - \frac{ax_b x_c}{1 + ax_4 x_4} \right), \quad g_{b4} = 0, \quad g_{44} = -\frac{1}{x_4 x_4(1 + ax_4 x_4)},
\]

\( b, c = 1, 2, 3 \),
from which \((g^{ij}) = (g_{ij})^{-1}\) is given by
\[
g^{bc} = x_A x_A (\delta^{bc} + ax_b x_c), \quad g^{b4} = 0, \quad g^{A4} = -x_A x_A (1 + ax_A x_A).
\]

We obtain easily the Christoffel symbols \(\{j^i_h\}\) of (1.2):
\[
\{j^i_h\} = \frac{1}{2} \sum_k g^{ik} \left( \frac{\partial g_{jk}}{\partial x_h} + \frac{\partial g_{kh}}{\partial x_j} - \frac{\partial g_{jh}}{\partial x_k} \right)
\]
as
\[
\{e^i_c\} = -ax_c \left( \delta_{bc} - \frac{ax_b x_c}{1 + ar^2} \right), \quad \{e^4_c\} = -\frac{1 + ax_A x_A}{x_A} \left( \delta_{bc} - \frac{ax_b x_c}{1 + ar^2} \right),
\]
\[
\{b^i 4\} = -\frac{1}{x_A^2} \delta^i_b, \quad \{b^4 4\} = 0,
\]
\[
\{4^i 4\} = 0, \quad \{4^4 4\} = -\frac{1 + 2ax_A x_A}{x_A(1 + ax_A x_A)}.
\]

The components \(R_{j_h k}^i\) of the curvature tensor:
\[
R_{j_h k}^i = \frac{\partial \{j^i_h\}}{\partial x_k} - \frac{\partial \{j^i_k\}}{\partial x_h} + \sum_l \{j^i_h\} \{l^k_j\} - \sum_l \{j^i_k\} \{l^h_j\}
\]
are computed by (1.3) as
\[
R_{a bc}^e = \delta_{b}^e g_{ac} - \delta_{c}^e g_{ab}, \quad R_{a bc}^4 = 0,
\]
\[
R_{A bc}^e = 0, \quad R_{A bc}^4 = 0, \quad R_{b^i 4c}^e = 0, \quad R_{b^4 4c} = g_{ce},
\]
\[
R_{i 4c}^e = -g_{i4} \delta^e_c, \quad R_{i 4c}^4 = 0,
\]
which are written simply as
\[
R_{j_h k}^i = \delta^i_{h} g_{jk} - \delta^i_{k} g_{jh}.
\]

We obtain the Ricci curvature \(R_{jk} = \sum_l R_{l}^j l k\) and the scalar curvature \(R = \sum_{j,k} g^{jk} R_{jk}\) as
\[
R_{jk} = 3g_{jk} \quad \text{and} \quad R = 12,
\]
which shows that the metric (1.1) satisfies the Einstein condition:
\[
R_{ij} = \frac{R}{4} g_{ij},
\]
Now, let $V = \sum v^i \frac{\partial}{\partial x^i}$ be a Killing field which satisfies the condition:

$$v_{i,j} + v_{j,i} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - 2 \sum_k \{^i_j\} v_k = 0.$$ 

By means of (1.2) and (1.3) this condition can be written as

(1.6) $$\frac{\partial v_b}{\partial x_c} + \frac{\partial v_c}{\partial x_b} + 2a \left( \delta_{bc} - \frac{ax_b x_c}{1 + ax^2} \right) \sum_e x_e v_e \delta_{bc} - \frac{ax_b x_c}{1 + ax^2} \right) v_4 = 0,$$

(1.7) $$\frac{\partial v_b}{\partial x_4} + \frac{\partial v_4}{\partial x_b} + \frac{2}{x_4} v_b = 0,$$

and

(1.8) $$\frac{\partial v_4}{\partial x_4} + \frac{1 + 2ax_4}{x_4(1 + ax_4 x_4)} v_4 = 0.$$

Integrating (1.8), we obtain easily

(1.9) $$v_4 = \frac{f}{x_4 \sqrt{1 + ax_4 x_4}}, \quad f = f(x_1, x_2, x_3).$$

Substituting this relation into (1.7) we obtain

$$\frac{\partial v_b}{\partial x_4} + \frac{2}{x_4} v_b + \frac{1}{x_4 \sqrt{1 + ax_4 x_4}} \frac{\partial f}{\partial x_b} = 0,$$

from which we obtain

$$\frac{\partial}{\partial x_4} (x_4 v_b) = -\frac{x_4}{\sqrt{1 + ax_4 x_4}} \frac{\partial f}{\partial x_b}$$

and integrating this relation we obtain

$$x_4 v_b = -\frac{\sqrt{1 + ax_4 x_4}}{a} \frac{\partial f}{\partial x_b} + f_b, \quad f_b = f_b(x_1, x_2, x_3),$$

i.e.

(1.10) $$v_b = -\frac{\sqrt{1 + ax_4 x_4}}{ax_4 x_4} \frac{\partial f}{\partial x_b} + \frac{f_b}{x_4 x_4}.$$

From (1.10) we obtain

$$\frac{\partial v_b}{\partial x_c} = -\frac{\sqrt{1 + ax_4 x_4}}{ax_4 x_4} \frac{\partial^2 f}{\partial x_b \partial x_c} + \frac{1}{x_4 x_4} \frac{\partial f_b}{\partial x_c}.$$
and substituting these relations into (1.6) we obtain the following conditions regarding integral free functions \( f(x_1, x_2, x_3) \) and \( f_b(x_1, x_2, x_3) \):

\[
(1.11) \quad -2\sqrt{1 + ax_4x_1} \frac{\partial^2 f}{\partial x_b \partial x_c} + \frac{\partial f_b}{\partial x_c} + \frac{\partial f_e}{\partial x_b} \\
+ 2 \left( \delta_{bc} - \frac{ax_bx_c}{1 + ar^2} \right) \left\{ \sqrt{1 + ax_4x_1} \left( f - \sum_e \frac{\partial f}{\partial x_e} x_e \right) + a \sum_e f_e x_e \right\} = 0,
\]

\( b, c = 1, 2, 3. \)

If we can find \( f, f_b \) satisfying (1.11), then we obtain the solution \( v_i \) satisfying (1.6)-(1.8). Noticing the indendency of variables, (1.11) can be replaced by

\[
(1.12) \quad \frac{\partial^2 f}{\partial x_b \partial x_c} = a \left( \delta_{bc} - \frac{ax_bx_c}{1 + ar^2} \right) \left( f - \sum_e \frac{\partial f}{\partial x_e} x_e \right),
\]

\[
(1.13) \quad \frac{\partial f_b}{\partial x_c} + \frac{\partial f_e}{\partial x_b} + 2a \left( \delta_{bc} - \frac{ax_bx_c}{1 + ar^2} \right) \sum_e f_e x_e = 0.
\]

We see that \( f \) and \( f_b, b = 1, 2, 3, \) can be treated separately.

§2. Solutions of the differential equations (1.12) and (1.13)

Supposing \( f(x_1, x_2, x_3) \) is analytic on \( x_1, x_2, x_3 \), we put

\[
f = \sum_{m=0}^{\infty} P_m(x_1, x_2, x_3),
\]

where \( P_m \) is a homogeneous polynomial of order \( m \) in \( x_1, x_2, x_3 \). Substituting this expression into (1.12), we obtain

\[
(1 + ar^2) \sum_{m=2}^{\infty} \frac{\partial^2 P_m}{\partial x_b \partial x_c} = a \left( 1 + ar^2 \right) \delta_{bc} - ax_bx_c \left( P_0 - \sum_{m=2}^{\infty} (m - 1)P_m \right),
\]

which we rewrite in considering the arrangement as

\[
(2.2) \quad \sum_{m=2}^{\infty} \frac{\partial^2 P_m}{\partial x_b \partial x_c} + ar^2 \sum_{m=2}^{\infty} \frac{\partial^2 P_m}{\partial x_b \partial x_c} = a \delta_{bc} \left( P_0 - \sum_{m=2}^{\infty} (m - 1)P_m \right)
\]

\[
+ a^2 \left( r^2 \delta_{bc} - x_bx_c \right) \left( P_0 - \sum_{m=2}^{\infty} (m - 1)P_m \right).
\]

Using the equalities

\[
(2.3) \quad \frac{\partial^2 P_m}{\partial x_b \partial x_c} = 2mr^{2m-4} \left( r^2 \delta_{bc} + 2(m - 1)x_bx_c \right), \quad m = 1, 2, 3, \ldots,
\]
we obtain \( P_m \) in turn up to \( m = 10 \) as follows:

\[
P_2 = \frac{a}{2} r^2, \quad P_3 = 0, \quad P_4 = -\frac{a^2}{8} r^4, \quad P_5 = 0, \quad P_6 = \frac{a^3}{16} r^6, \quad P_7 = 0, \quad P_8 = -\frac{5a^4}{128} r^8, \quad P_9 = 0, \quad P_{10} = \frac{7a^5}{256} r^{10}.
\]

Through the arguments determining these \( P_m \), we see that we can put

\[
P_{2m+1} = 0, \quad m = 1, 2, 3, \ldots
\]

and

\[
f = P_1 + \varphi(X), \quad X = r^2.
\]

Denoting the derivative of \( \varphi \) with respect to \( X \) by "\( \varphi' \)", we have

\[
\frac{\partial f}{\partial x_c} = \frac{\partial P_1}{\partial x_c} + 2\varphi' x_c,
\]

\[
\frac{\partial^2 f}{\partial x_b \partial x_c} = 2\varphi' \delta_{bc} + 4\varphi'' x_b x_c,
\]

and

\[
f - \sum_{e} \frac{\partial f}{\partial x_e} x_e = \varphi - 2\varphi' r^2.
\]

Substituting these into (1.12), we obtain

\[
(2\varphi' - a\varphi + 2a\varphi' X) \delta_{bc} + \left( 4\varphi'' + \frac{a^2}{1 + aX} (\varphi - 2\varphi' X) \right) x_b x_c = 0.
\]

Contracting this equality with \( c \) by multiplying with \( x_c \), we obtain

\[
(2\varphi' - a\varphi + 2a\varphi' X) x_b + (4\varphi'' + \frac{a^2}{1 + aX} (\varphi - 2\varphi' X)) r^2 x_b = 0,
\]

and hence

\[
2\varphi' - a\varphi + 2a\varphi' X + X \left( 4\varphi'' + \frac{a^2}{1 + aX} (\varphi - 2\varphi' X) \right) = 0.
\]

Substituting this expression into (2.4), we obtain

\[
\left\{ 4\varphi'' + \frac{a^2}{1 + aX} (\varphi - 2\varphi' X) \right\} (X \delta_{bc} - x_b x_c) = 0.
\]
Hence it must hold

\begin{equation}
4\varphi'' + \frac{a^2}{1 + aX} (\varphi - 2\varphi' X) = 0,
\end{equation}

\begin{equation}
2\varphi' - a\varphi + 2a\varphi' X = 0.
\end{equation}

From (2.6) we obtain by integration

\begin{equation}
\varphi = P_0 \sqrt{1 + aX} = P_0 \sqrt{1 + ar^2}.
\end{equation}

We can easily see that this \( \varphi \) satisfies (2.5). Thus we see that the general solution of (1.12) is given by

\begin{equation}
f(x_1, x_2, x_3) = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + P_0 \sqrt{1 + ar^2},
\end{equation}

where \( \lambda_1, \lambda_2, \lambda_3 \) and \( P_0 \) are integral constants.

Next, we shall treat (1.13). First, we put

\begin{equation}
f_b = \sum_{m=0}^{\infty} P_{bm},
\end{equation}

where \( P_{bm} \) is a homogeneous polynomial of order \( m \) in \( x_1, x_2, x_3 \). Substituting this expression into (1.13) and using the notation

\begin{equation}
Q_{m+1} = \sum_{e} P_{em} x_e
\end{equation}

for simplicity, we obtain

\[
\sum_{m=0}^{\infty} \left( \frac{\partial P_{bm}}{\partial x_c} + \frac{\partial P_{cm}}{\partial x_b} \right) + 2a \left( \delta_{bc} - \frac{ax_c x_e}{1 + ar^2} \right) \sum_{m=0}^{\infty} Q_{m+1} = 0,
\]

which we rewrite as, considering the arrangement,

\begin{equation}
\sum_{m=1}^{\infty} \left( \frac{\partial P_{bm}}{\partial x_c} + \frac{\partial P_{cm}}{\partial x_b} \right) + ar^2 \sum_{m=1}^{\infty} \left( \frac{\partial P_{bm}}{\partial x_c} + \frac{\partial P_{cm}}{\partial x_b} \right)
+ 2a \delta_{bc} \sum_{m=1}^{\infty} Q_m + 2a^2 \left( r^2 \delta_{bc} - x_b x_e \right) \sum_{m=1}^{\infty} Q_m = 0.
\end{equation}

From the terms of \( O^1 \)-order, we obtain

\begin{equation}
\frac{\partial P_{b1}}{\partial x_c} + \frac{\partial P_{c1}}{\partial x_b} = 0.
\end{equation}
From the terms of order 1 we obtain the relation:

\[
\frac{\partial P_{b2}}{\partial x_c} + \frac{\partial P_{b2}}{\partial x_b} + 2a_0 \delta_{bc} Q_1 = 0
\]

and, multiplying by \(x_c\) and contracting with respect to \(c\),

\[
P_{b2} + \frac{\partial Q_3}{\partial x_b} + 2ax_b Q_1 = 0
\]

and using the same way for \(b\)

\[
Q_3 = \frac{a}{2} r^2 Q_1.
\]

Going back to the previous equality we obtain

\[
P_{b2} = \frac{a}{2} (r^2 P_{b0} - 2x_b Q_1).
\]

We see easily that the above expression satisfies the first one.

Next, from the terms of order 2 we obtain the relation:

\[
\frac{\partial P_{b3}}{\partial x_c} + \frac{\partial P_{c3}}{\partial x_b} = \frac{a}{2} r^2 \left( \frac{\partial P_{b1}}{\partial x_c} + \frac{\partial P_{c1}}{\partial x_b} \right) + 2a \delta_{bc} Q_2 = 0,
\]

which becomes

\[
\frac{\partial P_{b3}}{\partial x_c} + \frac{\partial P_{c3}}{\partial x_b} = 0
\]

by means of (2.12) and \(Q_2 = \sum_b P_{b1} x_b = 0\). We obtain easily from these relations

\[
P_{b3} = 0 \quad \text{and} \quad Q_4 = 0.
\]

We obtain \(P_{b_m}\) in turn up to \(m = 8\) by analogous arguments as follows:

\[
P_{b4} = \frac{1}{8} a^2 r^2 (r^2 P_{b0} - 4x_b Q_1), \quad P_{b5} = 0,
\]

\[
P_{b6} = \frac{1}{16} a^3 r^4 (r^2 P_{b0} - 6x_b Q_1), \quad P_{b7} = 0,
\]

\[
P_{b8} = -\frac{5}{128} a^4 r^6 (r^2 P_{b0} - 8x_b Q_1).
\]

Through the arguments determining these \(P_m\), for any positive integer \(m\) we suppose that

\[
(2.13) \quad P_{b3} = P_{b5} = \cdots = P_{b(2m+1)} = 0, \\
P_{b(2n)} = (-1)^{n-1} k_n a^n r^{2n-2} (r^2 P_{b0} - 2nx_b Q_1), \quad n = 1, 2, 3, \ldots, m.
\]
From the terms of order $2m + 2$ of (2.11), we obtain
\[
\frac{\partial P_{b[2m+3]}^{}}{\partial x_c} + \frac{\partial P_{c[2m+3]}^{}}{\partial x_b} + a r^2 \left( \frac{\partial P_{b[2m+1]}^{}}{\partial x_c} + \frac{\partial P_{c[2m+1]}^{}}{\partial x_b} \right) + 2a \delta_{bc} Q_{2m+2} + 2a^2 (r^2 \delta_{bc} - x_b x_c) Q_{2m} = 0,
\]
which become by (2.13)
\[
\frac{\partial P_{b[2m+3]}^{}}{\partial x_c} + \frac{\partial P_{c[2m+3]}^{}}{\partial x_b} = 0.
\]
Multiplying this expression by $x_c$ and contracting with respect to $c$, we obtain
\[
(2m + 2) P_{b[2m+3]}^{(2m+4)} + \frac{\partial Q_{2m+4}^{}}{\partial x_b} = 0,
\]
which implies $Q_{2m+4} = 0$ and $P_{b[2m+3]} = 0$. Next, from the terms of order $2m + 1$, we obtain
\[
\frac{\partial P_{b[2m+2]}^{}}{\partial x_c} + \frac{\partial P_{c[2m+2]}^{}}{\partial x_b} + a r^2 \left( \frac{\partial P_{b[2m]}^{}}{\partial x_c} + \frac{\partial P_{c[2m]}^{}}{\partial x_b} \right) + 2a \delta_{bc} Q_{2m+1} + 2a^2 (r^2 \delta_{bc} - x_b x_c) Q_{2m-1} = 0.
\]
By means of (2.13) we obtain
\[
\frac{\partial P_{b[2m]}^{}}{\partial x_c} = (-1)^{m-1} k_m a^m r^{2m-4} 2 m \left\{ r^2 (P_{b0} x_c - P_{d0} x_b) - (r^2 \delta_{bc} + 2(m-1) x_b x_c) Q_1 \right\}
\]
and
\[
Q_{2m+1} = (-1)^m (2m - 1) k_m a^m r^{2m} Q_1,
Q_{2m-1} = (-1)^{m-1} (2m - 3) k_m^{m-1} a^{m-1} r^{2m-2} Q_1.
\]
Substituting these into the above expression, we obtain
\[
\frac{\partial P_{b[2m+2]}^{}}{\partial x_c} + \frac{\partial P_{c[2m+2]}^{}}{\partial x_b} + (-1)^{m+1} a^m r^{2m-2} \left\{ \left( (4m - 1) k_m - (2m - 3) k_{m-1} \right) x_b x_c \right\} Q_1 = 0.
\]
Multiplying this expression by $x_c$ and contracting with respect to $c$, we obtain
\[
(2m + 1) P_{b[2m+2]}^{(2m+3)} + \frac{\partial Q_{2m+3}^{}}{\partial x_b} + (-1)^m 2a^{m+1} r^{2m} (4m^2 - 1) k_m x_b Q_1 = 0,
\]
from which we obtain by the same way
\[ Q_{2m+3} = (-1)^{m+1} \frac{4m^2 - 1}{2(m+1)} k_m a^{m+1} r^{2m+2} Q_1 \]
and
\[ \frac{\partial Q_{2m+3}}{\partial x_b} = (-1)^{m+1} \frac{4m^2 - 1}{2(m+1)} k_m a^{m+1} r^{2m} (r^2 P_{b0} + 2(m+1)x_b Q_1). \]
Using these equalities, we obtain finally
\[ P_{b(2m+2)} = (-1)^m k_m \frac{2m-1}{2(m+1)} a^{m+1} r^{2m} (r^2 P_{b0} - 2(m+1)x_b Q_1) \]
and so we can put
\[ k_{m+1} = \frac{2m-1}{2(m+1)} k_m. \]
Thus, we have verified that (2.13) holds for all integers \( m > 0 \) and
\[ k_m = \frac{(2m-3)(2m-5)\cdots 1}{2m \cdot 2(m-1) \cdots 4} k_1 = \frac{(2m-3)!!}{2^m m!} \]
for \( m > 1 \), since \( k_1 = \frac{1}{2} \). Thus, we obtain the formulas:

\[ P_{b(2m)} = (-1)^{m-1} \frac{(2m-3)!!}{2^m m!} a^{m+2} r^{2m-2} (r^2 P_{b0} - 2mx_b Q_1), \quad m = 2, 3, \ldots \]

and
\[ P_{b2} = \frac{a}{2} (r^2 P_{b0} - 2x_b Q_1), \quad Q_1 = \sum e P_{e0} x_e. \]

Arranging the results in this section, we have the following theorem.

**Theorem 1.** For the spacetime on \( R^4_+ = R^3 \times R_+ \) with the metric (1.1) and (1.2) with \( a > 0 \), any Killing field \( V = \sum_i v^i \partial / \partial x_i \) is given by the formula:

\[ (v_b) = \frac{1}{x_4 x_4} \left\{ -\sqrt{1 + ax_4 x_4} \frac{\lambda + \sqrt{1 + ar^2}}{a} p - (\mu \times \bar{x}) \right. \\
- \frac{1}{\sqrt{1 + ar^2}} \left( p_0 \sqrt{1 + ax_4 x_4} + a (p \cdot \bar{x}) \right) \bar{x} \left\}, \quad v_4 = \frac{1}{x_4 \sqrt{1 + ax_4 x_4}} \left( (\lambda \cdot \bar{x}) + p_0 \sqrt{1 + ar^2} \right), \quad r^2 = (\bar{x} \cdot \bar{x}), \right. \]
where \( v_i = \sum_{j=1}^{4} g_{ij} v^j \) and \( \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \), \( p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \), \( \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} \), \( \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \), considered as vectors in \( \mathbb{R}^3 \) with the standard Euclidean metric: \( ds^2 = \sum_b dx_b dx_b \). \( \cdot \) and \( \times \) denote the inner product and the outer product of two vectors. Therefore \( V \) depends on 10 real constants \( p_0, \lambda_b, p_b, \mu_b, b = 1, 2, 3 \).

Proof. We have from (2.14)

\[
f_b = P_{b0} + P_{b1} + \frac{a}{2}(r^2 P_{b0} - 2x_b Q_1) + \sum_{m=2}^{\infty} (-1)^{m-1} \frac{(2m-3)!!}{2^m m!} \alpha^m r^{2m-2}(r^2 P_{b0} - 2mx_b Q_1) \]

\[
= P_{b1} + \left( 1 + \frac{a}{2} r^2 + \sum_{m=2}^{\infty} (-1)^{m-1} \frac{(2m-3)!!}{2^m m!} \alpha^m r^{2m-2} \right) P_{b0} - \left( a + \sum_{m=2}^{\infty} (-1)^{m-1} \frac{(2m-3)!!}{2^m m! (m-1)!} \alpha^m r^{2m-2} \right) x_b Q_1.
\]

Since we have

\[
(1 + t)^{\frac{1}{2}} = 1 + \frac{1}{2} t + \sum_{m=2}^{\infty} (-1)^{m-1} \frac{(2m-3)!!}{m! 2^m} t^m
\]

and

\[
(1 + t)^{-\frac{1}{2}} = 1 + \sum_{m=2}^{\infty} (-1)^{m-1} \frac{(2m-3)!!}{(m-1)! 2^{m-1}} t^{m-1},
\]

we obtain

\[
f_b = P_{b1} + (1 + ar^2)^{\frac{1}{2}} P_{b0} - a(1 + ar^2)^{-\frac{1}{2}} \left( \sum_e P_{e0} x_e \right) x_b.
\]

The \( 3 \times 3 \)-matrix \( \frac{\partial P_{b1}}{\partial x_c} \) is skew by (2.12), we denote it as

\[
\left( \frac{\partial P_{b1}}{\partial x_c} \right) = \begin{pmatrix} 0 & \mu_3 & -\mu_2 \\ -\mu_3 & 0 & \mu_1 \\ \mu_2 & -\mu_1 & 0 \end{pmatrix},
\]
then we have \((P_{b1}) = -(\mu \times \tilde{x})\). Setting \(P_{b0} = p_0\), we obtain

\[
(f_b) = -(\mu \times \tilde{x}) + \sqrt{1 + ar^2} p - \frac{a}{\sqrt{1 + ar^2}} (p \cdot \tilde{x}) \tilde{x}
\]

and

\[
f = \lambda \cdot \tilde{x} + p_0 \sqrt{1 + ar^2}, \quad p_0 = P_0.
\]

Finally from (1.10) and the above equalities we obtain

\[
(v_b) = -\sqrt{1 + ax_4x_4} \left( \frac{\lambda}{a} + \frac{ap_0}{\sqrt{1 + ar^2}} \tilde{x} \right)
\]

\[
+ \frac{1}{x_4^{x_4}} \left\{ -(\mu \times \tilde{x}) + \sqrt{1 + ar^2} p - \frac{a}{\sqrt{1 + ar^2}} (p \cdot \tilde{x}) \tilde{x} \right\}
\]

\[
= \frac{1}{x_4 \sqrt{1 + ax_4x_4}} \left\{ \sqrt{1 + ax_4x_4} \lambda + \sqrt{1 + ar^2} p - (\mu \times \tilde{x}) \right. 
\]

\[
- \frac{1}{\sqrt{1 + ar^2}} (p_0 \sqrt{1 + ax_4x_4} + a(p \cdot \tilde{x}) \tilde{x}) \right\}
\]

and

\[
v_4 = \frac{1}{x_4 \sqrt{1 + ax_4x_4}} \left( (\lambda \cdot \tilde{x}) + p_0 \sqrt{1 + ar^2} \right).
\]

Q.E.D.

Now we compute the norm of Killing field \(V\) given by (2.15):

\[(2.16) \quad N(V) = \sum_{i,j=1}^{4} g_{ij} v^i v^j = \sum_{i,j=1}^{4} g^{ij} v^i v^j.\]

Using the notations in Theorem 1 and setting \(\bar{v} = (v_b)\), we have

\[
N(V) = x_4 x_4 \sum_{b,c=1}^{3} (\bar{g}^{bc} + ax_b x_c) \bar{v}_b \bar{v}_c - x_4 x_4 (1 + ax_4 x_4) \bar{v}_4 \bar{v}_4
\]

and so

\[
N(V) = x_4 x_4 \left\{ (\bar{v} \cdot \bar{v}) + a(\bar{v} \cdot \tilde{x})^2 \right\} - \left( (\lambda \cdot \tilde{x}) + p_0 \sqrt{1 + ar^2} \right)^2
\]
\[
= \frac{1}{x_4 x_4} \left\{ \frac{1 + ax_4 x_4}{a^2} (\lambda \cdot \lambda) + (1 + ar^2)(p \cdot p) \\
+ \left( (\mu \times \tilde{x}) \cdot (\mu \times \tilde{x}) \right) + \frac{r^2}{1 + ar^2} \left( p_0 \sqrt{1 + ax_4 x_4} + a(p \cdot \tilde{x}) \right)^2 \\
- \frac{2\sqrt{1 + ax_4 x_4} \sqrt{1 + ar^2}}{a} (\lambda \cdot p) + \frac{2\sqrt{1 + ax_4 x_4}}{2} \left( (\lambda \times \mu) \cdot \tilde{x} \right) \\
+ \frac{2\sqrt{1 + ax_4 x_4}}{a \sqrt{1 + ar^2}} \left( p_0 \sqrt{1 + ax_4 x_4} + a(p \cdot \tilde{x}) \right) (\lambda \cdot \tilde{x}) \\
- \frac{2\sqrt{1 + ar^2}}{a} \left( p \times \mu \right) \cdot \tilde{x} - 2 \left( p_0 \sqrt{1 + ax_4 x_4} + a(p \cdot \tilde{x}) \right) (p \cdot \tilde{x}) \right\} \\
+ \frac{2\sqrt{1 + ax_4 x_4}}{a} \left( p_0 \sqrt{1 + ax_4 x_4} + a(p \cdot \tilde{x}) \right) \left( \frac{1}{a} (\lambda \cdot \tilde{x}) + \sqrt{1 + ar^2} (p \cdot \tilde{x}) \right)^2 \\
- \left( (\lambda \cdot \tilde{x}) + p_0 \sqrt{1 + ar^2} \right)^2 \\
\right\}
\]

which is arranged as follows

(2.17)

\[
x_4 x_4 N(v) = \frac{1 + ax_4 x_4}{a^2} (\lambda \cdot \lambda) + (1 + ar^2)(p \cdot p) + r^2(\mu \cdot \mu) \\
- (\mu \cdot \tilde{x})^2 + \frac{1}{a} (\lambda \cdot \tilde{x})^2 - a(p \cdot \tilde{x})^2 + \frac{2p_0 \sqrt{1 + ar^2}}{a} (\lambda \cdot \tilde{x}) \\
- 2p_0 \sqrt{1 + ax_4 x_4} (p \cdot \tilde{x}) + \frac{2\sqrt{1 + ax_4 x_4}}{a} ((\lambda \times \mu) \cdot \tilde{x}) \\
- 2\sqrt{1 + ar^2} \left( p \times \mu \right) \cdot \tilde{x} - 2\sqrt{1 + ar^2} \sqrt{1 + ax_4 x_4} (\lambda \cdot p) \\
+ p_0^2 (r^2 - x_4 x_4).
\]

**Example 1.** Case \( p_0 = 0, p = 0 \).

\[
\tilde{v} = (v_b) = \frac{1}{x_4 x_4} \left\{ - \frac{\sqrt{1 + ax_4 x_4}}{a} \lambda - (\mu \times \tilde{x}) \right\},
\]

\[
v_4 = \frac{1}{x_4 \sqrt{1 + ax_4 x_4}} (\lambda \cdot \tilde{x}),
\]

\[
N(V) = \frac{1}{x_4 x_4} \left\{ \frac{1 + ax_4 x_4}{a^2} (\lambda \cdot \lambda) + r^2(\mu \cdot \mu) - (\mu \cdot \tilde{x})^2 \\
+ \frac{1}{a} (\lambda \cdot \tilde{x})^2 + \frac{2\sqrt{1 + ax_4 x_4}}{a} ((\lambda \times \mu) \cdot \tilde{x}) \right\} \\
= \frac{1}{x_4 x_4} \left\{ \frac{\sqrt{1 + ax_4 x_4}}{a} \lambda + (\mu \times \tilde{x}) \right\}^2 + \frac{1}{a} (\lambda \cdot \tilde{x})^2 \right\},
\]
which implies that \( N(V) \geq 0 \) and \( N(V) = 0 \) is equivalent to
\[
\mu \times \bar{x} = -\frac{\sqrt{1 + \alpha x_4 x_4}}{a} \lambda
\]
and this relation implies \((\lambda \cdot \mu) = 0\). Hence, if \((\lambda \cdot \mu) \neq 0\), everywhere \( N(V) > 0 \), and so \( V \) is spacelike.

**Example 2.** Case \( p_0 \neq 0 \), \( \lambda = p = 0 \).
\[
\ddot{v} = (v_b) = \frac{1}{x_4 x_4} \left\{ -\left( \mu \times \bar{x} \right) - \frac{\sqrt{1 + \alpha x_4 x_4}}{\sqrt{1 + a r^2}} p_0 \bar{x} \right\},
\]
\[
v_4 = \frac{p_0 \sqrt{1 + a r^2}}{x_4 \sqrt{1 + \alpha x_4 x_4}},
\]
\[
N(V) = (\mu \times \bar{x})^2 + p_0^2 (r^2 - x_4 x_4),
\]
which shows that if \( r > x_4 \), \( V \) is spacelike.

**Example 3.** Case \( p_0 \neq 0 \), \( \lambda = p = \mu = 0 \).
\[
v_b = -\frac{p_0 \sqrt{1 + \alpha x_4 x_4}}{x_4 x_4 \sqrt{1 + a r^2}} \bar{x}_b, \quad v_4 = \frac{p_0 \sqrt{1 + a r^2}}{x_4 \sqrt{1 + \alpha x_4 x_4}}
\]
and
\[
v^i = -p_0 \sqrt{1 + a r^2} \sqrt{1 + \alpha x_4 x_4} x_i, \quad i = 1, \ldots, 4,
\]
\[
N(V) = \frac{p_0^2}{x_4 x_4} (r^2 - x_4 x_4).
\]

In the following sections, we shall investigate Killing fields of the spacetime with the Ot-metric (1.1) and (1.2), mainly noticing the Killing fields of the above examples and special pairs of two ones which construct a Lie algebra of dimension 2.

**§3. Special Killing fields**

We say a Killing field \( V \) given by (2.15) is *static*, if the Pfaff equation
\[
\sum_i v_i dx_i = 0
\]
is complete, that is, it admits locally a hypersurface satisfying this equation. As is well known, it is necessary and sufficient that the following equality holds
\[
\sum_{i=1}^4 v_i dx_i \wedge d \left( \sum_{j=1}^4 v_j dx_j \right) = 0.
\]
KILLING VECTOR FIELDS

Now, we denote the Killing field \( V \) of Example 3 in \( \S 2 \) with \( p_0 = -1 \) by \( \xi \) that is
\[
(3.1) \quad \xi^i = \sqrt{1 + ar^2} \sqrt{1 + ax_4 x_4}.
\]

**Theorem 2.** Killing field \( \xi \) is static.

**Proof.** We have
\[
\xi := \sum_{i=1}^{4} \xi_i dx_i = \frac{\sqrt{1 + ax_4 x_4}}{x_4 \sqrt{1 + ar^2}} \sum_{b=1}^{3} x_b dx_b - \frac{\sqrt{1 + ar^2}}{x_4 \sqrt{1 + ax_4 x_4}} dx_4,
\]
which is expressed only by \( r \) and \( x_4 \), since \( \sum x_b dx_b = r \, dr \). Hence, \( d\xi \) can be written as
\[
d\xi = \phi(r, x_4) \, dr \wedge dx_4,
\]
which implies the equality \( \xi \wedge d\xi = 0 \). Q.E.D.

Then, we take another Killing field \( V \) given by (2.15) and put
\[
\theta := \sum_{b} v_b dx_b + v_4 dx_4.
\]

We search for the condition that the system of Pfaff equations:
\[
\xi = 0 \quad \text{and} \quad \theta = 0
\]
is complete, that is, it admits locally a surface satisfying both equations. As is well known, it is necessary and sufficient that the following equalities hold:
\[
\xi \wedge \theta \wedge d\xi = 0 \quad \text{and} \quad \xi \wedge \theta \wedge d\theta = 0.
\]

From Theorem 2, the first equality holds. Regarding the second, we shall compute the three-form \( \theta \wedge d\theta \). For simplicity we use the notations
\[
L := 1 + ar^2, \quad M := 1 + ax_4 x_4 \quad \text{and} \quad d_2 \tilde{x} := \begin{pmatrix} dx_2 \wedge dx_3 \\ dx_3 \wedge dx_1 \\ dx_1 \wedge dx_2 \end{pmatrix},
\]
then \( \theta \) can be written as
\[
(3.2) \quad \theta = \frac{1}{x_4 x_4} \left[ -\frac{\sqrt{M}}{a} (\lambda \cdot d\tilde{x}) + \sqrt{L} (p \cdot d\tilde{x}) - ((\mu \times \tilde{x}) \cdot d\tilde{x}) - \frac{1}{\sqrt{L}} (p_0 \sqrt{M} + a (p \cdot \tilde{x})) r \, dr \right] + \frac{1}{x_4 \sqrt{M}} ((\lambda \cdot \tilde{x}) + p_0 \sqrt{L}) dx_4.
\]
from which we have

\[
d\theta = \frac{1}{a} \frac{\partial}{\partial d_4} \left( \frac{\sqrt{\mathcal{M}}}{x_4 x_4} (\lambda \cdot d_\mathbf{x}) \wedge dx_4 + d \frac{\sqrt{\mathcal{L}}}{x_4 x_4} \wedge (p \cdot d_\mathbf{x}) \right) \\
+ \frac{2}{(x_4)^3} dx_4 \wedge ((\mu \times \mathbf{x}) \cdot d_\mathbf{x}) - \frac{r}{\sqrt{\mathcal{L}}} d \frac{p_0 \sqrt{\mathcal{M}} + a(p \cdot \mathbf{x})}{x_4 x_4} \wedge dr \\
+ \frac{1}{x_4 \sqrt{\mathcal{M}}} \left( (\lambda \cdot d_\mathbf{x}) + \frac{ap_0 r}{\sqrt{\mathcal{L}}} dr \right) \wedge dx_4 - \frac{2}{x_4 x_4} (\mu \cdot d_2 \mathbf{x}) \\
= \left\{ \left( -\frac{2 \sqrt{\mathcal{M}}}{a(x_4)^3} + \frac{1}{x_4 \sqrt{\mathcal{M}}} \right) (\lambda \cdot d_\mathbf{x}) + \frac{2 \sqrt{\mathcal{L}}}{(x_4)^3} (p \cdot d_\mathbf{x}) - \frac{2}{(x_4)^3} ((\mu \times \mathbf{x}) \cdot d_\mathbf{x}) \right\} \\
+ \frac{r}{\sqrt{\mathcal{L}}} \left( \frac{ap_0}{x_4 \sqrt{\mathcal{M}}} - \frac{2p_0 \sqrt{\mathcal{M}}}{(x_4)^3} - \frac{2a(p \cdot \mathbf{x})}{(x_4)^3} \right) dr \\
+ \frac{1}{x_4 \sqrt{\mathcal{M}}} \left( (\lambda \cdot d_\mathbf{x}) + \frac{ap_0 r}{\sqrt{\mathcal{L}}} dr \right) \wedge dx_4 + \frac{2ar}{x_4 x_4 \sqrt{\mathcal{L}}} dr \wedge (p \cdot d_\mathbf{x}) \\
- \frac{2}{x_4 x_4} (\mu \cdot d_2 \mathbf{x}),
\]

and since we have

\[
d((\mu \times \mathbf{x}) \cdot d_\mathbf{x}) = 2(\mu \cdot d_2 \mathbf{x})
\]

which is arranged as

\[
(3.3) \quad \frac{x_4 x_4}{2} d\theta = \left\{ -\frac{1}{a x_4 \sqrt{\mathcal{M}}} (\lambda \cdot d_\mathbf{x}) + \frac{\sqrt{\mathcal{L}}}{x_4} (p \cdot d_\mathbf{x}) - \frac{1}{x_4} ((\mu \times \mathbf{x}) \cdot d_\mathbf{x}) \right\} \\
- \frac{r}{x_4 \sqrt{\mathcal{M}}} \left( \frac{p_0}{\sqrt{\mathcal{M}}} + a(p \cdot \mathbf{x}) \right) dr \wedge dx_4 + \frac{ar}{\sqrt{\mathcal{L}}} dr \wedge (p \cdot d_\mathbf{x}) - (\mu \cdot d_2 \mathbf{x}).
\]

Then, we obtain from (3.2) and (3.3)

\[
\frac{(x_4)^5}{2} \theta \wedge d\theta = \left[ -\frac{\sqrt{\mathcal{M}}}{a} (\lambda \cdot d_\mathbf{x}) + \sqrt{\mathcal{L}} (p \cdot d_\mathbf{x}) - ((\mu \times \mathbf{x}) \cdot d_\mathbf{x}) \right] \\
- \frac{r}{\sqrt{\mathcal{L}}} \left( (p_0 \sqrt{\mathcal{M}} + a(p \cdot \mathbf{x}) dr + \frac{x_4}{\sqrt{\mathcal{M}}} ((\lambda \cdot \mathbf{x}) + p_0 \sqrt{\mathcal{L}}) dx_4 \right) \\
\wedge \left[ \left\{ -\frac{1}{a \sqrt{\mathcal{M}}} (\lambda \cdot d_\mathbf{x}) + \sqrt{\mathcal{L}} (p \cdot d_\mathbf{x}) - ((\mu \times \mathbf{x}) \cdot d_\mathbf{x}) \right\} \wedge dx_4 + \frac{ar x_4}{\sqrt{\mathcal{L}}} dr \wedge (p \cdot d_\mathbf{x}) \\
- (x_4)^3 \theta \wedge (\mu \cdot d_2 \mathbf{x}),
\]

\[
\frac{(x_4)^5}{2} \theta \wedge d\theta = \left[ -\frac{\sqrt{\mathcal{M}}}{a} (\lambda \cdot d_\mathbf{x}) + \sqrt{\mathcal{L}} (p \cdot d_\mathbf{x}) - ((\mu \times \mathbf{x}) \cdot d_\mathbf{x}) \right] \\
- \frac{r}{\sqrt{\mathcal{L}}} \left( (p_0 \sqrt{\mathcal{M}} + a(p \cdot \mathbf{x}) dr + \frac{x_4}{\sqrt{\mathcal{M}}} ((\lambda \cdot \mathbf{x}) + p_0 \sqrt{\mathcal{L}}) dx_4 \right) \\
\wedge \left[ \left\{ -\frac{1}{a \sqrt{\mathcal{M}}} (\lambda \cdot d_\mathbf{x}) + \sqrt{\mathcal{L}} (p \cdot d_\mathbf{x}) - ((\mu \times \mathbf{x}) \cdot d_\mathbf{x}) \right\} \wedge dx_4 + \frac{ar x_4}{\sqrt{\mathcal{L}}} dr \wedge (p \cdot d_\mathbf{x}) \\
- (x_4)^3 \theta \wedge (\mu \cdot d_2 \mathbf{x}),
\]
which is arranged by using the relations:

\[(\mu \times \bar{x}) \cdot d\bar{x} \wedge (\mu \cdot d_2 \bar{x}) = 0, \quad (\lambda \cdot d\bar{x}) \wedge (\mu \cdot d_2 \bar{x}) = (\lambda \cdot \mu)dx_1 \wedge dx_2 \wedge dx_3,\]

\[(p \cdot d\bar{x}) \wedge (\mu \cdot d_2 \bar{x}) = (p \cdot \mu)dx_1 \wedge dx_2 \wedge dx_3, \quad r \, dr \wedge (\mu \cdot d_2 \bar{x}) =
\]

\[(\mu \cdot \bar{x})dx_1 \wedge dx_2 \wedge dx_3, \quad (\lambda \cdot d\bar{x}) \wedge (\mu \cdot d\bar{x}) = ((\lambda \times \mu) \cdot d_2 \bar{x})\]

after a little cumbersome computation as follows.

\[
\frac{(x_4)^3}{2} \theta \wedge d\theta = \left\{ -\frac{\sqrt{L}}{\sqrt{M}}(\lambda \cdot d\bar{x}) \wedge (p \cdot d\bar{x}) + \frac{1}{\sqrt{M}} (\lambda \cdot d\bar{x}) \wedge ((\mu \times \bar{x}) \cdot d\bar{x}) \right. \\
+ \frac{ar}{\sqrt{L} \sqrt{M}} ((p \cdot \bar{x})(\lambda \cdot d\bar{x}) - (\lambda \cdot \bar{x})(p \cdot d\bar{x}) - p_0(\mu \times \bar{x}) \cdot d\bar{x}) \wedge dr \\
+ \frac{r}{x_4 \sqrt{L}} \left\{ \sqrt{M} (\lambda \cdot d\bar{x}) \wedge (p \cdot d\bar{x}) - a(p \cdot d\bar{x}) \wedge ((\mu \times \bar{x}) \cdot d\bar{x}) \right\} \wedge dr \\
+ \frac{1}{x_4} \left\{ \frac{\sqrt{M}}{a} \lambda \cdot \mu \cdot d\bar{x}^2 + \frac{1}{\sqrt{L}} \left( p_0 \sqrt{M} + a(p \cdot \bar{x}) \right) (\mu \cdot \bar{x}) \right\} dx_1 \wedge dx_2 \wedge dx_3 \\
- \frac{1}{\sqrt{M}} ((\lambda \cdot \bar{x}) + p_0 \sqrt{L}) (\mu \cdot d_2 \bar{x}) \wedge dx_4.
\]

Next, since we have

\[\xi = \frac{\sqrt{M}}{x_4 x_4 \sqrt{L}} r \, dr - \frac{\sqrt{L}}{x_4 \sqrt{M}} dx_4 \]

by Theorem 1, we obtain from (3.4) the equality

\[\frac{1}{2} (x_4)^2 \xi \wedge \theta \wedge d\theta = \frac{\sqrt{M}}{\sqrt{L}} r \, dr \wedge \left\{ -\frac{\sqrt{L}}{\sqrt{M}} (\lambda \cdot d\bar{x}) \wedge (p \cdot d\bar{x}) \\
+ \frac{1}{\sqrt{M}} (\lambda \cdot d\bar{x}) \wedge ((\mu \times \bar{x}) \cdot d\bar{x}) \right\} \wedge dx_4 - \frac{x_4 \sqrt{L}}{\sqrt{M}} dx_4 \wedge \frac{r}{x_4 \sqrt{L}} \\
\left\{ \sqrt{M} (\lambda \cdot d\bar{x}) \wedge (p \cdot d\bar{x}) - a(p \cdot d\bar{x}) \wedge ((\mu \times \bar{x}) \cdot d\bar{x}) \right\} \wedge dr \\
- \frac{\sqrt{M}}{\sqrt{L}} (\bar{x} \cdot d\bar{x}) \wedge \frac{1}{\sqrt{M}} \left( (\lambda \cdot \bar{x}) + p_0 \sqrt{L} \right) (\mu \cdot d_2 \bar{x}) \wedge dx_4 \\
+ \frac{\sqrt{L}}{\sqrt{M}} \left\{ \frac{\sqrt{M}}{a} \lambda \cdot \mu \cdot d\bar{x}^2 + \frac{1}{\sqrt{L}} \left( p_0 \sqrt{M} + a(p \cdot \bar{x}) \right) (\mu \cdot \bar{x}) \right\} \right\} dx_1 \wedge \cdots \wedge dx_4 \]
\[
\begin{aligned}
&= \left\{ \frac{r}{\sqrt{L}} (\lambda \cdot d\tilde{x}) \land \left( (\mu \times \tilde{x}) \cdot d\tilde{x} \right) - \frac{ar}{\sqrt{M}} (p \cdot d\tilde{x}) \land \left( (\mu \times \tilde{x}) \cdot d\tilde{x} \right) \right\} \land dr \land dx_4 \\
&\quad + \left\{ \frac{\sqrt{L}}{a} (\lambda \cdot \mu) - \frac{L}{\sqrt{M}} (p \cdot \mu) + (\mu \cdot \tilde{x}) \left( \frac{a}{\sqrt{M}} (p \cdot \tilde{x}) - \frac{1}{\sqrt{L}} (\lambda \cdot \tilde{x}) \right) \right\} \\
&\quad dx_1 \land \cdots \land dx_4,
\end{aligned}
\]
which is reduced to
\begin{equation}
(3.5)
\xi \land \theta \land d\theta = \frac{2}{(x_4)^5} \left\{ \frac{(\lambda \cdot \mu)}{a\sqrt{1 + ar^2}} - \frac{(p \cdot \mu)}{\sqrt{1 + a^2x_4x_4}} \right\} dx_1 \land dx_2 \land dx_3 \land dx_4.
\end{equation}

From this equality, we obtain the following theorem.

**Theorem 3.** For the Killing field \( \theta \) given by (3.2), Pfaffian equation \( \theta = 0 \) forms a complete system with \( \xi = 0 \), if its constants \( \lambda, p, \mu \) and \( p_0 \) satisfy the following conditions: \( p_0 \neq 0 \) and

(i) \( \mu = 0 \), or

(ii) \( \mu \neq 0 \) and \( (\lambda \cdot \mu) = (p \cdot \mu) = 0 \),

different from \( \lambda = \mu = p = 0 \) which gives \( \theta = \xi \).

In the following we consider \( \xi \) and \( \theta \) with \( p_0 \neq 0 \) and \( \mu = 0 \). Here we denote \( \xi, \theta \) as contravariant vector fields by \( X = \sum X^i \partial / \partial x_i, Y = \sum Y^i \partial / \partial x_i \) respectively. By Example 3 and (2.15) we have

\[
X^i = \xi^i = \sqrt{L} \sqrt{M} x_i
\]
and

\[
Y^b = \sum_c g^{bc} v_c = x_4 x_4 \sum_c (\delta^{bc} + a x_b x_c) v_c
\]

\[
= \sum_c (\delta^{bc} + a x_b x_c) \left\{ -\frac{\sqrt{M}}{a} \lambda_c + \sqrt{L} p_c - \frac{1}{\sqrt{L}} (p_0 \sqrt{M} + a (p \cdot \tilde{x})) x_c \right\}
\]

\[
= -\frac{\sqrt{M}}{a} \lambda_b + \sqrt{L} p_b - \frac{1}{\sqrt{L}} (p_0 \sqrt{M} + a (p \cdot \tilde{x})) x_b
\]

\[
+ \left\{ -\sqrt{M} (\lambda \cdot \tilde{x}) + a \sqrt{L} (p \cdot \tilde{x}) - \frac{a r^2}{\sqrt{L}} (p_0 \sqrt{M} + a (p \cdot \tilde{x})) \right\} x_b
\]

\[
= -\frac{\sqrt{M}}{a} \lambda_b + \sqrt{L} p_b - \sqrt{M} (p_0 \sqrt{L} + (\lambda \cdot \tilde{x})) x_b,
\]

\[
Y^4 = g^{44} v_4 = -x_4 x_4 M \frac{1}{x_4 \sqrt{M}} (p_0 \sqrt{L} + (\lambda \cdot \tilde{x})) = -\sqrt{M} (p_0 \sqrt{L} + (\lambda \cdot \tilde{x})) x_4.
\]
From these expressions we compute the components of $[X, Y]$: 

$$[X, Y]^i = \sum_j \left( X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right).$$

First we have

$$[X, Y]^b = \sqrt{L} \sqrt{M} \sum_c x_c \left\{ \frac{a x_c}{\sqrt{L}} p_b - \sqrt{M} \left( \frac{a p_0 x_c}{\sqrt{L}} + \lambda_c \right) x_b - \sqrt{M} \left( p_0 \sqrt{L} + (\lambda \cdot \tilde{x}) \right) \delta_{bc} \right\}$$

$$- \sum_c \left\{ - \frac{\sqrt{M}}{a} \lambda_c + \sqrt{L} p_c - \sqrt{M} \left( p_0 \sqrt{L} + (\lambda \cdot \tilde{x}) x_c \right) \right\} \sqrt{M} \left( \frac{a x_c}{\sqrt{L}} + \sqrt{L} \delta_{bc} \right)$$

$$+ \sqrt{L} \sqrt{M} x_4 \left\{ - \frac{x_4}{\sqrt{M}} \lambda_b - \frac{a x_4}{\sqrt{M}} \left( p_0 \sqrt{L} + (\lambda \cdot \tilde{x}) \right) x_b \right\}$$

$$+ \sqrt{M} \left( p_0 \sqrt{L} + (\lambda \cdot \tilde{x}) \right) x_4 \sqrt{L} \left( \frac{a x_4}{\sqrt{M}} x_b, \right)$$

which is arranged as follows:

$$[X, Y]^b = \frac{\sqrt{L}}{a} \lambda_0 - \sqrt{M} \left( p_b + a (p \cdot \tilde{x}) x_b \right).$$

Next, we have

$$[X, Y]^4 = - \sqrt{L} \sqrt{M} \sum_c x_c \sqrt{M} \left( \frac{a p_0 x_c}{\sqrt{L}} + \lambda_c \right) x_4$$

$$- \sum_c \left\{ - \frac{\sqrt{M}}{a} \lambda_c + \sqrt{L} p_c - \sqrt{M} \left( p_0 \sqrt{L} + (\lambda \cdot \tilde{x}) x_c \right) \right\} \sqrt{M} \left( \frac{a x_4}{\sqrt{L}} x_4 \right)$$

$$- \sqrt{L} \sqrt{M} x_4 \left( p_0 \sqrt{L} + (\lambda \cdot \tilde{x}) \right) \left( \sqrt{M} + \frac{a x_4}{\sqrt{M}} \right)$$

$$+ \sqrt{M} \left( p_0 \sqrt{L} + (\lambda \cdot \tilde{x}) \right) x_4 \sqrt{L} \left( \sqrt{M} + \frac{a x_4}{\sqrt{M}} \right),$$

which is arranged as follows:

$$[X, Y]^4 = -a \sqrt{M} (p \cdot \tilde{x}) x_4.$$

From these expressions we obtain the relations

$$[X, Y]^b - Y^b - p_0 X^b = \frac{\sqrt{L} + \sqrt{M}}{a} \left( \lambda_0 - a p_0 \right) + \sqrt{M} \left( (\lambda - a p) \cdot \tilde{x} \right) x_b,$$

$$[X, Y]^4 - Y^4 - p_0 X^4 = \sqrt{M} \left( (\lambda - a p) \cdot \tilde{x} \right) x_4.$$
Theorem 4. For the Killing fields $X = \sum_i \sqrt{L} \sqrt{M} x_i \partial / \partial x_i$ and $Y = \sum_i Y^i \partial / \partial x_i$ as

$$Y^b = (\sqrt{M} - \sqrt{L}) p_b + a \sqrt{M} (p \cdot \vec{x}) x_b, \quad Y^i = a \sqrt{M} (p \cdot \vec{x}) x_i$$

we have the equality: $[X, Y] = Y$.

Proof. If we put $\lambda = ap$ in (3.6) and replace $Y + p_0 X$ by $-Y$, then we obtain $[X, Y] = Y$. Since $p_0$ is constant, $-Y - p_0 X$ is also a Killing field and its components become above. Q.E.D.

From Theorem 4 two vectors $X$ and $Y$ generate a Lie group of motion of dimension 2. We denote this new Killing field $Y$ by $\eta$ in the following.

§4. Integral submanifolds related with $\xi$ and $\eta$

By the definition of $\xi$ and $\eta$ in §3, we have

$$\xi_b = \frac{\sqrt{M}}{x_4 x_4 \sqrt{L}} x_b, \quad \xi_4 = - \frac{\sqrt{L}}{x_4 \sqrt{M}}, \quad \xi^i = \sqrt{L} \sqrt{M} x_i$$

and

$$\eta_b = \frac{1}{x_4 x_4} \left\{ (\sqrt{M} - \sqrt{L}) p_b + \frac{a}{\sqrt{L}} (p \cdot \vec{x}) x_b \right\}, \quad \eta_4 = - \frac{a}{x_4 \sqrt{M}} (p \cdot \vec{x}),$$

$$\eta^b = (\sqrt{M} - \sqrt{L}) p_b + a \sqrt{M} (p \cdot \vec{x}) x_b, \quad \eta^4 = a \sqrt{M} (p \cdot \vec{x}) x_4.$$

First, regarding Theorem 2, we integrate the Pfaff equation:

$$\sum_i \xi_i dx_i = \frac{\sqrt{M}}{x_4 x_4 \sqrt{L}} \sum_b x_b dx_b - \frac{\sqrt{L}}{x_4 \sqrt{M}} dx_4 = 0,$$

which is written as

$$\frac{\sqrt{1 + ax x_4}}{x_4 x_4 \sqrt{1 + ar^2}} r dr - \frac{\sqrt{1 + ar^2}}{x_4 \sqrt{1 + ax x_4}} dx_4 = 0.$$

From the above equality, we get

$$d \log(1 + ar^2) = d \log(1 + ax x_4)$$

and hence

$$1 + ax x_4 = c^2 (1 + ar^2)$$
where \( c > 0 \) is an integral constant. We denote this hypersurface by \( \Sigma_c \).

Second, regarding Theorem 3, we integrate the Pfaff equations:

\[
\sum_i \xi_i dx_i = 0 \quad \text{and} \quad \sum_i \eta_i dx_i = 0.
\]

From the first one we have (4.1) and \( c^2 r dr = x_4 dx_4 \). Using this we obtain

\[
\sum_i \eta_i dx_i = \frac{1}{x_4 x_4} \left\{ \left( \sqrt{M} - \sqrt{\mathcal{L}} \right) (p \cdot d\tilde{x}) + \frac{a}{\sqrt{\mathcal{L}}} (p \cdot \tilde{x}) r \, dr \right\} - \frac{a}{x_4 \sqrt{M}} (p \cdot \tilde{x}) dx_4
\]

\[
= \frac{1}{x_4 x_4} \left[ \left( (c - 1) \sqrt{L} (p \cdot d\tilde{x}) + \frac{a}{\sqrt{L}} (p \cdot \tilde{x}) r \, dr \right) - \frac{ac}{\sqrt{L}} (p \cdot \tilde{x}) r \, dr \right]
\]

\[
= \frac{(c - 1)}{x_4 x_4} \left[ \sqrt{L} (p \cdot d\tilde{x}) - \frac{a}{\sqrt{L}} (p \cdot \tilde{x}) r \, dr \right] = 0,
\]

from which we obtain \( c = 1 \) or

\[
\sqrt{L} (p \cdot d\tilde{x}) - \frac{a}{\sqrt{L}} (p \cdot \tilde{x}) r \, dr = 0.
\]

Integrating the above equation, we obtain \( (p \cdot \tilde{x})^2 = (1 + ar^2) \times \text{const.} \) Setting

\[
\tilde{p} = p/\sqrt{(p \cdot p)},
\]

we write the above equality as

\[
1 + ar^2 = c_1^2 (p \cdot \tilde{x})^2,
\]

where \( c_1 \) is an integral constant such that \( c_1 > \sqrt{a} \), since this equality can be written as

\[
(c_1^2 - a)u^2 - ar^2 = 1, \quad u = (p \cdot \tilde{x}), \quad r^2 = u^2 + v^2,
\]

and which is the equation of a rotating hyperbolic surface of order 2 with its center at the origin of \( R^3 \) and its axis is the line on \( p \). We denote this surface in \( R^3 \) by \( \Pi_{c_1} \). Hence, the solution of the Pfaff equation (4.2) is the intersection surface:

\[
\Gamma(c, c_1) = \Sigma_c \cap (\Pi_{c_1} \times R).
\]

For any point \( \tilde{x} \in \Pi_{c_1}, \{ \tilde{x} \} \times R \cap \Sigma_c \) is given by

\[
c^2 c_1^2 (\tilde{p} \cdot \tilde{x})^2 = 1 + ar_4 x_4.
\]

In order to get the value \( x_4 \), it is necessary and sufficient

\[
|\tilde{p} \cdot \tilde{x}| > \frac{1}{cc_1}
\]
which means that $\bar{x}$ lies outside of the closed domain between the two planes:

$$(\bar{p} \cdot \bar{x}) = \pm \frac{1}{\alpha_1}.$$ 

Third, regarding Theorem 4, we shall set up the surface generated by the tangent vector fields $X = \sum_i \xi^i \partial / \partial x_i$ and $Y = \sum_i \eta^i \partial / \partial x_i$. We see easily that the integral curves of $X$ are straight half lines starting the origin of $R^3 \times R_+$. Since $Y$ is written as

$$Y = a\sqrt{M} (p \cdot \bar{x}) x + \left( \sqrt{M} - \sqrt{L} \right) \begin{pmatrix} p \\ 0 \end{pmatrix},$$

we see that the integral surface through $x$ is the upper half plane $E_x^2$ through the half straight line joining the point $x$ and the origin and including the vector $p$.

**Theorem 5.** The integral curve of the vector field $Y = \sum_i \eta^i \partial / \partial x_i$ is an algebraic plane curve of order 4 on the plane $E_y^2$.

**Proof.** On the plane $E_y^2$ through a fixed point $y$ we denote any point $x \in E_y^2$ as

$$x = \lambda_1 p + \lambda_2 y$$

and consider $(\lambda_1, \lambda_2)$ as Descartes coordinate on $E_y^2$. Then we have

$$1 + ar^2 = 1 + a(\lambda_1 \lambda_1 |p|^2 + 2\lambda_1 \lambda_2 (p \cdot \bar{y}) + \lambda_2 \lambda_2 (\bar{y} \cdot \bar{y})) = L(\lambda_1, \lambda_2),$$

$$1 + ar^2 = 1 + a\lambda_2 \lambda_2 y_4 y_4 = M(\lambda_2),$$

$$(p \cdot \bar{x}) = \lambda_1 |p|^2 + \lambda_2 (p \cdot \bar{y}) = \frac{1}{2a} \frac{\partial L}{\partial \lambda_1}$$

and

$$\eta = (\sqrt{M} - \sqrt{L}) p + a\sqrt{M} (p \cdot \bar{x}) x = (\sqrt{M} - \sqrt{L}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\sqrt{M}}{2} \frac{\partial L}{\partial \lambda_1} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.$$ 

Hence the differential equation

$$\frac{dx}{dt} = \eta$$

becomes

$$\frac{d\lambda_1}{dt} = \sqrt{M} - \sqrt{L} + \frac{1}{2} \lambda_1 \sqrt{M} \frac{\partial L}{\partial \lambda_1}, \quad \frac{d\lambda_2}{dt} = \frac{1}{2} \lambda_2 \sqrt{M} \frac{\partial L}{\partial \lambda_1}.$$
form which we obtain
\[
\frac{d}{dt}(\sqrt{M} - \sqrt{L}) = -\frac{1}{2\sqrt{M}} \frac{\partial M}{\partial \lambda_2} \frac{d\lambda_2}{dt} - \frac{1}{2\sqrt{L}} \left( \frac{\partial L}{\partial \lambda_1} \frac{d\lambda_1}{dt} + \frac{\partial L}{\partial \lambda_2} \frac{d\lambda_2}{dt} \right)
\]
\[
= -\frac{1}{2\sqrt{L}} \frac{\partial L}{\partial \lambda_1} \left( \sqrt{M} - \sqrt{L} + \frac{1}{2} \lambda_1 \sqrt{M} \frac{\partial L}{\partial \lambda_1} \right) + \frac{1}{4} \left( \frac{1}{\sqrt{M}} \frac{\partial M}{\partial \lambda_2} - \frac{1}{\sqrt{L}} \frac{\partial L}{\partial \lambda_2} \right) \lambda_2 \sqrt{M} \frac{\partial L}{\partial \lambda_1}
\]
\[
= -\frac{1}{4\sqrt{L}} \left( \lambda_1 \frac{\partial L}{\partial \lambda_1} + \lambda_2 \frac{\partial L}{\partial \lambda_2} \right) \sqrt{M} \frac{\partial L}{\partial \lambda_1} + \frac{1}{2} (M - 1) \frac{\partial L}{\partial \lambda_1} - \frac{1}{2\sqrt{L}} \frac{\partial L}{\partial \lambda_1} (\sqrt{M} - \sqrt{L})
\]
\[
= (\sqrt{M} - \sqrt{L}) \frac{1}{2\sqrt{M}} \frac{\partial L}{\partial \lambda_1}
\]
and hence
\[
\frac{d}{dt} \log(\sqrt{M} - \sqrt{L}) = \frac{1}{2} \sqrt{M} \frac{\partial L}{\partial \lambda_1} = \frac{d}{dt} \log \lambda_2.
\]
Integrating the above equation, we obtain
\[
\sqrt{L} - \sqrt{M} = c_2 \lambda_2,
\]
where \(c_2\) is an integral constant. Then, we have
\[
\sqrt{L} + \sqrt{M} = \frac{L - M}{\sqrt{L} - \sqrt{M}} = \frac{L - M}{c_2 \lambda_2}
\]
and
\[
2\sqrt{M} = \frac{L - M}{c_2 \lambda_2} - c_2 \lambda_2 = \frac{L - M - (c_2 \lambda_2)^2}{c_2 \lambda_2},
\]
from which we obtain
\[
4M(c_2 \lambda_2)^2 = (L - M)^2 - 2(L - M)(c_2 \lambda_2)^2 + (c_2 \lambda_2)^4,
\]
which is also written as
\[
(L - M)^2 - 2(L + M)(c_2 \lambda_2)^2 + (c_2 \lambda_2)^4 = 0
\]
that is
\[
(4.5) \quad a^2 \left( \lambda_1 \lambda_1 \left| p \right|^2 + 2\lambda_1 \lambda_2 (p \cdot \hat{y}) + \lambda_2 \lambda_2 (y \cdot y) \right)^2 + (c_2 \lambda_2)^4
\]
\[
- 2a \left( \lambda_1 \lambda_1 \left| p \right|^2 + 2\lambda_1 \lambda_2 (p \cdot \hat{y}) + \lambda_2 \lambda_2 (y \cdot y) \right) (c_2 \lambda_2)^2 - 4(c_2 \lambda_2)^2 = 0,
\]
where we used the notations
\[
(y \mid y) = \sum_b y_b y_b - y_4 y_4, \quad (y \cdot y) = \sum_b y_b y_b + y_4 y_4.
\]

This expression shows the integral curve of the Killing field \( \eta \) is an algebraic plane curve of order 4 on \( E^2_y \).

Q.E.D.

**Note.** If we put \( \lambda_1 = 0, \lambda_2 = 1 \) in (4.5), we obtain
\[
(\circ_2)^4 - 2(a(y \cdot y) + 2)(\circ_2)^2 + a^2(y \mid y)^2 = 0.
\]

As a quadratic equation of \((\circ_2)^2\), its discriminant becomes
\[
4(a(y \cdot y) + 2)^2 - 4a^2(y \mid y)^2 = 4a^2 \{ (y \cdot y)^2 - (y \mid y)^2 \} + 16a(y \cdot y) + 16
\]
\[
= 16\{a^2(y \cdot y) + 1\} > 0.
\]

Hence, (4.6) gives two positive roots \((\circ_2)^2\). We see that there exist four solution curves through the point \( y \).

§5. **Another pair of Killing fields**

Now, we consider the second case in Theorem 3, the \( \theta \) given by (3.2) satisfies
\[
p_0 \neq 0, \quad \mu \neq 0 \quad \text{and} \quad (\lambda \cdot \mu) = (p \cdot \mu) = 0.
\]

Then, by (3.5) we have
\[
\xi \wedge \theta \wedge d\theta = 0
\]
and hence the pair of Pfaff equations:
\[
\xi = 0 \quad \text{and} \quad \theta = 0
\]
is completely integrable.

We consider the contravariant vector fields \( X \) and \( Y \) corresponding to \( \xi \) and \( \theta \), respectively. By (3.1) we have
\[
X^i = \sqrt{L} \sqrt{M} x_i
\]
and by (3.2) we have
\[
\theta_b = \frac{1}{x_4 x_4} \left[ -\frac{\sqrt{M}}{a} \lambda_b + \sqrt{L} p_b - (\mu \times \bar{x})_b - \frac{1}{\sqrt{L}} (p_0 \sqrt{M} + a(p \cdot \bar{x})) x_b \right],
\]
\[
\theta_4 = \frac{1}{x_4 \sqrt{M}} ((\lambda \cdot \bar{x}) + p_0 \sqrt{L}),
\]
and
\[ Y^b = \sum_c g^{bc} \theta_c = x_4 x_4 \sum_c (\delta^{bc} + ax_b x_c) \theta_c \]
\[ = -\frac{\sqrt{M}}{a} \lambda_b + \sqrt{L} p_b - (\mu \times \bar{x})_b - \frac{1}{\sqrt{L}} (p_0 \sqrt{M} + a(p \cdot \bar{x})) x_b \]
\[ + ax_b \left\{ -\frac{\sqrt{M}}{a} (\lambda \cdot \bar{x}) + \sqrt{L} (p \cdot \bar{x}) - \frac{r^2}{\sqrt{L}} (p_0 \sqrt{M} + a(p \cdot \bar{x})) \right\} \]
\[ = -\frac{\sqrt{M}}{a} \lambda_b + \sqrt{L} p_b - (\mu \times \bar{x})_b - \sqrt{M} (p_0 \sqrt{L} + (\lambda \cdot \bar{x})) x_b, \]
\[ Y^4 = -\sqrt{M} (p_0 \sqrt{L} + (\lambda \cdot \bar{x})) x_4. \]

If we use the notations
\[ \dot{\lambda} = \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}, \quad \dot{p} = \begin{pmatrix} p_1 \\ 0 \end{pmatrix}, \quad (\mu \times \bar{x})^* = \begin{pmatrix} (\mu \times \bar{x})_1 \\ 0 \end{pmatrix} \]
then we have
\[ (5.3) \]
\[ X = \sqrt{L} \sqrt{M} x, \quad Y = -\frac{\sqrt{M}}{a} \dot{\lambda} + \sqrt{L} \dot{p} - \sqrt{M} (p_0 \sqrt{L} + (\lambda \cdot \bar{x})) x - (\mu \times \bar{x})^*. \]

We shall compute \([X, Y]\) as follows. Since we have the equalities:
\[ [x, \dot{\lambda}] = -\dot{\lambda}, \quad (x \cdot \nabla L) = 2ar^2, \quad (x \cdot \nabla M) = 2ax_4 x_4, \]
\[ [x, (\mu \times \bar{x})] = 0, \]
we obtain
\[ [X, Y] \]
\[ = -\frac{1}{a} \left[ \sqrt{L} \sqrt{M} x, \sqrt{M} \dot{\lambda} \right] + \left[ \sqrt{L} \sqrt{M} x, \sqrt{L} \dot{p} \right] - \left[ \sqrt{L} \sqrt{M} x, \sqrt{M} (\lambda \cdot \bar{x}) x \right] \]
\[ - \left[ \sqrt{L} \sqrt{M} x, (\mu \times \bar{x}) \right] \]
\[ = \frac{\sqrt{L} \sqrt{M}}{a} \dot{\lambda} + \frac{\sqrt{M}}{a} \left( \dot{\lambda} \cdot \nabla (\sqrt{L} \sqrt{M}) x - \frac{\sqrt{L} \sqrt{M}}{a} (x \cdot \nabla \sqrt{M}) \dot{\lambda} - L \sqrt{M} \dot{p} \right) \]
\[ - \sqrt{L} \left( \dot{p} \cdot \nabla (\sqrt{L} \sqrt{M}) \right) x + \sqrt{L} \sqrt{M} (x \cdot \nabla \sqrt{L}) \dot{p} \]
\[ + \left\{ \left( \sqrt{M} (\lambda \cdot \bar{x}) x \cdot \nabla (\sqrt{L} \sqrt{M}) \right) - \left( \sqrt{L} \sqrt{M} x \cdot \nabla (\sqrt{M} (\lambda \cdot \bar{x})) \right) \right\} x \]
\[ + \left( (\mu \times \bar{x})^* \cdot \nabla (\sqrt{L} \sqrt{M}) \right) x \]
\[
\begin{align*}
&= \left\{ \frac{\sqrt{L}M}{a} - \frac{\sqrt{L}}{2a} \left( x \cdot \nabla M \right) \right\} \dot{\lambda} - \left\{ L\sqrt{M} - \frac{\sqrt{M}}{2} \left( x \cdot \nabla L \right) \right\} \dot{\rho} \\
&+ \left\{ \frac{\sqrt{M}}{a} \left( \dot{\lambda} \cdot \nabla \left( \sqrt{L} \sqrt{M} \right) \right) - \sqrt{L} \left( \dot{\rho} \cdot \nabla \left( \sqrt{L} \sqrt{M} \right) \right) \\
&+ \sqrt{M} (\lambda \cdot \ddot{x}) \left( x \cdot \nabla \left( \sqrt{L} \sqrt{M} \right) \right) - \sqrt{L} \sqrt{M} \left( x \cdot \nabla \left( \sqrt{M} (\lambda \cdot \ddot{x}) \right) \right) \\
&+ \left( (\mu \times \ddot{x}) \cdot \nabla \left( \sqrt{L} \sqrt{M} \right) \right) \right\} x \\
&= \frac{\sqrt{L}}{a} \dot{\lambda} - \sqrt{M} \dot{\rho} \\
&+ \left\{ \frac{M}{\sqrt{L}} (\lambda \cdot \ddot{x}) - a \sqrt{M} (p \cdot \ddot{x}) + \frac{aMr^2}{\sqrt{L}} (\lambda \cdot \ddot{x}) - \sqrt{L} M (\lambda \cdot \ddot{x}) \right\} x \\
&= \frac{\sqrt{L}}{a} \dot{\lambda} - \sqrt{M} \dot{\rho} - a \sqrt{M} (p \cdot \ddot{x}) x,
\end{align*}
\]

that is

\[(5.4) \quad [X,Y] = \frac{\sqrt{L}}{a} \dot{\lambda} - \sqrt{M} \dot{\rho} - a \sqrt{M} (p \cdot \ddot{x}) x.
\]

This expression shows that even though if we suppose that \( \lambda = ap \), \( X \) and \( Y \) could not generate a Lie algebra, then in fact we have

\[
Y = \left( \sqrt{L} - \sqrt{M} \right) \dot{\rho} - \sqrt{M} (p_0 \sqrt{L} + a(p \cdot \ddot{x})) x - (\mu \times \ddot{x}),
\]

\[ [X,Y] = \left( \sqrt{L} - \sqrt{M} \right) \dot{\rho} - a \sqrt{M} (p \cdot \ddot{x}) x, \]

from which we obtain the identity

\[(5.4^*) \quad [X,Y] - Y - p_0 X = (\mu \times \ddot{x}) \quad \text{with} \quad \lambda = ap.\]

Now, we shall solve the Pfaff equations (5.2). We already knew that the solution of \( \xi = 0 \) is given by

\[(5.5) \quad 1 + ax_4 x_4 = c^2 (1 + ar^2) \quad \text{or} \quad M = c^2 L,
\]

where \( c (> 0) \) is an integral constant. Using this relation, the equation

\[
\theta = \sum_b \theta_b dx_b + \theta_4 dx_4 = \frac{1}{x_4 x_4} \left[ -\frac{\sqrt{M}}{a} (\lambda \cdot d\ddot{x}) + \sqrt{L} (p \cdot d\ddot{x}) \\
- \frac{1}{\sqrt{L}} (p_0 \sqrt{M} + a(p \cdot \ddot{x}))r dr - ((\mu \times \ddot{x}) \cdot d\ddot{x}) \right] \\
+ \frac{1}{x_4 \sqrt{M}} (\lambda \cdot \ddot{x}) + p_0 \sqrt{L}) dx_4 = 0
\]
can be replaced by
\[
- \frac{c}{a} \sqrt{L} (\lambda \cdot d\tilde{x}) + \sqrt{L} (p \cdot d\tilde{x}) - \frac{1}{\sqrt{L}} \left( p_0 c \sqrt{L} + a (p \cdot \tilde{x}) \right) r \, dr \\
- \left( (\mu \times \tilde{x}) \cdot d\tilde{x} \right) + \frac{1}{c \sqrt{L}} \left( (\lambda \cdot \tilde{x}) + p_0 \sqrt{L} \right) x_4 dx_4 = 0.
\]

We suppose here that \( \lambda \) and \( p \) are independent as vectors of \( \mathbb{R}^3 \), that is \( \lambda \times p \neq 0 \). Then, by (5.1) we can put
\[
(5.6) \quad \mu = \mu_0 (\lambda \times p), \quad \mu_0 \neq 0,
\]
and so we have
\[
(\mu \times \tilde{x}) = \mu_0 \left( (\lambda \times p) \times \tilde{x} \right) = \mu_0 \left\{ (\lambda \cdot \tilde{x}) p - (p \cdot \tilde{x}) \lambda \right\} \\
\left( (\mu \times \tilde{x}) \cdot d\tilde{x} \right) = \mu_0 \left\{ (\lambda \cdot \tilde{x}) (p \cdot d\tilde{x}) - (p \cdot \tilde{x}) (\lambda \cdot d\tilde{x}) \right\}
\]

Then, the above equation becomes
\[
\sqrt{L} \left( (p \cdot d\tilde{x}) - \frac{c}{a} (\lambda \cdot d\tilde{x}) \right) - \frac{1}{\sqrt{L}} \left( p_0 c \sqrt{L} + a (p \cdot \tilde{x}) \right) r \, dr \\
- \mu_0 \left\{ (\lambda \cdot \tilde{x}) (p \cdot d\tilde{x}) - (p \cdot \tilde{x}) (\lambda \cdot d\tilde{x}) \right\} + \frac{c}{\sqrt{L}} \left( (\lambda \cdot \tilde{x}) + p_0 \sqrt{L} \right) r \, dr = 0,
\]
which is written as
\[
\sqrt{L} \left( \frac{\lambda \cdot d\tilde{x}}{a} - \frac{p \cdot d\tilde{x}}{c} \right) - \left( \frac{\lambda \cdot \tilde{x}}{a} - \frac{p \cdot \tilde{x}}{c} \right) d\sqrt{L} \\
- \frac{\mu_0}{c} (\lambda \cdot \tilde{x}) (p \cdot \tilde{x}) \left( \frac{\lambda \cdot d\tilde{x}}{\lambda \cdot \tilde{x}} - \frac{p \cdot d\tilde{x}}{p \cdot \tilde{x}} \right) = 0.
\]

For simplicity we set
\[
\frac{\lambda \cdot \tilde{x}}{a} = u, \quad \frac{p \cdot \tilde{x}}{c} = v,
\]
then the above equality becomes
\[
(5.7) \quad \omega := \sqrt{L} (du - dv) - (u - v) d\sqrt{L} - a \mu_0 (v du - u dv) = 0.
\]

Since the Pfaff equation (5.2) is completely integrable, we take an integral multiplier \( \Phi \), that is \( \Phi \omega \) is exact. Considering \( \Phi \) as function of \( u, v \) and
\[ z = \sqrt{L}, \text{ then we have} \]
\[ d(\Phi \omega) = d\Phi \wedge \omega + \Phi d\omega = \left( \frac{\partial \Phi}{\partial u} du + \frac{\partial \Phi}{\partial v} dv + \frac{\partial \Phi}{\partial z} d\sqrt{L} \right) \wedge \omega + \Phi d\omega \]
\[ = \left\{ -\left( \sqrt{L} - \alpha \mu_0 u \right) \frac{\partial \Phi}{\partial u} - \left( \sqrt{L} - \alpha \mu_0 v \right) \frac{\partial \Phi}{\partial v} + 2\alpha \mu_0 \Phi \right\} du \wedge dv \]
\[ + \left\{ -(u - v) \frac{\partial \Phi}{\partial u} - \left( \sqrt{L} - \alpha \mu_0 v \right) \frac{\partial \Phi}{\partial z} - 2\Phi \right\} du \wedge d\sqrt{L} \]
\[ + \left\{ -(u - v) \frac{\partial \Phi}{\partial v} + \left( \sqrt{L} - \alpha \mu_0 u \right) \frac{\partial \Phi}{\partial z} + 2\Phi \right\} dv \wedge d\sqrt{L}. \]

Hence, \( \Phi(u, v, z) \) must satisfy the following equalities:
\[ (z - \alpha \mu_0 u) \frac{\partial \Phi}{\partial u} + (z - \alpha \mu_0 v) \frac{\partial \Phi}{\partial v} - 2\alpha \mu_0 \Phi = 0, \]
\[ (u - v) \frac{\partial \Phi}{\partial u} + (z - \alpha \mu_0 v) \frac{\partial \Phi}{\partial z} + 2\Phi = 0, \]
\[ -(u - v) \frac{\partial \Phi}{\partial v} + (z - \alpha \mu_0 u) \frac{\partial \Phi}{\partial z} + 2\Phi = 0. \]

In order to solve (5.8) with respect to \( \Phi \), here we take a change of variables \( u, v \) and \( z \) to
\[ u^* = \alpha \mu_0 u - z, \quad v^* = \alpha \mu_0 v - z, \quad z^* = z, \]
then we have
\[ \frac{\partial \Phi}{\partial u} = \alpha \mu_0 \frac{\partial \Phi}{\partial u^*}, \quad \frac{\partial \Phi}{\partial v} = \alpha \mu_0 \frac{\partial \Phi}{\partial v^*}, \quad \frac{\partial \Phi}{\partial z} = -\frac{\partial \Phi}{\partial u^*} - \frac{\partial \Phi}{\partial v^*} + \frac{\partial \Phi}{\partial z^*} \]
and
\[ u - v = \frac{1}{\alpha \mu_0} (u^* - v^*). \]

(5.8) turns into respectively
\[ u^* \frac{\partial \Phi}{\partial u^*} + v^* \frac{\partial \Phi}{\partial v^*} + 2\Phi = 0, \]
\[ u^* \frac{\partial \Phi}{\partial u^*} + v^* \frac{\partial \Phi}{\partial v^*} + 2\Phi - u^* \frac{\partial \Phi}{\partial z^*} = 0, \]
\[ u^* \frac{\partial \Phi}{\partial u^*} + v^* \frac{\partial \Phi}{\partial v^*} + 2\Phi - u^* \frac{\partial \Phi}{\partial z^*} = 0, \]
which are equivalent to
\[ \frac{\partial \Phi}{\partial z^*} = 0 \quad \text{and} \quad u^* \frac{\partial \Phi}{\partial u^*} + v^* \frac{\partial \Phi}{\partial v^*} = -2\Phi, \]
Hence, we obtain the general solution of (5.8) given by
\[
\Phi = \frac{c_1}{(u^*)^2} + \frac{c_2}{(v^*)^2} + \frac{c_3}{u^*v^*} = \frac{c_1}{(a\mu_0 u - z)^2} + \frac{c_2}{(a\mu_0 v - z)^2} + \frac{c_3}{(a\mu_0 u - z)(a\mu_0 v - z)},
\]
where \(c_1, c_2, c_3\) and \(c_3\) are integral constants. Since we have
\[
\omega = z(du - dv) - (u - v)dz - a\mu_0 (v du - u dv)
= (z - a\mu_0 v)du - (z - a\mu_0 u)dv - (u - v)dz
= -v^* \frac{du^* + dz^*}{a\mu_0} + u^* \frac{dv^* + dz^*}{a\mu_0} - \frac{u^* - v^*}{a\mu_0} dz^* = -v^* \frac{du^* + u^* dv^*}{a\mu_0},
\]
which implies the relations
\[
\frac{1}{(u^*)^2} \omega = \frac{1}{a\mu_0} \frac{d}{d(u^*)} (\frac{v^*}{u^*}), \quad \frac{1}{(v^*)^2} \omega = -\frac{1}{a\mu_0} \frac{d}{d(v^*)} (\frac{u^*}{v^*}),
\]
\[
\frac{1}{u^* v^*} \omega = \frac{1}{a\mu_0} \frac{d}{d(u^*)} \log u^*.
\]
Thus, we obtain the general solution of (5.7) as follows. Setting \(F = F(x_1, x_2, x_3, c)\) by
\[
F := \frac{v^*}{u^*} = \frac{a\mu_0 v - z}{a\mu_0 u - z} = \frac{a\mu_0 (p \cdot \bar{x})/c - \sqrt{1 + ar^2}}{\mu_0 (\lambda \cdot \bar{x}) - \sqrt{1 + ar^2}},
\]
(5.7) is equivalent to
\[
d(c_1 F - c_2/F + c_3 \log F) = 0.
\]
We reach the following conclusion:

**Theorem 6.** The solutions of the pair of Pfaff equations
\[
\xi = 0 \quad \text{and} \quad \theta = 0
\]
with \(p_0 \neq 0, \lambda \times p \neq 0\) and \(\mu = \mu_0 (\lambda \times p), \mu_0 \neq 0\) are given by
\[
1 + ax_4 x_4 = c^2 (1 + ar^2),
\]
(5.9)
\[
a\mu_0 (p \cdot \bar{x}) - c\sqrt{1 + ar^2} = c_1 \left( \mu_0 (\lambda \cdot \bar{x}) - \sqrt{1 + ar^2} \right),
\]
where \(c > 0\) and \(c_1\) are integral constants.
Finally, we investigate the rest case: $\lambda \times p = 0$ in the previous argument, that is
\begin{equation}
(5.10) \quad p_0 \neq 0, \mu \neq 0 \quad \text{and} \quad (\lambda \cdot \mu) = (p \cdot \mu) = 0, \ (\lambda \times p) = 0.
\end{equation}
Choosing suitable coordinates $(x_1, x_2, x_3)$ in $\mathbb{R}^3$, we may put
\[
\lambda = \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix}, \quad p = \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} \quad \text{with} \ \mu \neq 0.
\]
Then, by (3.2) $\theta$ is expressed as
\[
\theta = \frac{1}{x_4x_1} \left[ -\frac{\sqrt{M}}{a} \lambda dx_1 + \sqrt{L} p dx_1 - \mu(x_3 dx_1 - x_1 dx_3) 
- \frac{1}{\sqrt{L}} (p_0 \sqrt{M} + ap x_1) r dr \right] + \frac{1}{x_4 \sqrt{M}} (\lambda x_1 + p_0 \sqrt{L}) dx_4.
\]
In order to solve the Pfaff equations (5.2), we can put
\[
M = c^2 L, \quad L = 1 + ar^2, \quad M = 1 + ax_4 x_1, \quad r^2 = x_1 x_1 + x_2 x_2 + x_3 x_3
\]
and so $\theta = 0$ turns into
\[
\left( -\frac{c \lambda}{a} + p \right) \sqrt{L} dx_1 - \mu(x_3 dx_1 - x_1 dx_3) - \frac{1}{\sqrt{L}} (cp_0 \sqrt{L} + ap x_1) r dr 
+ \frac{c}{\sqrt{L}} (\lambda x_1 + p_0 \sqrt{L}) r dr = 0,
\]
that is
\begin{equation}
(5.11) \quad -\frac{1}{a} (c \lambda - ap) \sqrt{L} dx_1 - \mu(x_3 dx_1 - x_1 dx_3) + \frac{1}{\sqrt{L}} (c \lambda - ap) x_1 r dr = 0.
\end{equation}
For simplicity we set
\[
A = c \lambda - ap \quad \text{and}
\]
\[
\omega := -\frac{A}{a} \sqrt{L} dx_1 - \mu(x_3 dx_1 - x_1 dx_3) + \frac{A}{\sqrt{L}} x_1 r dr,
\]
\[
= \left( \frac{A}{\sqrt{L}} x_1^2 - \frac{A}{a} \sqrt{L} - \mu x_3 \right) dx_1 + \frac{A}{\sqrt{L}} x_2 dx_2 + x_1 \left( \mu + \frac{A}{\sqrt{L}} x_3 \right) dx_3
\]
then we obtain
\[
d\omega = -\frac{Ar}{\sqrt{L}} dr \wedge dx_1 - 2 \mu dx_3 \wedge dx_1 + \frac{Ar}{\sqrt{L}} dx_1 \wedge dr
= \frac{2A}{\sqrt{L}} x_2 dx_1 \wedge dx_2 - 2 \left( \mu + \frac{A}{\sqrt{L}} x_3 \right) dx_3 \wedge dx_1.
Now, let $\Phi(x_1, x_2, x_3)$ be an integral multiplier for $\omega = 0$. Then, we obtain from the above expressions

\[
d(\Phi \omega) = \frac{\partial \Phi}{\partial x_1} dx_1 \wedge \omega + \frac{\partial \Phi}{\partial x_2} dx_2 \wedge \omega + \frac{\partial \Phi}{\partial x_3} dx_3 \wedge \omega + \Phi \, d\omega
\]

\[
= \frac{\partial \Phi}{\partial x_1} \left\{ \frac{A}{\sqrt{L}} x_1 x_2 dx_1 \wedge dx_2 - x_1 \left( \mu + \frac{A}{\sqrt{L}} x_3 \right) dx_3 \wedge dx_1 \right\}
+ \frac{\partial \Phi}{\partial x_2} \left\{ -\left( \frac{A}{\sqrt{L}} x_1^2 - \frac{A}{\sqrt{L}} x_2 \right) dx_1 \wedge dx_2
+ x_1 \left( \mu + \frac{A}{\sqrt{L}} x_3 \right) dx_2 \wedge dx_3 \right\}
+ \frac{\partial \Phi}{\partial x_3} \left\{ \left( \frac{A}{\sqrt{L}} x_1^2 - \frac{A}{\sqrt{L}} x_3 \right) dx_3 \wedge dx_1 - \frac{A}{\sqrt{L}} x_1 x_2 dx_2 \wedge dx_3 \right\}
+ \Phi \left\{ \frac{2A}{\sqrt{L}} x_2 dx_1 \wedge dx_2 - 2 \left( \mu + \frac{A}{\sqrt{L}} x_3 \right) dx_3 \wedge dx_1 \right\},
\]

from which we see that $\Phi$ must satisfy the following equalities:

\[
x_1 \left( \mu + \frac{A}{\sqrt{L}} x_3 \right) \frac{\partial \Phi}{\partial x_2} - \frac{A}{\sqrt{L}} x_1 x_2 \frac{\partial \Phi}{\partial x_3} = 0,
\]

\[
-x_1 \left( \mu + \frac{A}{\sqrt{L}} x_3 \right) \frac{\partial \Phi}{\partial x_1} + \left( \frac{A}{\sqrt{L}} x_1^2 - \frac{A}{\sqrt{L}} x_2 \right) \frac{\partial \Phi}{\partial x_3}
-2 \left( \mu + \frac{A}{\sqrt{L}} x_3 \right) \Phi = 0,
\]

\[\frac{A}{\sqrt{L}} x_1 x_2 \frac{\partial \Phi}{\partial x_1} - \left( \frac{A}{\sqrt{L}} x_1^2 - \frac{A}{\sqrt{L}} x_3 \right) \frac{\partial \Phi}{\partial x_2} + \frac{2A}{\sqrt{L}} x_2 \Phi = 0.\]

Since we have

\[
\frac{A}{\sqrt{L}} x_1^2 - \frac{A}{\sqrt{L}} x_2 \mu x_3 = \frac{A}{\sqrt{L}} \left( \frac{1}{a} + x_2 x_3 \right) - x_3 \left( \mu + \frac{A}{\sqrt{L}} x_3 \right)
\]

the above equalities turns into respectively

\[\left( \mu + \frac{A}{\sqrt{L}} x_3 \right) \frac{\partial \Phi}{\partial x_2} - \frac{A}{\sqrt{L}} x_2 \frac{\partial \Phi}{\partial x_3} = 0,
\]

\[
\left( \mu + \frac{A}{\sqrt{L}} x_3 \right) \left( x_1 \frac{\partial \Phi}{\partial x_1} + x_3 \frac{\partial \Phi}{\partial x_3} + 2\Phi \right) + \frac{A}{\sqrt{L}} \left( \frac{1}{a} + x_2 x_3 \right) \frac{\partial \Phi}{\partial x_3} = 0,
\]

\[
\frac{A}{\sqrt{L}} x_2 \left( x_1 \frac{\partial \Phi}{\partial x_1} + x_2 \frac{\partial \Phi}{\partial x_2} + 2\Phi \right) + \left( x_3 \left( \mu + \frac{A}{\sqrt{L}} x_3 \right) + \frac{A}{a \sqrt{L}} \right) \frac{\partial \Phi}{\partial x_2} = 0.
\]
Now, using the notations:
\[ u = Ax_3 + \mu \sqrt{L}, \quad A = c\lambda - ap, \quad B = a\mu^2 - A^2 \]
we change the variables \((x_1, x_2, x_3)\) to the new ones \((x_1^*, x_2^*, u^*)\) by
\[ x_1^* = x_1, \quad x_2^* = x_2x_2 + \frac{u^2}{B}, \quad u^* = x_2x_2 - \frac{u^2}{B}, \]
where we suppose \(B \neq 0\). Then, we have
\[
\frac{\partial u}{\partial x_1} = \frac{a\mu}{\sqrt{L}} x_1, \quad \frac{\partial u}{\partial x_2} = \frac{a\mu}{\sqrt{L}} x_2, \quad \frac{\partial u}{\partial x_3} = A + \frac{a\mu}{\sqrt{L}} x_3
\]
and
\[
\frac{\partial \Phi}{\partial x_1} = \Phi_{x_1^*} + \frac{2u}{B} \frac{a\mu x_1}{\sqrt{L}} \Phi_{x_2^*} - \frac{2u}{B} \frac{a\mu x_1}{\sqrt{L}} \Phi_{u^*} = \Phi_{x_1^*} + \frac{2a\mu}{B\sqrt{L}} x_1 (\Phi_{x_2^*} - \Phi_{u^*}),
\]
\[
\frac{\partial \Phi}{\partial x_2} = 2x_2 \left( 1 + \frac{a\mu}{B\sqrt{L}} \right) \Phi_{x_2^*} + 2x_2 \left( 1 - \frac{a\mu}{B\sqrt{L}} \right) \Phi_{u^*},
\]
\[
\frac{\partial \Phi}{\partial x_3} = \frac{2u}{B} \left( A + \frac{a\mu}{\sqrt{L}} x_3 \right) (\Phi_{x_2^*} - \Phi_{u^*}),
\]
where we consider \(\Phi\) as a function of \((x_1^*, x_2^*, u^*)\) and denote its partial derivatives with respect to the new variables as \(\Phi_{x_1^*}, \Phi_{x_2^*}, \Phi_{u^*}\). The first equality of (5.12) becomes
\[
u \frac{\partial \Phi}{\partial x_2} - Ax_2 \frac{\partial \Phi}{\partial x_3}
\]
\[
= 2ux_2 \left\{ \left( 1 + \frac{a\mu}{B\sqrt{L}} \right) \Phi_{x_2^*} + \left( 1 - \frac{a\mu}{B\sqrt{L}} \right) \Phi_{u^*} \right\}
\]
\[
- \frac{2Ax_2u}{B} \left( A + \frac{a\mu}{\sqrt{L}} x_3 \right) (\Phi_{x_2^*} - \Phi_{u^*})
\]
\[
= 2ux_2 \left\{ \left( 1 + \frac{a\mu}{B\sqrt{L}} \right) - \frac{A}{B} \left( A + \frac{a\mu}{\sqrt{L}} x_3 \right) \right\} \Phi_{x_2^*}
\]
\[
\quad + \left\{ 1 - \frac{a\mu}{B\sqrt{L}} + \frac{A}{B} \left( A + \frac{a\mu}{\sqrt{L}} x_3 \right) \right\} \Phi_{u^*}
\]
\[
= 2ux_2 \left\{ \left( 1 - \frac{A^2}{B} + \frac{a\mu}{B\sqrt{L}} (u - Ax_3) \right) \Phi_{x_2^*}
\quad + \left\{ \frac{B + A^2}{B} + \frac{a\mu}{B\sqrt{L}} (Ax_3 - u) \right\} \Phi_{u^*} \right\}
\]
\[
= 2ux_2 \left\{ \left( 1 - \frac{A^2}{B} + \frac{a\mu^2}{B} \right) \Phi_{x_2^*} + \left\{ \frac{a\mu^2}{B} + \frac{a\mu}{B\sqrt{L}} (-\mu \sqrt{L}) \right\} \Phi_{u^*} \right\}
\]
\[
= 4ux_2 \Phi_{x_2^*}, = 0,
\]
which implies

\[(5.13) \quad \Phi_{x^*} = \partial \Phi / \partial x_2^* = 0.\]

Using this equality, we have

\[
\begin{align*}
\frac{\partial \Phi}{\partial x_1} &= \Phi_{x^*_1} - \frac{2a\mu x_1}{B\sqrt{L}}\Phi_{u^*}, \\
\frac{\partial \Phi}{\partial x_2} &= 2x_2 \left( 1 - \frac{a\mu u}{B\sqrt{L}} \right) \Phi_{u^*}, \\
\frac{\partial \Phi}{\partial x_3} &= -\frac{2u}{B} \left( A + \frac{a\mu x_3}{\sqrt{L}} \right) \Phi_{u^*}.
\end{align*}
\]

Hence the second equality of (5.12') becomes

\[
u \left( x_1 \frac{\partial \Phi}{\partial x_1} + x_3 \frac{\partial \Phi}{\partial x_3} + 2\Phi \right) + A \left( \frac{1}{a} + x_2 x_2 \right) \frac{\partial \Phi}{\partial x_3}
= u x_1 \left( \Phi_{x^*_1} - \frac{2a\mu u}{B\sqrt{L}} x_1 \Phi_{u^*} \right)
- \left( u x_3 + A \left( \frac{1}{a} + x_2 x_2 \right) \right) 2u \left( A + \frac{a\mu}{\sqrt{L}} x_3 \right) \Phi_{u^*} + 2u \Phi
= \nu x_1 \Phi_{x^*_1}
- \left( \frac{2a\mu u^2}{B\sqrt{L}} x_1 \right) + 2u \left( A + \frac{1}{a} + x_2 x_2 \right) \left( A \mu x_3 + A \sqrt{L} \right) \Phi_{u^*}
+ 2u \Phi
= 0,
\]

which is equivalent to

\[(5.14) \quad x_1^* \Phi_{x^*_1} - 2H_2 \Phi_{u^*} + 2\Phi = 0,\]

where we set

\[H_2 = \frac{1}{B\sqrt{L}} \left\{ a\mu x_1^2 + \left( u x_3 + A \left( \frac{1}{a} + x_2 x_2 \right) \right) (a\mu x_3 + A \sqrt{L}) \right\}.
\]

The third equality of (5.12') becomes

\[
\begin{align*}
A x_2 \left( x_1 \frac{\partial \Phi}{\partial x_1} + x_2 \frac{\partial \Phi}{\partial x_2} + 2\Phi \right) + \left( u x_3 + A \left( \frac{1}{a} + x_2 x_2 \right) \right) \frac{\partial \Phi}{\partial x_3}
= A x_2 \left( \Phi_{x^*_1} - \frac{2a\mu u}{B\sqrt{L}} \Phi_{u^*} \right) + \left( A x_2^2 + u x_3 + A \right) 2x_2 \left( 1 - \frac{a\mu u}{B\sqrt{L}} \right) \Phi_{u^*}
+ 2A x_2 \Phi
= A x_2 \Phi_{x^*_1} - 2x_2 \left( \frac{a\mu u x_1^2}{B\sqrt{L}} + \left( A x_2^2 + u x_3 + A \right) \left( \frac{a\mu u}{B\sqrt{L}} - 1 \right) \right) \Phi_{u^*}
+ 2A x_2 \Phi
= 0,
\end{align*}
\]
which is equivalent to
\[ (5.15) \quad x_1^* \Phi_{x_1^*} - 2H_3 \Phi_{u^*} + 2 \Phi = 0, \]

where we set
\[ H_3 = \frac{1}{B \sqrt{L}} \left\{ a \mu x_1^2 + \left( x_2^2 + \frac{ux_3}{A} + \frac{1}{a} \right) (a \mu u - B \sqrt{L}) \right\}. \]

From (5.14) and (5.15) we obtain the equality
\[ (H_2 - H_3) \Phi_{u^*} = 0. \]

Since we have
\[ B \sqrt{L} (H_2 - H_3) = \left( ux_3 + A \left( \frac{1}{a} + x_2 x_2 \right) \right) (a \mu x_3 + A \sqrt{L}) \]
\[ - \left( x_2^2 + \frac{ux_3}{A} + \frac{1}{a} \right) (a \mu u - B \sqrt{L}) \]

and
\[ \left( x_2^2 + \frac{ux_3}{A} + \frac{1}{a} \right) (a \mu u - B \sqrt{L}) = \frac{1}{A} \left( ux_3 + A \left( \frac{1}{a} + x_2 x_2 \right) \right) (a \mu x_3 + A \sqrt{L} - B \sqrt{L}) \]
\[ = \left( ux_3 + A \left( \frac{1}{a} + x_2 x_2 \right) \right) (a \mu x_3 + A \sqrt{L}), \]

we obtain \( H_2 = H_3 \). Therefore (5.14) and (5.15) are identical.

Now we shall express \( H_2 \) by \( x_1^*, x_2^* \) and \( u^* \). We have
\[ H_2 = \frac{1}{B \sqrt{L}} \left[ a \mu x_1^2 + a \mu x_3^2 + (A \mu (1 + ax_2^2) + A u \sqrt{L}) x_3 \right. \]
\[ + \frac{A^2}{a} (1 + ax_2^2) \sqrt{L} \]
\[ = \frac{1}{B \sqrt{L}} \left[ a \mu x_1^2 + a \mu x_3^2 + (\mu (1 + ax_2^2) + u \sqrt{L}) (u - \mu \sqrt{L}) \right. \]
\[ + \frac{A^2}{a} (1 + ax_2^2) \sqrt{L} \]
\[ = \frac{1}{B \sqrt{L}} \left[ a \mu x_1^2 + a \mu x_3^2 + \mu u (1 + ax_2^2) - \mu u (1 + ax_1^2 + ax_2^2 + ax_3^2) \right. \]
\[ + \left\{ u^2 - \mu^2 (1 + ax_2^2) + \frac{A^2}{a} (1 + ax_2^2) \right\} \sqrt{L} \]
\[ = \frac{1}{B} \left\{ u^2 - \left( \mu^2 - \frac{A^2}{a} \right) (1 + ax_2^2) \right\} = \frac{1}{B} u^2 - \frac{1}{a} (1 + ax_2^2) = -\frac{1}{a} - u^*. \]
Thus, the equality (5.14) turns into

(5.14') \[ x_1^* \Phi_{x_1^*} + 2 \left( \frac{1}{a} + u^* \right) \Phi_{u^*} + 2 \Phi = 0. \]

If we take \( v = \sqrt{u^* + \frac{1}{a}} \) in place of \( u^* \), then we have

\[ 2 \left( \frac{1}{a} + u^* \right) \Phi_{u^*} = 2v^2 \Phi_v, \quad \frac{1}{2 \sqrt{u^* + 1/a}} = v \Phi_v \]

and hence (5.14') can be replaced by

(5.14'') \[ x_1^* \Phi_{x_1^*} + v \Phi_v + 2 \Phi = 0, \]

whose solution is given by

\[ \Phi = \frac{c_1}{x_1 x_1} + \frac{c_2}{uv} + \frac{c_3}{x_1 v}, \quad v = \sqrt{\frac{1}{a} + x_2 x_2 - \frac{uu}{B}} \]

\[ u = Ax_3 + \mu \sqrt{L}, \]

where \( c_1, c_2 \) and \( c_3 \) are integral constants.

Finally we shall integrate the Pfaff equation

\[ \omega = 0. \]

Since we have

\[ \frac{1}{x_1 x_1} \omega = - \frac{A}{a} \sqrt{L} \frac{dx_1}{x_1^2} - \mu \left( \frac{x_3}{x_1^2} \frac{dx_1}{dx_3} - \frac{1}{x_1} \frac{dx_3}{dx_1} \right) + \frac{A}{a} \sqrt{L} \frac{1}{x_1} dr \]

\[ = \frac{A}{a} \sqrt{L} \frac{d}{x_1} + \mu \left( x_3 d \frac{1}{x_1} + \frac{1}{x_1} \frac{dx_3}{dx_1} \right) + \frac{A}{a} \sqrt{L} \frac{1}{x_1} d \left( \frac{A}{a} \frac{\sqrt{L}}{x_1} + \mu \frac{x_3}{x_1} \right), \]

the solution of the above Pfaff equation is given by

\[ \frac{A}{a} \sqrt{L} + \mu x_3 = c_1 x_1 \quad \text{or} \quad A^2 (1 + ar^2) = a^2 (c_1 x_1 - \mu x_3)^2, \]

which is given by the original coordinates \( x_i \) by

\[ (c |\lambda| - a |p|)^2 (1 + ar^2) = \frac{a^2}{|\lambda|^2} \left\{ c_1 (\lambda \cdot \tilde{x}) - ((\lambda \times \mu) \cdot \tilde{x}) \right\}^2, \]

where \( \lambda \) is supposed \( \lambda \neq 0 \) and \( c_1 \) is an integral constant.

**Theorem 7.** The solutions of the pair of Pfaff equations

\[ \xi = 0 \quad \text{and} \quad \theta = 0 \]

with \( p_0 \neq 0, \mu \neq 0, \lambda \neq 0, \mu \neq 0, (\lambda \times p) = 0, (\lambda \cdot \mu) = (p \cdot \mu) = 0 \) and

\( a |\mu|^2 \neq (c |\lambda| - a |p|)^2 \), are given by

\[ 1 + ax_4 x_4 = c^2 (1 + ar^2), \]

\[ |\lambda|^2 (c |\lambda| - a |p|)^2 (1 + ar^2) = a^2 \left\{ c_1 (\lambda \cdot \tilde{x}) - ((\lambda \times \mu) \cdot \tilde{x}) \right\}^2. \]
References


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