SUT Journal of Mathematics
Vol. 35, No. 1 (1999), 37–79

# Asymptotic behavior for large time of solutions to the nonlinear nonlocal Schrödinger equation on half-line

**Elena I. Kaikina, Pavel I. Naumkin and Ilya A. Shishmarev**

(Received December 7, 1998)

**Abstract.** We study the following initial-boundary value problem for the nonlinear nonlocal Schrödinger equation

$$\text{(NNS)} \qquad u_t + \mathbf{N}(u) + bu + \mathbf{K}u = 0, \quad (t,x) \in \mathbf{R}^+ \times \mathbf{R}^+,$$

$$u(0,x) = \bar{u}(x), \quad x > 0, \quad \partial_x^{j-1} u(0,t) = \tilde{u}_j(t), \quad t > 0, \quad \text{for} \quad j = 1, ..., n,$$

with the compatibility condition $\partial_x^{j-1}\bar{u}(0) = \tilde{u}_j(0), j = 1, 2..., n$, where $n = [\frac{\alpha}{2}]$, $[s]$ denotes the largest integer less than $s$, $b \geq 0$, the nonlinear term $\mathbf{N}(u) = ia(t)|u|^\rho u$, $\rho > 1$, the coefficient $a(t) \in \mathbf{C}^1$ and $\mathbf{K}$ is the pseudodifferential operator on the half line $\mathbf{R}^+$ of order $\alpha > 1$. We prove that if $x^\delta \bar{u} \in \mathbf{L}^1$, with $0 < \delta < \frac{1}{2}$ and the norm $\|\bar{u}\|_{\mathbf{X}}$ of the initial data and the norms $\|\tilde{u}_j\|_{\mathbf{Y}}$, $j = 1, ..., n$ of the boundary data are sufficiently small then there exists a unique solution $u \in \mathbf{C}([0,\infty); \mathbf{L}^2) \cap \mathbf{C}(\mathbf{R}^+; \mathbf{W}_2^{[\alpha]-1} \cap \mathbf{C}^{[\alpha]-1})$ of the initial-value problem (NNS). Here $\mathbf{X} = \mathbf{W}_\infty^{[\alpha]} \cap \mathbf{W}_1^{[\alpha]+1}$ and $\mathbf{Y} = \mathbf{W}_\infty^1 \cap \mathbf{W}_1^2$ and $\mathbf{W}_p^k$ is the Sobolev space with the norm $\|\phi\|_{\mathbf{W}_p^k} = \|(1-\partial_x^2)^{k/2}\phi(x)\|_{\mathbf{L}^p}$. We also find the large time asymptotics of the solutions.

*AMS* 1991 *Mathematics Subject Classification.* 35Q55.

*Key words and phrases.* Nonlinear Nonlocal Schrödinger Equation, Large Time Asymptotics.

## §1. Introduction

In this paper we study the following initial-boundary value problem for the nonlinear nonlocal Schrödinger equation

$$\begin{cases} u_t + \mathbb{N}(u) + bu + \mathbb{K}u = 0, \quad (t,x) \in \mathbf{R}^+ \times \mathbf{R}^+, \\ u(x,0) = \bar{u}(x), \quad x > 0, \\ \partial_x^{j-1} u(0,t) = \tilde{u}_j(t), \quad t > 0, \quad j = 1, ..., n \end{cases} \tag{1.1}$$

with the compatibility condition $\partial_x^{j-1}\bar{u}(0) = \tilde{u}_j(0)$, $j = 1, ..., n$, where $n = \left[\frac{\alpha}{2}\right]$, $[s]$ is the largest integer less than $s$, $b \geq 0$, the nonlinear term $\mathbb{N}(u) =$

$ia(t)|u|^\rho u$, $\rho > 1$, the coefficient $a(t) \in \mathbf{C}^1$ and $\mathbb{K}$ is the pseudodifferential operator defined by the inverse Laplace transformation as follows

$$\mathbb{K}u = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px}K(p)\left(\hat{u}(p,t) - \sum_{j=1}^{[\alpha]}\frac{\partial_x^{j-1}u(0,t)}{p^j}\right)dp, \qquad (1.2)$$

where $\hat{u}(p,t) = \int_0^\infty e^{-px}u(x,t)dx$ is the Laplace transform with respect to $x$ of the function $u(x,t)$, $K(p)$ is the symbol of the operator $\mathbb{K}$ and $\alpha$ is the order of the operator $\mathbb{K}$, that is the infimum of real numbers $\alpha$ such that the estimate $|K(p)| \leq C|p|^\alpha$ is valid for all $|p| \geq 1$ in the right half-complex plane $\Re p \geq 0$. Note that if the symbol $K(p)$ is an integer power of $p$ then the operator $\mathbb{K}$ is a differential operator. We suppose that the symbol $K(p)$ of the operator $\mathbb{K}$ is an analytic one-valued function, defined in the right half-complex plane $\Re p \geq 0$ and the derivative $K'(p)$ does not have zeros in $\Re p > 0$. Initial-boundary value problem (1.1) is of great interest from the physical point of view, since it describes many physical phenomena, such as the focusing of the laser beams, waves on water and others [5]. In the present paper we are interested in the case, when the symbol of the operator $\mathbb{K}$ has the following form $K(p) = Ep^\alpha$, where the constant $E$ is such that the operator $\mathbb{K}$ is dissipative, i. e. $\Re K(p) > 0$ for all $p$ on the imaginary axis $\Re p = 0$. The dissipation condition implies that $\alpha$ is not equal to an odd integer. Also we assume that p$\alpha > 1$. (We denote by $p^\alpha$ the main branch of the complex analytic function so that $1^\alpha = 1$. We make a cut along the negative real axis $(-\infty, 0)$ in the complex plane of variable $p$.) Note that in the particular case $\alpha = 2$, $b = 1$, $a = 1$ and $\rho = 2$ problem (1.1) contains the initial-value problem for the well-known Landau-Ginzburg equation $u_t + i|u|^2u + u - u_{xx} = 0$ (see [5]). Existence, uniqueness and some qualitative properties of the solutions to the Cauchy problems for some classes of nonlinear nonlocal dissipative equations were studied in [3] - [5]. Large time asymptotic behavior of solutions to the Cauchy problem for dissipative and conservative nonlinear nonlocal equations was studied in [5] - [6].

To state our results in this paper we give the following notations.

Let us denote $\mathbf{X} = \mathbf{W}_\infty^{[\alpha]} \cap \mathbf{W}_1^{[\alpha]+1}$ and $\mathbf{Y} = \mathbf{W}_\infty^1 \cap \mathbf{W}_1^2$, where $\mathbf{W}_p^k$ is the Sobolev space with the norm $\|\phi(x)\|_{\mathbf{W}_p^k} = \left\|\left(1-\partial_x^2\right)^{k/2}\phi(x)\right\|_{\mathbf{L}^p}$. We also introduce the following function space:

$$\mathbf{Z}_T = \left\{\phi(x,t) \in C\left((0,T]; \mathbf{W}_2^{[\alpha]-1} \cap \mathbf{C}^{[\alpha]-1}\right) : \|\phi\|_{\mathbf{Z}_T} < \infty\right\}$$

with the norm

$$\|\phi\|_{\mathbf{Z}_T} = \|\phi\|_{\mathbf{L}^2} + \sup_{t\in[0,T]}\sum_{j=0}^{[\alpha]-1} t^{\delta_j}\left(\left\|\partial_x^j\phi\right\|_{\mathbf{L}^2} + \left\|\partial_x^j\phi\right\|_{\mathbf{L}^\infty}\right),$$

where $\delta_j = \max\left(0, \frac{j-n+\gamma}{\alpha}\right)$, $\gamma > 0$ is small enough and $n = \left[\frac{\alpha}{2}\right]$. By the same letter $C$ we denote different positive constants.

Now we state our results. First of all we formulate the local existence of solutions to the initial-boundary value problem (1.1). We consider the generalized solutions of the initial-boundary problem (1.1), that is we multiply equation (1.1) by any function $\varphi \in \mathbf{C}^2([0,T] \times (0,\infty))$ such that $\varphi(x,T) = 0$ and $\partial_x^j \varphi(0,t) = 0$, $j = 0,1,2$ and integrate by parts in the domain $[0,T] \times (0,\infty)$. Then the linear operator $\mathbb{K}$ has a sense since we can represent it in the following form

$$\mathbb{K}u = \partial_x^3 \frac{1}{2\pi i} \int_{\Re p=0,|p|>1} e^{px} \frac{K(p)}{p^3} \left( \hat{u}(p,t) - \sum_{j=1}^{[\alpha]} \frac{\partial_x^{j-1} u(0,t)}{p^j} \right) dp$$
$$+ \frac{1}{2\pi i} \int_{\Re p=0,|p|\le 1} e^{px} K(p) \left( \hat{u}(p,t) - \sum_{j=1}^{[\alpha]} \frac{\partial_x^{j-1} u(0,t)}{p^j} \right) dp.$$

Whence we see that the integrals converge uniformly with respect to $x > 0$ since $\frac{K(p)}{p^{[\alpha]+2}} \in \mathbf{L}^2((-i\infty,-i] \cup [i,i\infty))$ and

$$p^{[\alpha]-1} \left( \hat{u}(p,t) - \sum_{j=1}^{[\alpha]} \frac{\partial_x^{j-1} u(0,t)}{p^j} \right) \in \mathbf{C}\left(\mathbf{R}^+; \mathbf{L}^2\right)$$

for the solution $u(x,t) \in \mathbf{C}\left(\mathbf{R}^+; \mathbf{W}_2^{[\alpha]-1}\right)$.

**Theorem 1.1.** *Let $\alpha > 1$ and be not equal to an odd integer. Suppose that the initial data $\bar{u}(x) \in \mathbf{X}$ and the boundary data $\tilde{u}_j(t) \in \mathbf{Y}$, $j = 1,...,[\frac{\alpha}{2}]$ (in the case $\alpha \ge 2$.) Then for some $T > 0$ there exists a unique solution $u(x,t) \in \mathbf{Z}_T$ of the initial-boundary value problem (1.1).*

Now we give some sufficient conditions for the global existence of the solutions. First we consider the case, when the asymptotics of solutions for large time is determined by decaying properties of the boundary data.

**Theorem 1.2.** *Let $\alpha \ge 2$ and be not equal to an odd integer. Let the coefficient in the nonlinearity $a(t) \in \mathbf{C}^1(\mathbf{R}^+)$ satisfy the estimate $|a(t)| \le C(1+t)^{-\eta}$ for all $t > 0$, where $\eta \in \mathbf{R}$. Let the initial data $\bar{u} \in \mathbf{X}$ and the norm $\|\bar{u}\|_{\mathbf{X}} \le \epsilon$, and the boundary data $\tilde{u}_j$ satisfy the following conditions $\sum_{j=1}^{[\frac{\alpha}{2}]} \|\tilde{u}_j\|_{\mathbf{Y}} \le \epsilon$ and*

$$\tilde{u}_j = A_j e^{-bt} t^{-\chi - \frac{j-1}{\alpha}} + e^{-bt} \phi_j(t) \tag{1.3}$$

*for* $j = 1, ..., n$, *where* $|\phi_j(t)| \le C\epsilon t^{-\chi - \frac{j-1}{\alpha} - \gamma}$ *and* $|\phi_1'(t)| = O\left(\epsilon t^{-\chi-\gamma}\right)$, $0 < \chi < \frac{1}{\alpha}$, *if* $b > 0$ *and* $\max\left(0, \frac{1-\eta+1/\alpha}{\rho+1}\right) < \chi < \frac{1}{\alpha}$ *if* $b = 0$. *We suppose that the constants* $\epsilon > 0$ *and* $\gamma > 0$ *are small enough. Then there exists a unique solution* $u(x,t) \in \mathbf{C}\left([0,\infty); \mathbf{L}^2 \cap \mathbf{L}^\infty\right) \cap \mathbf{C}\left(\mathbf{R}^+; \mathbf{W}_2^{[\alpha]-1} \cap \mathbf{C}^{[\alpha]-1}\right)$ *of the initial-boundary value problem (1.1). Moreover this solution has the following asymptotics for large time uniformly with respect to* $x > 0$

$$u(x,t) = \frac{e^{-bt}}{2\pi i t^\chi} \sum_{j=1}^{n} A_j G_j\left(xt^{-1/\alpha}\right) + O\left(e^{-bt} t^{-\chi-\gamma}\right), \tag{1.4}$$

*where* $G_j(q) = \int_{-i\infty}^{i\infty} dy y^{\alpha-j} \int_0^1 dz e^{yq - Ey^\alpha(1-z)} z^{-\chi - \frac{j-1}{\alpha}}$, $E = e^{i\pi\left[\frac{\alpha+1}{2}\right]}$.

*Remark 1.1.* In the case $A_j = 0$ the asymptoic formula (1.4) gives only the estimate of the solution: $\|u\|_{\mathbf{L}^\infty} \le Ce^{-bt} t^{-\chi-\gamma}$.

In the following theorem we consider the case, when the boundary data decay with time sufficiently rapidly and we show that the character of the large time asymptotics of the solutions is defined by the initial data.

**Theorem 1.3.** *Let* $\alpha > 1$ *and be not equal to an odd integer. Let the coefficient in the nonlinearity* $a(t) \in \mathbf{C}^1(\mathbf{R}^+)$ *satisfy the estimate* $|a(t)| \le C(1+t)^{-\eta}$ *for all* $t > 0$, *where* $\eta \in \mathbf{R}$. *Suppose that* $\eta > 1 - \rho/\alpha$ *if* $b = 0$ *and* $\eta \in \mathbf{R}$ *if* $b > 0$. *Let the initial data* $\bar{u} \in \mathbf{X}$ *be such that* $x^\delta \bar{u} \in \mathbf{L}^1$, *with* $0 < \delta < 1/2$ *and the norm* $\|\bar{u}\|_{\mathbf{X}} \le \epsilon$, *where* $\epsilon > 0$ *is sufficiently small. Let the boundary data* $\tilde{u}_j \in \mathbf{Y}$ *for* $j = 1, ..., [\frac{\alpha}{2}]$, *(in the case* $\alpha \ge 2$*) satisfy condition (1.3) with* $\chi = \frac{1}{\alpha}$ *and the following estimates* $\sum_{j=1}^n \|\tilde{u}_j\|_{\mathbf{Y}} \le \epsilon$ *and* $|\hat{\tilde{u}}_j'(\xi)| = O\left(\epsilon|\xi|^{-2}\right)$ *for all* $|\xi| > 1$, $\Re\xi = 0$. *When* $\alpha$ *is integer we also suppose that* $|\hat{\tilde{u}}_j''(\xi)| = O\left(\epsilon|\xi|^{-2}\right)$ *for all* $|\xi| > 1$, $\Re\xi = 0$. *Then there exists the unique solution* $u(x,t) \in \mathbf{C}\left([0,\infty); \mathbf{L}^2\right) \cap \mathbf{C}\left(\mathbf{R}^+; \mathbf{W}_2^{[\alpha]-1} \cap \mathbf{C}^{[\alpha]-1}\right)$ *of the initial-boundary value problem (1.1).* (*In the case* $\alpha \ge 2$ *we have* $u(x,t) \in \mathbf{C}\left([0,\infty); \mathbf{L}^2 \cap \mathbf{L}^\infty\right) \cap \mathbf{C}\left(\mathbf{R}^+; \mathbf{W}_2^{[\alpha]-1} \cap \mathbf{C}^{[\alpha]-1}\right)$.) *This solution has the following asymptotics for large time uniformly with respect to* $x > 0$

$$u(x,t) = e^{-bt} t^{-\frac{1}{\alpha}} \sum_{j=0}^{N} B_j G_j\left(x/t^{1/\alpha}\right) + O\left(e^{-bt} t^{-\frac{1}{\alpha}-\gamma}\right), \tag{1.5}$$

*where* $N = [\alpha]$, *if* $\alpha$ *is not integer and* $N = \alpha - 1$, *if* $\alpha$ *is integer,* $G_0(q) = \int_{-i\infty}^{i\infty} e^{qy - Ey^\alpha} dy$ *and*

$$G_j(q) = \int_{-i\infty}^{i\infty} dy y^{\alpha-j} \int_0^1 dz e^{yq - Ey^\alpha(1-z)} z^{-\frac{j}{\alpha}}.$$

$E = e^{i\pi[\frac{\alpha+1}{2}]}$, *the constants* $B_j$ *we define below in Section 5.*

As an example of application of our theory we consider the initial-boundary value problem for the well-known Landau-Ginzburg equation $u_t + i|u|^2u + u - u_{xx} = 0$. In this case we have $K(p) = p^2$, $b = 1$, $a = 1$ and $\rho = 2$. Since $\alpha = 2$ we need only one boundary condition, so we get the following initial-boundary value problem

$$\begin{cases} u_t + i|u|^2u + u - u_{xx} = 0, \quad (t,x) \in \mathbf{R}^+ \times \mathbf{R}^+, \\ u(x,0) = \bar{u}(x), \quad x > 0, \\ u(0,t) = \tilde{u}(t), \quad t > 0. \end{cases} \tag{1.6}$$

We suppose that the initial data $\bar{u}(x) \in \mathbf{W}^2_\infty \cap \mathbf{W}^3_1$ and the boundary data $\tilde{u} \in \mathbf{W}^1_\infty \cap \mathbf{W}^2_1$ satisfy the compatibility condition $\bar{u}(0) = \tilde{u}(0)$. Then for some time $T > 0$ the initial-boundary value problem (1.6) has a unique solution $u(x,t) \in \mathbf{C}\left((0,T]; \mathbf{W}^1_2 \cap \mathbf{C}^1\right) \cap \mathbf{C}\left([0,T]; \mathbf{L}^2 \cap \mathbf{L}^\infty\right)$. If we assume in addition that the initial and boundary data are small enough and the boundary data have the following asymptotics for large time $\tilde{u}(t) = Ae^{-t}t^{-\chi} + O\left(e^{-t}t^{-\chi-\gamma}\right)$, where $0 \le \chi < \frac{1}{2}$ then there exists a unique global in time solution $u(x,t) \in \mathbf{C}\left([0,\infty); \mathbf{L}^2 \cap \mathbf{L}^\infty\right) \cap \mathbf{C}\left(\mathbf{R}^+; \mathbf{W}^1_2 \cap \mathbf{C}^1\right)$ of the initial-boundary value problem (1.6) and this solution has the following large time asymptotics uniformly with respect to $x > 0$

$$u(x,t) = \frac{Ae^{-t}}{2\pi i t^\chi} G\left(x/\sqrt{t}\right) + O\left(e^{-t}t^{-\chi-\gamma}\right),$$

where $G(q) = \int_{-i\infty}^{i\infty} ydy \int_0^1 e^{yq+y^2(1-z)} z^{-\chi} dz$. Finally if we suppose that the boundary and initial data are small enough and the boundary data decay with time more rapidly $\tilde{u}(t) = Ae^{-t}t^{-\frac{1}{2}} + O\left(e^{-t}t^{-\frac{1}{2}-\gamma}\right)$, then the character of the asymptotic behavior of solutions is defined by the initial data

$$u(x,t) = e^{-t}t^{-\frac{1}{2}} \sum_{j=0}^{1} B_j G_j(x/\sqrt{t}) + O\left(e^{-t}t^{-\frac{1}{2}-\gamma}\right),$$

where $G_0(q) = \int_{-i\infty}^{i\infty} e^{qy+y^2} dy$ and $G_1(q) = \int_{-i\infty}^{i\infty} dyy \int_0^1 e^{yq+y^2(1-z)} \frac{dz}{\sqrt{z}}$. The constants $B_j$ are defined below in Section 5.

We organize our paper as follows. In section 2 we consider the linear initial-boundary value problem corresponding to nonlinear problem (1.1). We discuss an important question on the amount of the necessary boundary data, to be posed for the correct resolution of the initial-value problem. In Theorem 2.1 we prove the local existence of solutions to the linear problem. Section 3 is devoted to the proof of Theorem 1.1. In sections 4 and 5 we prove Theorem 1.2 and Theorem 1.3 respectively.

## §2 Linear problem

In this section we consider the following linear initial-boundary value problem

$$\begin{cases} u_t + bu + \mathbb{K}u = f(x,t), \quad (t,x) > 0, \\ u(x,0) = \bar{u}(x), \quad x > 0, \\ \partial_x^{j-1} u(t,0) = \tilde{u}_j, \quad t > 0, \quad j = 1,2,...,n, \end{cases} \tag{2.1}$$

with the compatibility conditions $\partial_x^{j-1}\bar{u}(0) = \tilde{u}_j(0)$, $j = 1,...,n$, where $\mathbb{K}$ is the pseudodifferential operator defined by formula (1.2). We explain below the choice of the integer $n = [\frac{\alpha}{2}]$. Taking the Laplace transformation of equations of system (2.1) we get the following initial-value problem in the right-half complex plane $\Re p \geq 0$

$$\begin{cases} \hat{u}_t(p,t) + (b + K(p))\hat{u}(p,t) - K(p)\sum_{j=1}^{[\alpha]} p^{-j}\partial_x^{j-1}u(0,t) = \hat{f}(p,t), \\ \hat{u}(p,0) = \hat{\bar{u}}(p), \\ \partial_x^{j-1}u(0,t) = \tilde{u}_j(t), \quad t > 0, \quad j = 1,...,n. \end{cases} \tag{2.2}$$

Integrating (2.2) with respect to time $t$, we obtain the following representation for the Laplace transform of the solution

$$\hat{u}(p,t) = e^{-(K(p)+b)t}\hat{\bar{u}}(p) + \int_0^t e^{-(K(p)+b)(t-\tau)} \left( K(p) \sum_{j=1}^{[\alpha]} \partial_x^{j-1} u(0,\tau) p^{-j} + \hat{f}(p,\tau) \right) d\tau. \tag{2.3}$$

For the existence of the inverse Laplace transformation it is sufficient that the following condition is valid $|\hat{u}(p,t)| \leq C(1+|p|)^\beta$, for all $\Re p \geq 0$, with some $\beta \in \mathbf{R}$ (see [7]). Note that in the domains, where $\Re(K(p)+b) \geq 0$ formula (2.3) does not give any exponential growth of $\hat{u}$ with respect to $p$. But in the region $\Re(K(p)+b) < 0$ we can not deduce from (2.3) that $\hat{u}(p,t)$ does not grow exponentially in $p$. So in the domain $\Re(K(p)+b) < 0$, $\Re p \geq 0$, $|p| \geq C > 0$ we rewrite the solution (2.3) in the form

$$\hat{u}(p,t) = e^{-(K(p)+b)t}\left(\hat{\bar{u}}(p) + \int_0^\infty e^{(K(p)+b)\tau}\left(K(p)\sum_{j=1}^{[\alpha]}\frac{\partial_x^{j-1}u(0,\tau)}{p^j} + \hat{f}(p,\tau)\right)d\tau\right) - \int_t^{+\infty} e^{(K(p)+b)(\tau-t)}\left(K(p)\sum_{j=1}^{[\alpha]}p^{-j}\partial_x^{j-1}u(0,\tau) + \hat{f}(p,\tau)\right)d\tau, \tag{2.4}$$

whence we see that to exclude the exponential growth of the Laplace transform $\hat{u}(p,t)$ it is sufficient to satisfy the equality

$$\hat{\bar{u}}(p) + \int_0^\infty e^{(K(p)+b)\tau}\left(K(p)\sum_{j=1}^{[\alpha]}\partial_x^{j-1}u(0,\tau)p^{-j} + \hat{f}(p,\tau)\right)d\tau = 0 \tag{2.5}$$

in domains where $\Re(K(p)+b) < 0$, $\Re p \geq 0$, $|p| \geq C > 0$. Therefore by virtue of equation (2.5) we can find some of the boundary functions $\partial_x^j u(0,\tau)$ involved in the definition of the operator $\mathbb{K}$. Indeed, suppose that there are exactly $m$ domains in the right half-complex plane $\Re p \geq 0$, where $\Re(K(p)+b) < 0$, $|p| \geq C > 0$. Making a change of the variable $-K(p) - b = \xi$ we transform each of these domains to the right half-complex plane $\Re\xi > 0$. Since this transformation is conformal, there exist $m$ different inverse functions $\phi_l(\xi) = (K+b)^{-1}(\xi) = p$, $l = 1, 2, ..., m$, which transform right half-complex plane $\Re\xi > 0$ to these $m$ domains, where $\Re(K(p)+b) < 0$. Then condition (2.5) can be written as a system of $m$ equations in the half-complex plane $\Re\xi > 0$

$$\frac{1}{\xi+b}\hat{\bar{u}}(\phi_l(\xi)) - \sum_{j=1}^{[\alpha]} \frac{\left(\widehat{\partial_x^{j-1} u(0,t)}\right)(\xi)}{\phi_l^j(\xi)} - \frac{1}{\xi+b}\hat{\hat{f}}(\phi_l(\xi),\xi) = 0, \tag{2.6}$$

where $l = 1, 2, ..., m$ and $\hat{\bar{u}}(\phi_l(\xi)) = \int_0^{+\infty} e^{-\phi_l(\xi)x}\bar{u}(x)dx$,
$\left(\widehat{\partial_x^{j-1} u(0,t)}\right)(\xi) = \int_0^{+\infty} e^{-\xi t}\partial_x^{j-1}u(0,t)dt$,

$$\hat{\hat{f}}(\phi_l(\xi),\xi) = \int_0^{+\infty}\int_0^{\infty} e^{-\phi_l(\xi)x-\xi t} f(x,t)dxdt$$

are the Laplace transforms with respect to space and time for the initial data $\bar{u}(x)$, boundary values $\partial_x^{j-1}u(0,t)$ and the source $f(x,t)$ respectively. We have defined in the problem (2.1) the first $n = [\alpha] - m$ boundary functions $\partial_x^{j-1}u(0,t) = \tilde{u}_j(t)$, $j = 1, 2, ..., n$. So we can determine the remainder $m$ boundary functions $v_j(t) = \partial_x^{n+j-1}u(0,t)$, $j = 1, 2, ..., m$, from the linear system

$$\frac{1}{\xi+b}\hat{\bar{u}}(\phi_l(\xi)) - \sum_{j=1}^{n} \frac{\hat{\tilde{u}}_j(\xi)}{\phi_l^j(\xi)} - \frac{1}{\xi+b}\hat{\hat{f}}(\phi_l(\xi),\xi) = \sum_{j=1}^{m} \frac{\hat{v}_j(\xi)}{\phi_l^{n+j}(\xi)}, \tag{2.7}$$

where $l = 1, 2, ..., m$. For example, if we consider the case $K(p) = Ep^\alpha$, with $E = e^{i\pi[\frac{\alpha+1}{2}]}$, we have $m = [\frac{\alpha+1}{2}]$ domains in the complex half-plane $\Re p > 0$, $|p| \geq b^{1/\alpha}$, where $\Re(K(p)+b) < 0$. Indeed, we define $p^\alpha = |p|^\alpha e^{i\phi\alpha}$, where $\phi \in (-\pi, \pi]$. Therefore the condition

$$\Re(K(p)+b) = |p|^\alpha \cos\left(\phi\alpha + \pi\left[\frac{\alpha+1}{2}\right]\right) + b < 0 \quad \text{yields}$$

$\phi \in \left(\frac{\pi}{\alpha}\left(2l - 1 - \left[\frac{\alpha+1}{2}\right]\right) - \frac{1}{\alpha}\arccos\frac{b}{|p|^\alpha}, \frac{\pi}{\alpha}\left(2l - 1 - \left[\frac{\alpha+1}{2}\right]\right) + \frac{1}{\alpha}\arccos\frac{b}{|p|^\alpha}\right)$, where $l$ is a natural. Since we also assume that $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ we obtain

$l \in \left[1+\frac{1}{2\pi}\arccos\frac{b}{|p|^\alpha}-\frac{1}{2}\left\{\frac{\alpha+1}{2}\right\}-\frac{1}{4}, 1+\left[\frac{\alpha+1}{2}\right]-\frac{1}{2\pi}\arccos\frac{b}{|p|^\alpha}+\frac{1}{2}\left\{\frac{\alpha+1}{2}\right\}-\frac{3}{4}\right)$. Whence we see that $l=1,2,...,\left[\frac{\alpha+1}{2}\right]$ and we have $m=\left[\frac{\alpha+1}{2}\right]$ domains in the complex half-plane, where $|p|>b^{1/\alpha}$ and $\Re(K(p)+b)<0$. We define $m$ inverse functions $\phi_l(\xi)=r_l(b+\xi)^{\frac{1}{\alpha}}$, where $r_l=e^{\frac{2\pi i}{\alpha}(l-1)}$, $l=1,...,[\frac{\alpha+1}{2}]$ (we made a cut along the negative axis $(-\infty,-b)$ in the complex plane of the variable $\xi$ and chose the main branch of the function $(b+\xi)^{\frac{1}{\alpha}}$). The determinant of system (2.7) is the Wronskian

$$W(\xi)=\det\begin{pmatrix}\phi_1^{-n-1}(\xi) & \dots & \phi_1^{-[\alpha]}(\xi)\\ \dots\dots & \dots & \dots\dots\\ \phi_m^{-n-1}(\xi) & \dots & \phi_m^{-[\alpha]}(\xi)\end{pmatrix}=\frac{\det\tilde{W}}{(\xi+b)^{\frac{m(2n+m+1)}{2\alpha}}}\prod_{j=1}^m r_j^{-[\alpha]},$$

where the matrix

$$\tilde{W}=\begin{pmatrix}r_1^{m-1} & \dots & 1\\ \dots\dots & \dots & \\ r_m^{m-1} & \dots & 1\end{pmatrix}. \tag{2.8}$$

It is not equal to zero since all the constants $r_l$ are different. Now we obtain the following result.

**Theorem 2.1.** *Let $\alpha>1$ be not equal to an odd number. Then for some $T>0$ there exists a unique solution $u(x,t)\in\mathbf{Z}_T$ of the initial-boundary value problem (2.1) such that*

$$\|u\|_{\mathbf{Z}_T}\le C\lambda,$$

*where $\lambda=\|\bar{u}\|_{\mathbf{X}}+T^\mu\sup_{t\in[0,T]}t^{\bar{\gamma}}\|f\|_{\mathbf{L}^1}$ for the case $\alpha\in(1,2)$ and $\lambda=\|\bar{u}\|_{\mathbf{X}}+\sum_{j=1}^n\|\tilde{u}_j\|_{\mathbf{Y}}+T^\mu\sup_{t\in[0,T]}t^{\bar{\gamma}}\|f\|_{\mathbf{W}_1^1}$ for the case $\alpha\ge 2$; here $n=\left[\frac{\alpha}{2}\right]$, $\mu=1-\frac{1}{\alpha}-\bar{\gamma}>0$, $\bar{\gamma}>0$ is small enough. We assume that the initial data $\bar{u}$, boundary data $\tilde{u}_j$, $j=1,2,...,n$ and the force $f(x,t)$ in (2.1) are such that the value $\lambda<\infty$.*

Before proving Theorem 2.1 we consider the following function

$$F(x,t)=\int_{-i\infty}^{+i\infty}e^{\xi t-x\Theta\xi^\mu}\xi^{-\beta}d\xi$$

for $x\ge 0$, $t\in\mathbf{R}\backslash 0$, where $\Theta$ is a complex constant such that $\Re(\pm i)^\mu\Theta>0$ and $0<\beta<1$, $\mu>0$. First of all we prove an estimate and the Hölder condition for the function $F(x,t)$ with respect to $x\ge 0$ and $t\in\mathbf{R}\backslash 0$.

**Lemma 2.1.** *We have the following estimates*

$$|F(x,t)| \le C|t|^{\beta-1}, |F(x,t) - F(y,t)| \le C|x-y|^{\omega}|t|^{\beta-1-\omega\mu}$$

*and*

$$|F(x,t) - F(x,\tau)| \le C|t-\tau|^{\nu}(|\tau|^{\beta-1-\nu} + |t|^{\beta-1-\nu}|)$$

*for all* $x, y \ge 0, t, \tau \in \mathbf{R}\backslash 0$, *where* $0 \le \nu < \min(1, \frac{\beta}{\mu})$, $0 \le \omega < \min(1, \frac{\beta}{\mu})$.

*Proof.* In the case $x > 1$ we can differentiate with respect to $x$ and $t$ to obtain

$$F_t(x,t) = \int_{-i\infty}^{+i\infty} e^{\xi t - x\Theta\xi^{\mu}}\xi^{1-\beta}d\xi$$

and

$$F_x(x,t) = -\Theta\int_{-i\infty}^{+i\infty} e^{\xi t - x\Theta\xi^{\mu}}\xi^{\mu-\beta}d\xi,$$

whence using the estimate $\left|\xi^{\sigma}e^{-x\Theta\xi^{\mu}}\right| \le C|\xi|^{-2}$ for $|\xi| > 1$ and any $\sigma \in \mathbf{R}$ we easily get $|F(x,t)| + |F_x(x,t)| + |F_t(x,t)| \le C$ uniformly with respect to $x > 1$.

Now consider the case $0 \le x \le 1$. If $t, \tau > 0$ we change the variable of integration $\xi = q/t$ and denote $z = x/t^{\mu}$. Then we divide the domain of integration in three parts as follows $F(x,t) = t^{\beta-1}(F_1(z) + F_2(z) + F_3(z))$, where

$$F_1(z) = \int_{-i}^{+i} e^{q - z\Theta q^{\mu}}q^{-\beta}dq, \quad F_2(z) = \int_{i}^{+i\infty} e^{q - z\Theta q^{\mu}}q^{-\beta}dq$$

and $F_3(z) = \int_{-i\infty}^{-i} e^{q-z\Theta q^{\mu}}q^{-\beta}dq$. It is easy to see that $|F_1(z)| < C$ and

$$|F_1'(z)| = C\left|\int_{-i}^{+i} e^{q - z\Theta q^{\mu}}q^{-\beta+\mu}dq\right| \le C \quad \text{for all} \quad z \ge 0.$$

Therefore $F_1(z)$ satisfies the Hölder condition. Let $z \ge z' > 0$. Integration by parts with respect to $q$ yields

$$|F_2(z)| \le \left|e^{q - z\Theta q^{\mu}}q^{-\beta}\Big|_i^{i\infty}\right|$$
$$+C\left|\int_i^{i\infty} e^{q-\Theta q^{\mu}z}z\frac{dq}{q^{1+\beta-\mu}}\right| + C\left|\int_i^{i\infty} e^{q-\Theta q^{\mu}z}\frac{dq}{q^{1+\beta}}\right|$$

and

$$|F_2(z) - F_2(z')| \le \left| e^{q-z'\Theta q^\mu}(e^{-\Theta q^\mu (z-z')} - 1) q^{-\beta} \Big|_i^{i\infty} \right|$$
$$+C \left| \int_i^{i\infty} e^{q-\Theta q^\mu z'} z'(1 - e^{-\Theta q^\mu (z-z')}) q^{\mu-1-\beta} dq \right|$$
$$+(z-z')C \left| \int_i^{i\infty} e^{q-\Theta q^\mu z} \frac{dq}{q^{1+\beta-\mu}} \right|$$
$$+C \left| \int_i^{i\infty} e^{q-\Theta q^\mu z'} (1 - e^{-\Theta q^\mu (z-z')}) \frac{dq}{q^{1+\beta}} \right|.$$

Now using estimates $\left| z' q^\mu e^{-\Theta q^\mu z'} \right| \le C$, $\left| 1 - e^{-\Theta q^\mu (z-z')} \right| \le C \min(1, |q|^\mu |z - z'|)$ and $\left| (z-z') q^\mu e^{-\Theta q^\mu z} \right| \le C$ we get for $F_2$ the following estimates $|F_2(z)| \le C$ and

$$|F_2(z) - F_2(z')| \le C(z-z')^\nu \left( 1 + \int_1^\infty q^{\mu\nu - 1 - \beta} dq \right) \le C|z - z'|^\nu,$$

since $0 \le \nu < \min(1, \beta/\mu)$. The integral $F_3(z)$ is considered analogously. Thus we get $|F(x,t)| \le Ct^{\beta-1}$, $|F(x,t) - F(y,t)| \le C|x-y|^\omega |t|^{\beta-1-\omega\mu}$ and $|F(x,t) - F(x,\tau)| \le C|t-\tau|^\nu (\tau^{\beta-1-\nu} + t^{\beta-1-\nu})$ for all $0 \le x \le 1, t > 0, \tau > 0$, since $|t^\sigma - \tau^\sigma| \le C|t-\tau|^\nu (t^{\sigma-\nu} + \tau^{\sigma-\nu})$ for any $\sigma \in \mathbf{R}$.

The case $t, \tau < 0$ is considered similarly. And in the cases $t > 0, \tau < 0$ or $t < 0, \tau > 0$ we have $|t - \tau| = |t| + |\tau|$ so it is sufficient to use the estimate $|F(x,t)| \le C|t|^{\beta-1}$ to obtain $|F(x,t) - F(x,\tau)| \le C\Big( |t|^{\beta-1-\nu} + |\tau|^{\beta-1-\nu} \Big) |t - \tau|^\nu$. Lemma 2.1 is proved. □

*Proof of Theorem 2.1.* As we already know the solution of problem (2.1) can be represented by the Laplace transformation in the following manner

$$\hat{u}(p,t) = e^{-(K(p)+b)t} \hat{\tilde{u}}(p)$$
$$+ \int_0^t e^{-(K(p)+b)(t-\tau)} \left( K(p) \left( \sum_{j=1}^n \frac{\tilde{u}_j(\tau)}{p^j} + \sum_{j=1}^m \frac{v_j(\tau)}{p^{n+j}} \right) + \hat{f}(p,\tau) \right) d\tau, \tag{2.9}$$

and the Laplace transforms $\hat{v}_j(\xi)$ of the boundary values $v_j(t) = \partial_x^{n+j-1} u(0,t)$ of the solution are defined from the system (2.7)

$$\hat{v}_j(\xi) = \frac{h}{(\xi+b)^{1-\frac{n+j}{\alpha}}} \sum_{l=1}^m r_l^{[\alpha]} M_{j,l} \Bigg( \hat{\tilde{u}}(\phi_l(\xi))$$
$$-(\xi+b) \sum_{k=1}^n \frac{\hat{\tilde{u}}_k(\xi)}{\phi_l^k(\xi)} - \hat{\hat{f}}(\phi_l(\xi), \xi) \Bigg), \tag{2.10}$$

where $h = (\det\tilde{W})^{-1}$ and $M_{j,l}$ are the algebraic minors of the matrix $\tilde{W}$ (see (2.8)) and $\hat{\hat{f}}(x,t)$ is the Laplace transform with respect to the space and time of the force $f$:

$$\hat{\hat{f}}(\phi_l(\xi),\xi) = \int_0^\infty e^{-\phi_l(\xi)x}dx \int_0^T e^{-\xi\tau} f(x,\tau)d\tau.$$

We remind that we consider the symbol $K(p) = Ep^\alpha$ with $E = e^{i\pi[\frac{\alpha+1}{2}]}$, so $m = [\frac{\alpha+1}{2}]$, $\phi_l(\xi) = r_l(\xi+b)^{1/\alpha}$, and $r_l = e^{\frac{i}{\alpha}(2\pi l - \pi[\frac{\alpha-1}{2}])}$ are some constants, such that $\Re\phi_l(\xi) > 0$ on the imaginary axis $\xi \in (-i\infty, +i\infty)$. (Note that the sum $\sum_1^0$, which appears in (3.3) and below in the case $\alpha \in (1,2)$ we assume to be identically zero.) We denote

$$\lambda = \|\bar{u}\|_{\mathbf{X}} + T^\mu \sup_{t\in[0,T]} t^{\bar{\gamma}}\|f\|_{\mathbf{L}^1}$$

for the case $\alpha \in (1,2)$ and

$$\lambda = \|\bar{u}\|_{\mathbf{X}} + \sum_{j=1}^n \|\tilde{u}_j\|_{\mathbf{Y}} + T^\mu \sup_{t\in[0,T]} t^{\bar{\gamma}}\|f\|_{\mathbf{W}_1^1}$$

for the case $\alpha \geq 2$, where $\mu = 1 - \frac{1}{\alpha} - \bar{\gamma} > 0$, $\bar{\gamma} > 0$ is small enough.

First of all we estimate the boundary values $v_j = \partial_x^{n+j-1}u(0,t)$ of the solution.

**Lemma 2.2.** *The following estimate*

$$\sup_{t\in[0,T]} \sum_{j=1}^m t^{\frac{j-1}{\alpha}}|v_j| \leq C\lambda$$

*is valid.*

*Proof.* Integrating by parts $n+m$ times in the Laplace transform of the initial data and one time in the Laplace transform of the boundary data and using the compatibility conditions $\tilde{u}_j(0) = \partial_x^{j-1}\bar{u}(0)$ for $j = 1,...,n$, we get

$$\begin{aligned}
&\hat{\bar{u}}(\phi_l) - (\xi+b)\sum_{k=1}^n \frac{\hat{\tilde{u}}_k(\xi)}{\phi_l^k} \\
&= \sum_{k=1}^m \frac{\partial_x^{n-1+k}\bar{u}(0)}{\phi_l^{n+k}} + \frac{1}{\phi_l^{n+m}}\int_0^{+\infty} e^{-\phi_l(\xi)x}\partial_x^{n+m}\bar{u}(x)dx \\
&- \sum_{k=1}^n \phi_l^{-k}\int_0^{+\infty} e^{-\xi\tau}\tilde{u}_k'(\tau)d\tau - \frac{b}{\xi}\sum_{k=1}^n \frac{\tilde{u}_k(0) + \int_0^{+\infty} e^{-\xi\tau}\tilde{u}_k'(\tau)d\tau}{\phi_l^{k+1}(\xi)} \\
&= \sum_{k=1}^m \frac{\partial_x^{n-1+k}\bar{u}(0)}{\phi_l^{n+k}} + O\left(\lambda|\xi+b|^{-\frac{n+1+m}{\alpha}}\right)
\end{aligned}$$

for $|\xi|>1$, since $\frac{1}{\xi}=O\left(|\xi+b|^{-1}\right)$. Whence via (2.10) we get

$$\hat{v}_j(\xi)=\sum_{k=1}^{m}\frac{C_{k,j}\partial_x^{n-1+k}\bar{u}(0)}{(\xi+b)^{1+\frac{k-j}{\alpha}}}-\sum_{l=1}^{m}hM_{j,l}r_l^{[\alpha]}I_{j,l}+O\left(\lambda|\xi+b|^{-1-\frac{m+1-j}{\alpha}}\right) \tag{2.11}$$

for $j=1,...,m$, where we denote

$$I_{j,l}=(\xi+b)^{\frac{n+j}{\alpha}-1}\hat{\hat{f}}(\phi_l(\xi),\xi)$$

and

$$C_{k,j}=h\sum_{l=1}^{m}r_l^{m-k}M_{j,l}$$

are some constants. In the case $\alpha\geq 2$ integration by parts with respect to $x$ of (2.1) yields

$$I_{j,l}=\frac{1}{(\xi+b)^{1-\frac{n-1+j}{\alpha}}}\left(\hat{f}(0,\xi)+\int_0^{+\infty}dxe^{-\phi_l(\xi)x}\int_0^T e^{-\xi\tau}f_x(x,\tau)d\tau\right).$$

Since $\sup_{t\in[0,T]}t^{\bar{\gamma}}\|f\|_{\mathbf{W}_1^1}<\infty$ it is easy to see that

$$\hat{f}(0,\xi)=\int_0^T e^{-\xi\tau}f(0,\tau)d\tau=O\left(\frac{\lambda}{|\xi|}\right)$$

and therefore from (2.11) we get for $|\xi|\geq 1,\xi\in(-i\infty,i\infty)$

$$\begin{aligned}\hat{v}_j(\xi)&=\sum_{k=1}^{m}\frac{C_{k,j}\partial_x^{n-1+k}\bar{u}(0)}{(\xi+b)^{1+\frac{k-j}{\alpha}}}\\&+\frac{h}{(\xi+b)^{1-\frac{n-1+j}{\alpha}}}\sum_{l=1}^{m}M_{j,l}r_l^{[\alpha]}\int_0^{+\infty}dxe^{-\phi_l(\xi)x}\int_0^T e^{-\xi\tau}f_x(x,\tau)d\tau\\&+O\left(\lambda|\xi+b|^{-1-\frac{m-j+1}{\alpha}}\right).\end{aligned} \tag{2.12}$$

Making a change of variables $\xi=\xi'+b$ and then $\xi'(t-\tau)=q$ by Lemma 2.1 we have (prime we omit)

$$\begin{aligned}&\left|\int_{-i\infty}^{i\infty}\frac{e^{\xi t}}{(\xi+b)^{1-\frac{n-1+j}{\alpha}}}d\xi\int_0^{+\infty}dxe^{-x\phi_l(\xi)}\int_0^T e^{-\xi\tau}f_x(x,\tau)d\tau\right|\\&\leq e^{bT}\int_0^T d\tau\int_0^{+\infty}dx|f_x(x,\tau)|\left|\int_{-i\infty}^{+i\infty}\frac{e^{\xi(t-\tau)-xr_l\xi^{1/\alpha}}}{\xi^{1-\frac{n-1+j}{\alpha}}}d\xi\right|\\&\leq C\int_0^T d\tau|t-\tau|^{-\frac{n-1+j}{\alpha}}\int_0^{\infty}dx|f_x(x,\tau)|\left|\int_{-i\infty}^{+i\infty}\frac{e^{q-zr_l(\mathrm{sign}(t-\tau))^{\frac{1}{\alpha}}q^{1/\alpha}}}{q^{1-\frac{n-1+j}{\alpha}}}dq\right|\\&\leq CT^{\mu}\sup_{t\in[0,T]}t^{\bar{\gamma}}\|f_x\|_{\mathbf{L}^1},\end{aligned} \tag{2.13}$$

where $\mu = 1 - \frac{1}{\alpha} - \bar{\gamma}$, $z = x|t-\tau|^{-\frac{1}{\alpha}}$, $j, l = 1, ..., m$. From (2.10) it is easy to see that $|\hat{v}_j| \leq C\lambda|\xi|^{\frac{n+j}{\alpha}-1}$ for $|\xi| \leq 1$. Therefore substitution of (2.13) into (2.12) yields

$$|v_j(t)| = C\left|\int_{-i\infty}^{+i\infty} e^{\xi t}\hat{v}_j(\xi)d\xi\right| \leq C\|\bar{u}\|_{\mathbf{X}}\sum_{k=1}^{m}\left|\int_{|\xi|>1,\Re\xi=0} e^{\xi t}\xi^{-1-\frac{k-j}{\alpha}}d\xi\right|$$
$$+CT^{\mu}\sup_{t\in[0,T]} t^{\bar{\gamma}}\|f_x\|_{\mathbf{L}^1} + \lambda\int_{|\xi|>1,\Re\xi=0} O\left(|\xi+b|^{-1-\frac{m-j+1}{\alpha}}\right)d\xi$$
$$+\left|\int_{-i}^{i} e^{\xi t}\hat{v}_j(\xi)d\xi\right| \leq C\lambda t^{-\frac{j-1}{\alpha}}$$

for all $t \in (0, T]$, $j = 1, ..., m$.

We now consider the case $\alpha \in (1, 2)$. Denoting $I_{1,1} = I$ in (2.9), we have

$$\left|\int_{-i\infty}^{i\infty} e^{\xi t}I(\xi)d\xi\right| \leq C\int_0^T d\tau\int_0^{+\infty} dx|f(x,\tau)|\left|\int_{-i\infty}^{+i\infty} e^{\xi(t-\tau)-\xi^{\frac{1}{\alpha}}x}\xi^{1-\frac{1}{\alpha}}d\xi\right|$$
$$\leq C\int_0^T d\tau\|f(\cdot,\tau)\|_{\mathbf{L}^1}\sup_{x>0}|F(x,t-\tau)|.$$

By virtue of Lemma 2.1 we have $|F(x,t)| \leq C|t|^{-\frac{1}{\alpha}}$ for $t \in \mathbf{R}\backslash 0$. Therefore we get

$$\left|\int_{-i\infty}^{i\infty} e^{\xi t}I(\xi)d\xi\right| \leq C\sup_{t\in[0,T]} t^{\bar{\gamma}}\|f\|_{\mathbf{L}^1}\int_0^T \tau^{-\bar{\gamma}}|t-\tau|^{-\frac{1}{\alpha}}d\tau \leq C\lambda.$$

Thus we easily obtain

$$|v_1(t)| = C\left|\int_{-i\infty}^{+i\infty} e^{\xi t}\hat{v}_1(\xi)d\xi\right| \leq C\|\bar{u}\|_{\mathbf{X}}\left|\int_{|\xi|>1,\Re\xi=0} e^{\xi t}\xi^{-1}d\xi\right|$$
$$+C\lambda + C\lambda\int_{|\xi|>1,\Re\xi=0} O\left(|\xi+b|^{-1-\frac{1}{\alpha}}\right)d\xi + \left|\int_{-i}^{i} e^{\xi t}\hat{v}_1(\xi)d\xi\right| \leq C\lambda.$$

Lemma 2.2 is proved. □

In the following lemma we obtain the Hölder conditions for the boundary values $v_j(t) = \partial_x^{n-1+j}u(0,t)$ of the solution.

**Lemma 2.3.** *We have* $|v_j(\tau)-v_j(t)| \le C\lambda(t-\tau)^{\nu} t^{-\nu-\frac{j-1}{\alpha}}$ *for all* $0<t/2\le \tau\le t\le T$ *and* $j=1,...,m$, *where* $\nu\in(0,\frac{m-j+1}{\alpha})$.

*Proof.* Taking the inverse Laplace transformation of (2.12) we have in the case $\alpha\ge 2$

$$\begin{aligned} |v_j(\tau)-v_j(t)| &\le C\sum_{k=1}^{m}|\partial_x^{n-1+k}\bar{u}(0)|\left|\tau^{\frac{k-j}{\alpha}}-t^{\frac{k-j}{\alpha}}\right| \\ &+C(t-\tau)\|\bar{u}\|_{\mathbf{X}}\sum_{k=1}^{m}\left|\int_{-i}^{i}\frac{d\xi}{\xi^{\frac{k-j}{\alpha}}}\right| \\ +C\sum_{l=1}^{m}\int_0^T d\tau'\int_0^{\infty}|f_x(x,\tau')|dx&\left|\int_{|\xi|\ge1,\Re\xi=0}\frac{(e^{\xi(\tau-\tau')}-e^{\xi(t-\tau')})e^{-\phi_l(\xi)x}d\xi}{(\xi+b)^{1-\frac{n-1+j}{\alpha}}}\right| \\ &+C\lambda\left|\int_{|\xi|\ge1}(e^{\xi\tau}-e^{\xi t})|\xi+b|^{-1-\frac{m-j+1}{\alpha}}d\xi\right| \\ &+C\int_{-i}^{i}|e^{\xi\tau}-e^{\xi t}||\hat{v}_j(\xi)|d\xi = C\sum_{j=1}^{5}J_j. \end{aligned} \tag{2.14}$$

We estimate each summand in (2.14). We have $|\tau^{\eta}-t^{\eta}|\le Ct^{\eta-\nu}(t-\tau)^{\nu}$ for $0<t/2\le\tau\le t\le T$, where $\eta\in\mathbf{R}$ and $0\le\nu\le1$. Therefore taking $\eta=\frac{k-j}{\alpha}$ we see that the first two summands in (2.14) are less than

$$C\|\bar{u}\|_{\mathbf{X}}(t-\tau)^{\nu}\sum_{k=1}^{m}t^{\frac{k-j}{\alpha}-\nu}\le C\lambda(t-\tau)^{\nu}t^{-\nu-\frac{j-1}{\alpha}}.$$

Using Lemma 2.1 with $\beta=1-\frac{n-1+j}{\alpha}$, $\mu=\frac{1}{\alpha}$, $\Theta=r_l$ and $0\le\nu<\frac{m+1-j}{\alpha}$ we get for the third summand in (2.14)

$$\begin{aligned} J_3 = \sum_{l=1}^{m}\int_0^T d\tau'\int_0^{\infty}|f_x(x,\tau')|dx&\left|\int_{|\xi|\ge1,\Re\xi=0}\frac{(e^{\xi(\tau-\tau')}-e^{\xi(t-\tau')})e^{-x\phi_l(\xi)}d\xi}{(\xi+b)^{1-\frac{n-1+j}{\alpha}}}\right| \\ \le C\sum_{l=1}^{m}\int_0^T d\tau'\int_0^{+\infty}&dx|f_x(x,\tau')||F(x,\tau-\tau')-F(x,t-\tau')| \\ +CT^{1-\bar{\gamma}}(t-\tau)\sup_{t\in[0,T]}t^{\bar{\gamma}}\|f_x\|_{\mathbf{L}^1}&\le C(t-\tau)^{\nu}\sup_{t\in[0,T]}t^{\bar{\gamma}}\|f\|_{\mathbf{W}_1^1}\Big(T^{1-\bar{\gamma}}+ \\ \int_0^T\left(|t-\tau'|^{-\frac{n-1+j}{\alpha}-\nu}+|\tau-\tau'|^{-\frac{n-1+j}{\alpha}-\nu}\right)d\tau'\Big)&\le C\lambda(t-\tau)^{\nu}. \end{aligned}$$

For the last two summands in (2.14) we easily obtain

$$|J_4 + J_5| \leq \lambda \int_{|\xi|\geq 1, \Re\xi=0} \frac{\left|e^{\xi\tau} - e^{\xi t}\right| |d\xi|}{|\xi + b|^{1+\frac{m-j+1}{\alpha}}}$$
$$+ \int_{-i}^{i} |e^{\xi\tau} - e^{\xi t}||\hat{v}_j(\xi)||d\xi| \leq C\lambda(t-\tau)^{\nu}$$

since $0 \leq \nu < \frac{m-j+1}{\alpha}$.

In the case $\alpha \in (1,2)$ we have by virtue of (2.11)

$$|v_1(\tau) - v_1(t)| \leq C \int_0^T d\tau' \int_0^{+\infty} |f(x,\tau')| F(x,\tau-\tau') - F(x,t-\tau')|dx$$
$$+C\lambda \int_{|\xi|>1} (e^{\xi\tau} - e^{\xi t})|\xi + b|^{-1-\frac{1}{\alpha}} d\xi + C \int_{-i}^{i} |e^{\xi\tau} - e^{\xi t}||\hat{v}_1(\xi)|d\xi,$$

where $F(x,t) = \int_{-i\infty}^{i\infty} e^{\xi t - \xi^{\frac{1}{\alpha}} x} \xi^{\frac{1}{\alpha}-1} d\xi$. Using Lemmas 2.1 and 2.2 we obtain for $\tau \in [t/2, t]$

$$|v_1(\tau) - v_1(t)| \leq C\lambda(t-\tau)^{\nu} t^{-\nu}$$
$$+C \sup_{t\in[0,T]} t^{\bar{\gamma}} \|f\|_{\mathbf{L}^1} (t-\tau)^{\nu} \int_0^T \tau^{\bar{\gamma}} (|\tau - \tau'|^{-\frac{1}{\alpha}-\nu} + |t-\tau'|^{-\frac{1}{\alpha}-\nu}) d\tau'$$
$$\leq C\lambda(t-\tau)^{\nu} t^{-\nu},$$

where $\nu \in (0, \frac{1}{\alpha})$. Thus we get the desired estimate of Lemma 2.3. □

We now prove the following asymptotic representation

$$\hat{u}(p,t) - \sum_{j=1}^{s} \frac{\partial_x^{j-1} u(0,t)}{p^j} = p^{-s-1} \partial_x^s u(0,t) + O\left(\lambda |p|^{-s-1-\gamma} t^{-\delta_s}\right) \tag{2.15}$$

for $|p| > 1$, $\Re(K(p) + b) > 0$, where $\partial_x^{k-1} u(0,t) = \tilde{u}_k$, for $k = 1, ..., n$ and $\partial_x^{k-1} u(0,t) = v_{k-n}$,for $k = n+1, ..., [\alpha] - 1$,

$\delta_s = \max(0, \frac{s-n+\gamma}{\alpha})$, $s = 0, 1, ..., [\alpha] - 1$ and $n = [\frac{\alpha}{2}]$. We rewrite the representation (2.9) of the Laplace transform of the solution in the form

$$\begin{aligned}\hat{u}(p,t) &= e^{-(K(p)+b)t}\hat{\hat{u}}(p) + K(p)\sum_{j=1}^{n} p^{-j}\int_0^t e^{-(K(p)+b)(t-\tau)}\tilde{u}_j(\tau)d\tau \\ &+K(p)\sum_{j=1}^{s-n+1} p^{-n-j}v_j(t)\int_0^t e^{-(K(p)+b)(t-\tau)}d\tau \\ &+K(p)\sum_{j=1}^{s-n+1} p^{-n-j}\int_0^t e^{-(K(p)+b)(t-\tau)}(v_j(\tau)-v_j(t))d\tau \\ &+K(p)\sum_{j=s-n+2}^{m} p^{-n-j}\int_0^t e^{-(K(p)+b)(t-\tau)}v_j(\tau)d\tau \\ &+\int_0^t e^{-(K(p)+b)(t-\tau)}\hat{f}(p,\tau)d\tau.\end{aligned} \tag{2.16}$$

Integrating by parts in the second and the third summands of the right-hand side of (2.16) we get

$$\hat{u}(p,t) = \sum_{j=1}^{n}\frac{\tilde{u}_j(t)}{p^j} + \sum_{j=1}^{s-n+1}\frac{v_j(t)}{p^{n+j}} + R(p,t), \tag{2.17}$$

where

$$\begin{aligned}R(p,t) &= e^{-(K(p)+b)t}\left(\hat{\hat{u}}(p) - \sum_{j=1}^{n}\frac{\tilde{u}_j(0)}{p^j} - \sum_{j=1}^{s-n+1}\frac{v_j(t)}{p^{n+j}}\right) \\ &-\frac{b}{K(p)+b}\left(\sum_{j=1}^{n}\frac{\tilde{u}_j(t)}{p^j} + \sum_{j=1}^{s-n+1}\frac{v_j(t)}{p^{n+j}}\right) + \frac{be^{-K(p)t-bt}}{K(p)+b}\left(\sum_{j=1}^{n}\frac{\tilde{u}_j(0)}{p^j} + \sum_{j=1}^{s-n+1}\frac{v_j(t)}{p^{n+j}}\right) \\ &-\frac{K(p)}{K(p)+b}\sum_{j=1}^{n}\int_0^t e^{-(K(p)+b)(t-\tau)}\tilde{u}_j'(\tau)d\tau \\ &+K(p)\sum_{j=1}^{s-n+1} p^{-n-j}\int_0^t e^{-(K(p)+b)(t-\tau)}(v_j(\tau)-v_j(t))d\tau \\ &+K(p)\sum_{j=s-n+2}^{m} p^{-n-j}\int_0^t e^{-(K(p)+b)(t-\tau)}v_j(\tau)d\tau \\ &+\int_0^t e^{-(K(p)+b)(t-\tau)}\hat{f}(p,\tau)d\tau = \sum_{j=1}^{7} I_j.\end{aligned} \tag{2.18}$$

The sum $\sum_{j=1}^{s-n+1}$ is identically zero when $s = 0, 1, ..., n-1$. We have the following inequality $e^{-\Re(K(p)+b)t} \leq \frac{C}{(|p|^\alpha t)^\nu}$ for all $p \in (-i\infty, +i\infty)$ and any $\nu \geq 0$. Therefore choosing $\nu = \max(0, \frac{s-n+\gamma}{\alpha})$ and $\nu_j = \max(0, \frac{s-n+1-j+\gamma}{\alpha})$ from the compatibility conditions and estimate of Lemma 2.2 we get for the first summand in (2.18)

$$\begin{aligned} |I_1| &\leq \frac{e^{-\Re(K(p)+b)t}}{|p|^{n+1}} \left( \left| \partial_x^n \bar{u}(0) \right| + \left| \int_0^{+\infty} e^{-px} \partial_x^{n+1} \bar{u}(x) dx \right| + \sum_{j=1}^{s-n+1} \frac{|v_j(t)|}{|p|^{j-1}} \right) \\ &\leq \frac{\lambda}{|p|^{n+1+\nu\alpha} t^\nu} + C\lambda \sum_{j=1}^{s-n+1} \frac{1}{t^{\frac{j-1}{\alpha}+\nu_j} |p|^{n+j+\nu_j \alpha}} \leq O(\lambda |p|^{-s-1-\gamma} t^{-\delta_s}). \end{aligned} \tag{2.19}$$

By virtue of Lemma 2.2 we easily get for the second and third summands in (2.18)

$$|I_2| + |I_3| = O\left(\lambda |p|^{-s-1-\gamma} t^{-\delta_s}\right). \tag{2.20}$$

Since $\Re(K(p) + b) > 0$ we obtain for forth summand in (2.18)

$$|I_4| \leq C\lambda \left| \sum_{j=1}^{n} p^{-j} \int_0^t e^{-\Re K(p)(t-\tau)} d\tau \right| = O\left(\lambda |p|^{-s-1-\gamma}\right). \tag{2.21}$$

Applying Lemma 2.3 with $\nu = \nu_j = \max(0, \frac{s-n+1-j+\gamma}{\alpha}) < \frac{m-j+1}{\alpha}$ in domain $0 < t/2 < \tau < t \leq T$ and using inequality $e^{-\Re K(p)t/2} < C(|K(p)|t)^{-1-\nu_j}$ in the domain $0 < \tau \leq t/2$ via Lemma 2.2 making a change of variables $y = |p|^\alpha (t - \tau)$ we get for the fifth summand in (2.18)

$$\begin{aligned} I_5 &\leq C \sum_{j=1}^{s-n+1} |p|^{\alpha-n-j} \int_0^{t/2} e^{-\Re(K(p)+b)(t-\tau)} (|v_j(\tau)| + |v_j(t)|) d\tau \\ &+ C\lambda \sum_{j=1}^{s-n+1} t^{-\nu_j - \frac{j-1}{\alpha}} |p|^{\alpha-n-j} \int_{t/2}^{t} e^{-\Re(K(p)+b)(t-\tau)} (t-\tau)^{\nu_j} d\tau \\ &\leq C\lambda \sum_{j=1}^{s-n+1} |p|^{-n-j-\nu_j\alpha} t^{-1-\nu_j} \int_0^{t/2} \tau^{\frac{1-j}{\alpha}} d\tau \\ &+ C\lambda t^{-\delta_s} \sum_{j=1}^{s-n+1} |p|^{-n-j-\nu_j\alpha} \int_0^{+\infty} e^{-\Theta y^\alpha} y^{\nu_j} dy = O\left(\lambda |p|^{-s-1-\gamma} t^{-\delta_s}\right), \end{aligned} \tag{2.22}$$

where $\Theta = \min\left(\Re\left(i^\alpha E\right), \Re\left((-i)^\alpha E\right)\right) > 0$. Via Lemma 2.2 and the inequality $e^{-\Re(K(p)+b)(t-\tau)} \leq C|p|^{-\tilde{\nu}_j \alpha} (t-\tau)^{-\tilde{\nu}_j}$ with $\tilde{\nu}_j = \frac{s+\alpha-n+1-j+\gamma}{\alpha}$ we have for $I_6$

in (2.18)

$$|I_6| \leq C\lambda \sum_{j=s-n+2}^{m} |p|^{\alpha-n-j-\alpha\tilde{\nu}_j-\alpha} \int_0^t (t-\tau)^{-\tilde{\nu}_j} \tau^{\frac{1-j}{\alpha}} d\tau = O\left(\frac{\lambda}{|p|^{s+1+\gamma} t^{\delta_s}}\right). \tag{2.23}$$

In the case $\alpha \geq 2$ integration by parts in the Laplace transform of right-hand side of (2.1) yields

$$|\hat{f}(p,t)| = \left| p^{-1} \left( f(0,t) + \int_0^{+\infty} e^{-px} f_x(x,t) dx \right) \right| \leq C\lambda |p|^{-1} t^{-\bar{\gamma}}$$

for all $|p| \geq 1, p \in (-i\infty, i\infty)$ and also $\left\| \hat{f}(p,t) \right\|_{\mathbf{L}^\infty} \leq C\lambda t^{-\bar{\gamma}}$. Whence we obtain

$$|I_7| \leq C\lambda |p|^{-1} \int_0^t e^{-\Re(K(p)+b)(t-\tau)} t^{-\bar{\gamma}} d\tau = O\left(\lambda |p|^{-s-1-\gamma}\right). \tag{2.24}$$

In the case $\alpha \in (1,2)$ we easily obtain

$$|I_7| \leq \sup_{t \in [0,T]} t^{\bar{\gamma}} \|f(x,t)\|_{\mathbf{L}^1} \int_0^t \tau^{-\bar{\gamma}} e^{-\Re(K(p)+b)(t-\tau)} d\tau \leq C\lambda |p|^{-\alpha}.$$

Substitution of estimates (2.19) - (2.24) into representation (2.18) yields the estimate $R(p,t) = O\left(\lambda |p|^{-s-1-\gamma} t^{-\delta_s}\right)$. Therefore from (2.17) we have (2.15). Now from Lemma 2.2 and (2.9) we obtain

$$\begin{aligned} \sup_{p \in [-i,i]} |\hat{u}(p,t)| \leq C \Bigg( \|\hat{\hat{u}}(p)\|_{\mathbf{L}^\infty} + \sum_{k=1}^{[\alpha]} \int_0^t |\partial_x^{k-1} u(0,\tau)| d\tau \\ + \int_0^t \|\hat{f}(p,\tau)\|_{\mathbf{L}^\infty} d\tau \Bigg) \leq C\lambda. \end{aligned} \tag{2.25}$$

We prove now that the solution $u(x,t)$ is given by the inverse Laplace transform of $\hat{u}(p,t)$

$$u(x,t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{px} \hat{u}(p,t) dp.$$

Using (2.15) and (2.25) we easily see that the integral is converging. Let us prove now that $u(x,t) = 0$ for $x < 0$. Since $\hat{u}(p,t)$ is analytic function in the right-half complex plane $\Re p > 0$ we get

$$\int_{-i\infty}^{i\infty} e^{px} \hat{u}(p,t) dp = -\lim_{R \to \infty} \int_{\Gamma_R} e^{px} \hat{u}(p,t) dp,$$

where $\Gamma_R$ is a circumference $p = Re^{i\phi}$, $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $R > 0$. Denote $\Gamma_R = \Gamma_1 + \Gamma_2$, where $\Gamma_1 = \{p = Re^{i\phi}, \phi \in \left(-\frac{\pi}{2} + \varkappa, \frac{\pi}{2} - \varkappa\right)\}$, and $\Gamma_2 = \{p = Re^{i\phi}, \phi \in \left(-\frac{\pi}{2}, -\frac{\pi}{2} + \varkappa\right) \cup \left(\frac{\pi}{2} - \varkappa, \frac{\pi}{2}\right)\}$, here $\varkappa > 0$ is such that $\Re(K(p) + b) > 0$ for $p \in \Gamma_2$ (such small values $\varkappa$ exist since $\Re(K(p) + b) > 0$ on the imaginary axis $p \in (-i\infty, i\infty)$.) By (2.3) - (2.5) we have $|\hat{u}(p,t| \leq C(1 + |p|)^\beta$ for $\Re p > 0$, with some $\beta > 0$. Since $|e^{px}| < C \exp(-\sin(\varkappa R|x|)) < C(\varkappa R|x|)^{-1-\beta-\gamma}$ for $x < 0$, $p \in \Gamma_1$, $\gamma > 0$ we have with $\varkappa = R^{-\frac{\gamma}{2(1+\beta+\gamma)}}$

$$\left| \lim_{R\to\infty} \int_{\Gamma_1} e^{px} \hat{u}(p,t) dp \right| \leq \lim_{R\to\infty} \int_{\Gamma_1} (1 + |p|)^\beta \exp(-\sin(\varkappa R|x|))|dp|$$
$$\leq C \lim_{R\to\infty} (\varkappa |x|)^{-1-\beta-\gamma} R^{-\gamma} = 0.$$

Since $\Re(K(p) + b) > 0$ for $p \in \Gamma_2$ from estimate (2.15) we obtain $\hat{u}(p,t) = \frac{u(0,t)}{p} + O\left(\lambda |p|^{-1-\gamma}\right)$. Therefore

$$\left| \lim_{R\to\infty} \int_{\Gamma_2} e^{px} \hat{u}(p,t) dp \right| \leq C \lim_{R\to\infty} (\varkappa |u(0,t)| + \lambda \varkappa R^{-\gamma}) = 0.$$

Thus $\int_{-i\infty}^{i\infty} e^{px} \hat{u}(p,t) dp = 0$ for all $x < 0$. As we show below (see (2.28) the function $\hat{u}(p,t) \in \mathbf{L}^2(-i\infty, i\infty)$ therefore by the Fourier transformation theory we see that the inverse Laplace transform is given by the formula $\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{px} \hat{u}(p,t) dp$ (see [7]). Via (2.15), (2.25) and Lemma 2.2 we have

$$\partial_x^j u = \frac{1}{2\pi} \int_{-i\infty}^{+i\infty} e^{px} p^j \left( \hat{u}(p,t) - \sum_{k=1}^{j} \frac{\partial_x^{k-1} u(0,t)}{p^k} \right) dp$$
$$= \partial_x^j u(0,t) \frac{1}{2\pi} \int_{|p|\geq 1, \Re p = 0} e^{px} \frac{dp}{p}$$
$$+ Ct^{-\delta_j} \int_{|p|\geq 1, \Re p = 0} e^{px} O(\lambda |p|^{-1-\gamma}) dp + C\lambda t^{-\delta_j}.$$

Whence we see that the derivatives $\partial_x^j u(x,t)$, $j = 0, 1, ..., [\alpha] - 1$ are continuous with respect to $x$ and the boundary data are fulfilled

$\partial_x^{j-1} u(x,t) \to \tilde{u}_j(t)$ as $x \to 0$ for all $j = 0, 1, ..., \frac{[\alpha]}{2}$, and $t > 0$. Moreover we have the estimate

$$\sup_{t\in[0,T]} \sum_{j=1}^{[\alpha]-1} t^{\delta_j} \|\partial_x^j u\|_{\mathbf{L}^2}$$
$$\leq C \sum_{j=1}^{[\alpha]-1} \sup_{t\in[0,T]} \left( \lambda + \left( \int_{|p|>1, \Re p = 0} O\left(\frac{\lambda}{|p|^2}\right) |dp| \right)^{\frac{1}{2}} \right) \leq C\lambda \tag{2.26}$$

and

$$\sup_{t\in[0,T]}\sum_{j=1}^{[\alpha]-1} t^{\delta_j}\|\partial_x^j u\|_{\mathbf{L}^\infty} \le C\sum_{j=0}^{[\alpha]-1}\sup_{t\in[0,T]}\left(\lambda + t^{\delta_j}\left|\int_{-i\infty}^{i\infty} e^{-px}\partial_x^j u(0,t)\frac{dp}{p}\right|\right.$$
$$\left. + \int_{|p|>1,\Re p=0} O\left(\lambda|p|^{-1-\gamma}\right)|dp|\right) \le C\lambda, \tag{2.27}$$

where $\gamma > 0$ is a small constant. From (2.9), Lemma 2.2 and the conditions of the theorem, we easily obtain

$$\|u\|_{\mathbf{L}^2} \le C\|G\|_{\mathbf{L}^1}\|\bar{u}\|_{\mathbf{L}^2} + C\sum_{j=0}^{[\alpha]}\int_0^T \left|\partial_x^{j-1}u(0,\tau)\right|(t-\tau)^{-1+\frac{j}{\alpha}-\frac{1}{2\alpha}}d\tau$$
$$+C\sup_{t\in[0,T]} t^{\bar{\gamma}}\|f(x,t)\|_{\mathbf{L}^1}\int_0^T \tau^{-\bar{\gamma}}(t-\tau)^{-\frac{1}{2\alpha}}d\tau \le C\lambda, \tag{2.28}$$

where $G(x,t) = \int_{-i\infty}^{i\infty} e^{px-Ep^\alpha t}dp$. Via (2.26) - (2.28) we obtain

$$\|u\|_{\mathbf{Z}_T} = \|u\|_{\mathbf{L}^2} + \sup_{t\in[0,T]}\sum_{j=0}^{[\alpha]-1} t^{\delta_j}\left(\left\|\partial_x^j u(x,t)\right\|_{\mathbf{L}^\infty} + \left\|\partial_x^j u(x,t)\right\|_{\mathbf{L}^2}\right) \le C\lambda.$$

We now prove uniqueness of the solution. Consider two different solutions $u_1$ and $u_2$. Then the difference $u_1 - u_2$ satisfy the linear problem (2.1) with $f = 0$, $\bar{u} = 0$ and $\tilde{u}_j = 0$. Then by estimate (2.28) we get $\|u_1 - u_2\|_{\mathbf{L}^2} = 0$, hence $u_1 = u_2$. Theorem 2.1 is proved. □

## §3 Local existence and uniqueness for the nonlinear problem

*Proof of Theorem 1.1.* We prove local existence of solutions by the contraction principle. We define $u$ as a solution of the following linear problem

$$\begin{cases} u_t + \mathbb{N}(w) + bu + \mathbb{K}u = 0, \quad t > 0, x > 0, \\ u(x,t)\Big|_{t=0} = \bar{u}(x), \quad x > 0 \\ \partial u_x^{j-1}(x,t)\Big|_{x=0} = \tilde{u}_j(t), \quad t > 0, \quad j = 1,...,n, \end{cases} \tag{3.1}$$

with the compatibility conditions $\tilde{u}_j(0) = \partial_x^{j-1}\bar{u}(0)$ for $j = 1,...,n$, where $\mathbb{N}(w) = ia(t)|w|^\rho w$ is known since $w(x,t)$ is fixed from the space $\mathbf{Z}_T$ and satisfies the initial and boundary conditions of the problem (3.1), $n = [\frac{\alpha}{2}]$

(in the case $\alpha \in (1,2)$ the boundary data are absent.) Note that the initial-boundary value problem (3.1) defines a mapping $u = \mathbb{M}(w)$ and we will show that $\mathbb{M}(w)$ is the contraction mapping from $\mathbf{Z}_T$ to $\mathbf{Z}_T$.

As we know from Theorem 2.1 problem (3.1) has a unique solution which can be represented by the Laplace transformation in the following manner

$$\hat{u}(p,t) = e^{-(K(p)+b)t}\hat{\tilde{u}}(p) + \int_0^t e^{-(K(p)+b)(t-\tau)}\left(K(p)\left(\sum_{j=1}^{n}\frac{\tilde{u}_j(\tau)}{p^j} + \sum_{j=1}^{m}\frac{v_j(\tau)}{p^{n+j}}\right) + \widehat{\mathbb{N}(w)}(p,\tau)\right)d\tau, \tag{3.2}$$

where the Laplace transforms $\hat{v}_j(\xi)$ of the boundary values $v_j(t) = \partial_x^{n+j-1}u(0,t)$ of the solution are defined from the system (2.7) with $\hat{\hat{f}} = \widehat{\widehat{\mathbb{N}(w)}}$:

$$\hat{v}_j(\xi) = \frac{h}{(\xi+b)^{1-\frac{n+j}{\alpha}}}\sum_{l=1}^{m} r_l^{[\alpha]} M_{j,l}\Bigg(\hat{\tilde{u}}(\phi_l(\xi)) - (\xi+b)\sum_{k=1}^{n}\frac{\hat{\tilde{u}}_k(\xi)}{\phi_l^k(\xi)} - \widehat{\widehat{\mathbb{N}(w)}}(\phi_l(\xi),\xi)\Bigg), \tag{3.3}$$

where $h = (\det\tilde{W})^{-1}$ and $M_{j,l}$ are the algebraic minors of the matrix $\tilde{W}$ (see (2.8)); and $\widehat{\widehat{\mathbb{N}(w)}}$ is the Laplace transform of the nonlinearity with respect to the space and time

$$\widehat{\widehat{\mathbb{N}(w)}}(\phi_l(\xi),\xi) = \int_0^\infty e^{-\phi_l(\xi)x}dx\int_0^T e^{-\xi\tau}\mathbb{N}(w)(x,\tau)d\tau.$$

Here $m = [\frac{\alpha+1}{2}]$, $\phi_l(\xi) = r_l(\xi+b)^{1/\alpha}$, $r_l = e^{\frac{i}{\alpha}(2\pi l - \pi[\frac{\alpha-1}{2}])}$ are some constants such that $\Re\phi_l(\xi) > 0$ on the imaginary axis $\xi \in (-i\infty, +i\infty)$. (Note that the sum $\sum_1^0$ which appears in (3.3) and below in the case $\alpha \in (1,2)$ we assume to be identically zero.) Since $w \in \mathbf{Z}_T$ we have for the nonlinear term in the case $\alpha \geq 2$

$$\begin{aligned}\sup_{t\in[0,T]} t^{\bar{\gamma}}\int_0^{+\infty}|\partial_x\mathbb{N}(w)(x,\tau)|dx &\leq C\sup_{t\in[0,T]}|a(t)|t^{\bar{\gamma}}\|\partial_x w\|_{\mathbf{L}^2}\|w\|_{\mathbf{L}^\infty}^{\rho-1}\|w\|_{\mathbf{L}^2}\\ &\leq C\|w\|_{\mathbf{L}^\infty}^{\rho-1}\|w\|_{\mathbf{L}^2}\|w\|_{\mathbf{Z}_T}.\end{aligned} \tag{3.4}$$

and in the case $\alpha > 1$

$$\begin{aligned}\sup_{t\in[0,T]} t^{\bar{\gamma}}\int_0^{+\infty}|\mathbb{N}(w)(x,\tau)|dx &\leq C\sup_{t\in[0,T]}|a(t)|t^{\bar{\gamma}}\|w\|_{\mathbf{L}^\infty}^{\rho-1}\|w\|_{\mathbf{L}^2}^2\\ &\leq C\sup_{t\in[0,T]} t^{\bar{\gamma}}\|w\|_{\mathbf{L}^\infty}^{\rho-1}\|w\|_{\mathbf{L}^2}\|w\|_{\mathbf{Z}_T}.\end{aligned} \tag{3.5}$$

Via Theorem 2.1 we have the following estimate for the solution

$$\|u\|_{\mathbf{Z}_T} = \|u\|_{\mathbf{L}^2} + \sup_{t\in[0,T]} \sum_{j=0}^{[\alpha]-1} t^{\delta_j} \left( \|\partial_x^j u\|_{\mathbf{L}^2} + \|\partial_x^j u\|_{\mathbf{L}^\infty} \right) \le C\lambda, \tag{3.6}$$

where $\lambda = \|\bar{u}\|_{\mathbf{X}} + \sum_{j=1}^{n} \|\tilde{u}_j\|_{\mathbf{Y}} + T^\mu \sup_{t\in[0,T]} t^{\bar{\gamma}} \|\mathbb{N}(w)\|_{\mathbf{W}_1^1}$ if $\alpha \ge 2$ and $\lambda = \|\bar{u}\|_{\mathbf{X}} + T^\mu \sup_{t\in[0,T]} t^{\bar{\gamma}} \|\mathbb{N}(w)\|_{\mathbf{L}^1}$ if $\alpha \in (1,2)$. Here $\delta_j = \max(0, \frac{j-n+\gamma}{\alpha})$, $\mu = 1 - \frac{1}{\alpha} - \bar{\gamma} > 0$, $\bar{\gamma} = \frac{(\rho-1)\gamma}{\alpha} > 0$, $\gamma > 0$ is small enough. Choosing sufficiently small $T > 0$ we obtain

$$\|u\|_{\mathbf{Z}_T} = \|u\|_{\mathbf{L}^2} + \sup_{t\in[0,T]} \sum_{j=0}^{[\alpha]-1} t^{\delta_j} \left( \left\|\partial_x^j u(x,t)\right\|_{\mathbf{L}^\infty} + \left\|\partial_x^j u(x,t)\right\|_{\mathbf{L}^2} \right) \le C \left( \|\bar{u}\|_{\mathbf{X}} + \sum_{j=1}^{n} \|\tilde{u}_j\|_{\mathbf{Y}} \right).$$

Thus the mapping $\mathbb{M}$ transforms the closed ball in the space $\mathbf{Z}_T$ with a center at the origin and a radius $C(\|\bar{u}\|_{\mathbf{X}} + \sum_{j=1}^{n} \|\tilde{u}_j\|_{\mathbf{Y}})$ to itself. Analogously we can prove the estimate $\|u - \tilde{u}\|_{\mathbf{Z}_T} < \frac{1}{2}\|w - \tilde{w}\|_{\mathbf{Z}_T}$. Indeed, the initial and boundary data for the function $u - \tilde{u}$ are equal to zero. So choosing small $T > 0$ such that $\|u - \tilde{u}\|_{\mathbf{L}^\infty}^{\rho-1} \|u - \tilde{u}\|_{\mathbf{L}^2} T^\mu \le 1/2$ we have $\|u - \tilde{u}\|_{\mathbf{Z}_T} \le \frac{1}{2}\|w - \tilde{w}\|_{\mathbf{Z}_T}$ from (3.4). Therefore the mapping $\mathbb{M}$ is a contraction mapping and there exists the unique solution $u(x,t)$ of the initial-boundary value problem (1.1). Theorem 1.1 is proved. □

*Remark 3.1.* If we can obtain the following a-priori estimate of the solution $\|u\|_{\mathbf{L}^2} + \|u\|_{\mathbf{L}^\infty} \le \infty$ for some time interval $(0,T]$, then via estimate (3.4) by the standard continuation argument we can prove existence of unique solution $u \in \mathbf{C}((0,T], \mathbf{Z}_T)$.

*Remark 3.2* From (3.21) and (3.22) we see that if the norm of the initial data $\|\bar{u}\|_{\mathbf{X}} < \epsilon$ and the norm of the boundary data $\sum_{j=1}^{n} \|\tilde{u}_j\|_{\mathbf{Y}} < \epsilon$, then there exists a time $T > 1$ such that the solution is also sufficiently small $\sup_{t\in[0,T]}(t^\delta \|u\|_{\mathbf{L}^\infty} + \|u\|_{\mathbf{L}^2}) < C\epsilon$, where $\delta = \max(0, \frac{\gamma - [\alpha/2]}{\alpha})$, $\gamma > 0$ is small enough.

## §4 Asymptotics determined by the boundary data

This section is devoted to the proof of Theorem 1.2. Here we consider the initial-boundary value problem (1.1) with $\alpha \ge 2$ to be not equal to an odd number and with small initial and boundary data $\|\bar{u}\|_{\mathbf{X}} < \epsilon$, and

$\sum_{j=1}^{n}\|\tilde{u}_j\|_{\mathbf{Y}} \le \epsilon$. Also we suppose that the boundary data have the following asymptotics $\tilde{u}_j(t) = A_j e^{-bt} t^{-\chi-\frac{j-1}{\alpha}} + O\left(\epsilon e^{-bt} t^{-\chi-\frac{j-1}{\alpha}-\gamma}\right)$ as $t \to \infty$ for all $j = 1, ..., n$, where the coefficients $A_j$ do not equal to zero simultaneously and are small: $|A_j| \le \epsilon$, the value $\epsilon > 0$ is sufficiently small. Here $b \ge 0$ and $0 < \chi < \frac{1}{\alpha}$, if $b > 0$ and $\frac{\max(0,1-\eta+\frac{1}{\alpha})}{\rho+1} < \chi < \frac{1}{\alpha}$, if $b = 0$, $\eta \ge 0$.

Before proving Theorem 1.2 we give some preliminary estimates in Lemmas 4.1 - 4.7. First we consider the following function

$$G(x) = \int_{-i\infty}^{i\infty} e^{xy} y^{\delta} dy \int_0^1 e^{-Ey^{\alpha}(1-z)} z^{\beta} dz,$$

where $-1 < \beta < 0$, $-1 < \delta \le \alpha - 1$ and $E$ is a constant such that $\Re(E(\pm i)^{\alpha}) > 0$. In the next lemma we prove that the function $G(x)$ is bounded for all $x \ge 0$.

**Lemma 4.1.** *There exist a constant $C > 0$ such that $|G(x)| \le C$ for all $x > 0$ and there exists a limit $\lim_{x\to+0} G(x)$.*

*Proof.* We write the following representation

$$G(x) = \int_0^{1/2} dz z^{\beta} \int_{-i\infty}^{i\infty} e^{yx - Ey^{\alpha}(1-z)} y^{\delta} dy + \int_{-i}^{i} e^{yx} y^{\delta} dy \int_{1/2}^{1} e^{-Ey^{\alpha}(1-z)} z^{\beta} dz$$
$$+ \int_{|y|\ge 1, \Re y = 0} e^{yx} y^{\delta} dy \int_{1/2}^{1} e^{-Ey^{\alpha}(1-z)} z^{\beta} dz = J_1 + J_2 + J_3.$$

Denoting $\theta = \min(\Re E(-i)^{\alpha}, \Re E i^{\alpha}) > 0$, changing $y = iq$ we get

$$|J_1| \le 2\int_0^{1/2} dz z^{\beta} \int_0^{+\infty} q^{\delta} e^{-\theta q^{\alpha}/2} dq \le C$$

and

$$|J_2| \le 2\int_0^1 q^{\delta} dq \int_{1/2}^1 z^{\beta} dz \le C$$

since $-1 < \beta$ and $-1 < \delta$. Also note that $J_1$ and $J_2$ are continuous with respect to $x \ge 0$. Integrating by parts with respect to $z$, we get

$$|J_3| \le C\left|\int_{|y|>1, \Re y = 0} e^{xy} y^{\delta-\alpha} dy\right| + C\left|\int_{|y|>1, \Re y = 0} e^{xy - \frac{Ey^{\alpha}}{2}} y^{\delta-\alpha} dy\right|$$
$$+ C\int_{|y|>1, \Re y = 0} |y|^{\delta-\alpha-\nu\alpha} dy \int_{1/2}^{1} z^{\beta-1}(1-z)^{-\nu} dz \le C,$$

where $\frac{\delta+1-\alpha}{\alpha} < \nu < 1$. Also we see that each term in the representation of $J_3$ has a limit for $x \to +0$. $\Big($ In the case $\delta - \alpha = -1$ by virtue of the identity

$$\mathrm{VP} \int_{|y|>1, \Re y=0} e^{xy} \frac{dy}{y} = 2\pi i - 2i \lim_{\epsilon \to +0} \int_{\epsilon}^{1} \frac{\sin xy}{y} dy$$

we integrate by parts to get

$$\left| \mathrm{VP} \int_{|y|>1, \Re y=0} e^{xy} \frac{dy}{y} - 2\pi i \right| \le 2|x| \left( 1 - \lim_{\epsilon \to +0} \epsilon \log \epsilon \right) = 2|x|,$$

whence $\lim_{x \to +0} \mathrm{VP} \int_{|y|>1, \Re y=0} e^{xy} \frac{dy}{y} = 2\pi i. \Big)$ Lemma 4.1 is proved. □

Now we consider the function $F(x,t) = \int_{-i\infty}^{+i\infty} e^{\xi t - \Theta \xi^{\mu} x} d\xi$ for all $x > 0$ and $t > 0$, where $\mu \in (0,1)$, $\Theta$ is a constant such that $\Re \Theta (\pm i)^{\mu} > 0$.

**Lemma 4.2.** *We have the following estimate* $|F(x,t)| \le C t^{-\delta\mu - 1} x^{\delta}$ *for all* $x > 0, t > 0$ *and for any* $\delta \in \left[ -\frac{1}{\mu}, 1 \right]$.

*Proof.* Making a change of the variable $\xi t = q$ and denoting $z = x t^{-\mu}$ we get

$$F(x,t) = \frac{1}{t} \int_{-i\infty}^{i\infty} e^{q - \Theta q^{\mu} z} dq.$$

Integration by parts in the case $0 < z < 1$ yields

$$|F| \le C z t^{-1} \left| \int_{-i\infty}^{i\infty} e^{q - \Theta q^{\mu} z} q^{\mu - 1} dq \right| \le C z t^{-1} \le C t^{-1} z^{\delta},$$

where $\delta \in \left[ -\frac{1}{\mu}, 1 \right]$. In the case $z > 1$ making a change of the variable $y = q z^{1/\mu}$ we obtain

$$|F| \le C t^{-1} z^{-1/\mu} \left| \int_{-i\infty}^{i\infty} e^{y z^{-1/\mu} - \Theta y^{\mu}} dy \right| \le C t^{-1} z^{-1/\mu} \le C t^{-1} z^{\delta},$$

with any $\delta \in \left[ -\frac{1}{\mu}, 1 \right]$. Lemma 4.2 is proved. □

**Lemma 4.3.** *Suppose that* $\|\bar{u}\|_{\mathbf{L}^1} \le \epsilon$ *and* $\sum_{j=1}^{n}\|\tilde{u}_j\|_{\mathbf{W}_1^1} \le \epsilon$ *and* $\hat{\tilde{u}}'_j(\xi) = O\left(\frac{\epsilon}{|\xi|^2}\right)$ *for* $|\xi| > 1,\ j = 1,...,n$. *We assume that the following estimates for the solution of (1.1) are valid*

$$\sup_{t>0}(1+t)^{\chi}e^{bt}\left((1+t)^{-\frac{1}{2\alpha}}\|u\|_{\mathbf{L}^2} + \|u\|_{\mathbf{L}^\infty}\right) \le \epsilon_1. \tag{4.1}$$

*Then for the solutions* $v_j$ *of system (2.7) we have*

$$\sup_{t>1} t^{\frac{n+j}{\alpha}}e^{bt}v_j(t) \le C(\epsilon + \epsilon_1^{\rho+1}) \tag{4.2}$$

*for* $j = 1,...,m$.

*Proof.* According to (3.3) we write the solutions $v_j(t)$ of system (2.7) in the form

$$v_j(t) = \frac{h}{2\pi i}\sum_{l=1}^{m} r_l^{[\alpha]}M_{j,l}(I_1 + I_2 + I_3), \tag{4.3}$$

where

$$I_1 = \int_{-i\infty}^{i\infty} d\xi e^{\xi t}(\xi+b)^{-1+\frac{n+j}{\alpha}}\int_0^{+\infty} e^{-\phi_l(\xi)x}\bar{u}(x)dx,$$

$$I_2 = \sum_{k=1}^{n}\int_{-i\infty}^{i\infty} e^{\xi t}(\xi+b)^{\frac{n+j}{\alpha}}\phi_l^{-k}(\xi)\hat{\tilde{u}}_k(\xi)d\xi$$

and

$$I_3 = \int_{-i\infty}^{i\infty} d\xi e^{\xi t}(\xi+b)^{\frac{n+j}{\alpha}-1}\int_0^{+\infty} d\tau\int_0^{+\infty} dx\mathbb{N}(u)(x,\tau)e^{-\xi\tau-\phi_l(\xi)x}.$$

Now we estimate each summand of (4.3). Since $\bar{u} \in \mathbf{L}^1$ and $\Re r_l(\pm i)^{\frac{1}{\alpha}} > 0$ making a change of the variable of integration $y = \xi + b$ in the first summand $I_1$ we have by virtue of Lemma 4.2 with $\delta = 0$ and Lemma 3.1

$$I_1 = e^{-bt}\int_0^{+\infty}\bar{u}(x)dx\int_{-i\infty}^{+i\infty} e^{yt - r_l y^{\frac{1}{\alpha}}x}y^{\frac{n+j}{\alpha}-1}dy = O\left(\epsilon e^{-bt}t^{-\frac{n+j}{\alpha}}\right), \tag{4.4}$$

for all $j, l = 1,...,m$. Integrating by parts in the second summand $I_2$ in (4.3) we get

$$\begin{aligned} I_2 &= \sum_{k=1}^{n}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\hat{\tilde{u}}_k(\xi)(\xi+b)^{\frac{n+j}{\alpha}}\phi_l^{-k}(\xi)d\xi \\ &= t^{-1}\sum_{k=1}^{n}\Bigg(e^{\xi t}\frac{\hat{\tilde{u}}_k(\xi)(\xi+b)^{\frac{n+j}{\alpha}}}{\phi_l^k}\Bigg|_{-i\infty}^{i\infty} - \int_{-i\infty}^{i\infty} e^{\xi t}\Bigg(\frac{C\hat{\tilde{u}}_k(\xi)}{(\xi+b)^{1-\frac{n+j}{\alpha}}\phi_l^k(\xi)} \\ &+ \frac{C\hat{\tilde{u}}_k(\xi)}{(\xi+b)^{1-\frac{n+j-k}{\alpha}}} + \frac{C\hat{\tilde{u}}'_k(\xi)(\xi+b)^{\frac{n+j}{\alpha}}}{\phi_l^k(\xi)}\Bigg)d\xi\Bigg) = O\left(\epsilon e^{-bt}t^{-1}\right), \end{aligned} \tag{4.5}$$

since $\hat{\hat{u}}_k(\xi) = O(\epsilon|\xi|^{-1})$ and $\hat{\hat{u}}_k'(\xi) = O(\epsilon|\xi|^{-2})$ for all $|\xi| > 1, \Re\xi = 0$, $k = 1,...,n$. Via (4.1) we have

$$\int_0^{+\infty}(1+x^{-\gamma})|\mathbb{N}(u)(x,t)|dx \le |a(t)|\|u\|_{\mathbf{L}^\infty}^{\rho-1}\left(\|u\|_{\mathbf{L}^2}^2 + \|u\|_{\mathbf{L}^\infty}^2\right) \\ \le \epsilon_1^{\rho+1}e^{-bt(\rho+1)}(1+t)^{-\chi(\rho+1)+1/\alpha-\eta}, \tag{4.6}$$

where $\gamma > 0$ is small enough. Therefore interchanging the order of integration and making a change of the variable of integration $y = (\xi + b)(t-\tau)$ and using the condition $\chi(\rho+1) - \frac{1}{\alpha} + \eta > 1$ if $b = 0$ by virtue of Lemma 3.1 we have for the third summand $I_3$ in (4.3)

$$|I_3| \le Ce^{-bt}\int_0^{+\infty}|t-\tau|^{-\frac{n+j}{\alpha}}d\tau\int_0^{+\infty}e^{b\tau}|\mathbb{N}(u)(x,\tau)|dx \\ \left|\int_{-i\infty}^{i\infty}e^{y-r_l y^{\frac{1}{\alpha}}x(t-\tau)^{-1/\alpha}}y^{\frac{n+j}{\alpha}-1}dy\right| \\ \le C\epsilon_1^{\rho+1}e^{-bt}\int_0^{+\infty}\frac{e^{-b\rho\tau}d\tau}{|t-\tau|^{\frac{n+j}{\alpha}}(1+\tau)^{\chi(\rho+1)-1/\alpha+\eta}}d\tau = O(\epsilon_1^{\rho+1}e^{-bt}t^{-\frac{n+j}{\alpha}}) \tag{4.7}$$

for $j = 1,...,m$, if $\alpha \neq [\alpha]$ and for $j = 1,...,m-1$, if $\alpha = [\alpha]$. In the case $\alpha = [\alpha]$ for $j = m$ using (4.6) and Lemma 4.2 with $\Theta = r_l, l = 1,...,m$ and $\delta = \gamma$ in the domain $|t-\tau| \le 1$ and $\delta = 0$ in the domain $|t-\tau| > 1$ we have for $I_3$

$$I_3 = Ce^{-bt}\int_0^{+\infty}d\tau\int_0^{+\infty}F(x,t-\tau)e^{b\tau}\mathbb{N}(u)(x,\tau)dx \\ \le Ce^{-bt}\bigg(\int_{|t-\tau|\le 1}\frac{d\tau}{|t-\tau|^{1-\frac{\gamma}{\alpha}}}\int_0^{+\infty}\frac{|\mathbb{N}(u)(x,\tau)|}{x^\gamma}dx \\ + \int_{|t-\tau|>1}\frac{d\tau}{|t-\tau|}\int_0^{+\infty}|\mathbb{N}(u)(x,\tau)|dx\bigg) = O\left(e^{-bt}\epsilon_1^{\rho+1}t^{-1}\right).$$

Substitution of (4.4), (4.5) and (4.7) into (4.3) yields estimate (4.2). Lemma 4.3 is proved. □

**Lemma 4.4.** *Let the initial data $\bar{u} \in \mathbf{L}^1$ be small: $\|\bar{u}\|_{\mathbf{L}^1} \le \epsilon$. Then*

$$\left\|\int_{-i\infty}^{i\infty}e^{px-(K(p)+b)t}\hat{\bar{u}}(p)dp\right\|_{\mathbf{L}^s} = O\left(\epsilon e^{-bt}(1+t)^{-\frac{1}{\alpha}+\frac{1}{s\alpha}}\right)$$

*for all $t > 1$, where $s = 2, +\infty$.*

*Proof.* Making a change of the variable $iy = pt^{\frac{1}{\alpha}}$ we have

$$\left\| \int_{-i\infty}^{i\infty} e^{px-(K(p)+b)t}\hat{\tilde{u}}(p)dp \right\|_{\mathbf{L}^\infty}$$
$$\leq e^{-bt}t^{-\frac{1}{\alpha}}\|\hat{\tilde{u}}\|_{\mathbf{L}^\infty}\int_0^{+\infty} e^{-\Theta y^\alpha}dy = O\left(\epsilon e^{-bt}(1+t)^{-\frac{1}{\alpha}}\right),$$

and

$$\left\| \int_{-i\infty}^{i\infty} e^{px-(K(p)+b)t}\hat{\tilde{u}}(p)dp \right\|_{\mathbf{L}^2} \leq e^{-bt}\left(\int_{-i\infty}^{i\infty} e^{-2Ep^\alpha t}|\hat{\tilde{u}}|^2 dp\right)^{\frac{1}{2}}$$
$$\leq \frac{Ce^{-bt}\epsilon}{(1+t)^{\frac{1}{2\alpha}}}\left(\int_0^\infty e^{-2\Theta y^\alpha}dy\right)^{\frac{1}{2}} < \frac{Ce^{-bt}\epsilon}{(1+t)^{\frac{1}{2\alpha}}}$$

since $\Theta = \min(\Re E(\pm i)^\alpha) > 0$. Lemma 4.4 is proved. $\square$

**Lemma 4.5.** *Let the boundary data have the large time representation*

$$\tilde{u}_j(t) = A_j e^{-bt}t^{-\chi-\frac{j-1}{\alpha}} + e^{-bt}\phi_j(t) \tag{4.8}$$

*for $j = 1, ..., n$ and $t > 0$, where $|\phi_j(t)| \leq C\epsilon t^{-\chi-\frac{j-1}{\alpha}-\gamma}$ and*
*$\phi_1'(t) = O\big(\epsilon t^{-\chi-\frac{j-1}{\alpha}-\gamma}\big)$, $A_j$ are some constants which do not equal to zero simultaneously and $|A_j| \leq \epsilon$, here $\gamma, \epsilon > 0$ are small enough, $\chi$ is defined at the beginning of this section. Then we have the following estimate*

$$\left\| \int_{-i\infty}^{i\infty} dpe^{px}K(p)p^{-j}\int_0^t e^{-(K(p)+b)(t-\tau)}\tilde{u}_j(\tau)d\tau \right\|_{\mathbf{L}^s} = O\left(\epsilon e^{-bt}t^{-\chi-\frac{1}{s\alpha}}\right)$$

*and the asymptotics*

$$\begin{aligned} I &\equiv \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dpe^{px}K(p)p^{-j}\int_0^t e^{-(K(p)+b)(t-\tau)}\tilde{u}_j(\tau)d\tau \\ &= \frac{Ee^{-bt}}{2\pi i}t^{-\chi}A_jG_j\left(\frac{x}{t^{1/\alpha}}\right) + O(\epsilon e^{-bt}t^{-\chi-\gamma}), \end{aligned} \tag{4.9}$$

*for all $t > 1$ uniformly with respect to $x$, where*

$$G_j(q) = \int_{-i\infty}^{i\infty} e^{yq}y^{\alpha-j}dy\int_0^1 e^{-Ey^\alpha(1-z)}z^{-\chi-\frac{j-1}{\alpha}}dz,$$

*and $E$ is the constant from the definition of the symbol $K(p)$ of the pseudo-differential operator (such that $\Re E(\pm i)^\alpha > 0$.)*

*Proof.* Substitution of (4.8) into $I$ yields

$$I \equiv \frac{EA_je^{-bt}}{2\pi i}\int_{-i\infty}^{i\infty} dpe^{px}p^{\alpha-j}\int_0^t e^{-Ep^\alpha(t-\tau)}\tau^{-\chi-\frac{j-1}{\alpha}}d\tau + R(x,t), \tag{4.10}$$

where

$$R(x,t)=\epsilon e^{-bt}\int_{-i\infty}^{i\infty}e^{px}p^{\alpha-j}dp\int_0^t e^{-Ep^{\alpha}(t-\tau)}\phi_j(\tau)d\tau.$$

Since $|\phi_j(t)|\le C\epsilon t^{-\chi-\frac{j-1}{\alpha}-\gamma}$ and $\chi<\frac{1}{\alpha}$ making a change of the variable of integration $p(t-\tau)^{1/\alpha}=y$ we get for $j=2,...,n$

$$|R(x,t)|\le C\epsilon e^{-bt}\int_0^t\frac{d\tau}{\tau^{\chi+\frac{j-1}{\alpha}+\gamma}(t-\tau)^{1-\frac{j-1}{\alpha}}}\int_0^{+\infty}e^{-\Theta y^{\alpha}}y^{\alpha-j}dy$$
$$=O\big(\epsilon e^{-bt}t^{-\chi-\gamma}\big),$$

where $\Theta=\min(\Re E(i)^{\alpha},\Re E(-i)^{\alpha})>0$. In the case $j=1$ making a change of variables $\tau=tz$ and $pt^{1/\alpha}=y$ we have

$$R(x,t)=e^{-bt}\int_0^{\frac{1}{2}}\phi_1(tz)dz\int_{-i\infty}^{i\infty}e^{yq}y^{\alpha-1}e^{-Ey^{\alpha}(1-z)}dy$$
$$+e^{-bt}\int_{-i}^{i}e^{yq}y^{\alpha-1}dy\int_{\frac{1}{2}}^{1}e^{-Ey^{\alpha}(1-z)}\phi_1(tz)dz$$
$$+e^{-bt}\int_{|y|\ge1}e^{yq}y^{\alpha-1}dy\int_{\frac{1}{2}}^{1}e^{-Ey^{\alpha}(1-z)}\phi_1(tz)dz,$$

where $q=xt^{-\frac{1}{\alpha}}$. By analogy with the proof of Lemma 4.1 we get for $\nu>0$ is small

$$|R(x,t)|\le Ce^{-bt}\bigg(\int_0^1|\phi_1(tz)|d\tau+\int_0^{\frac{1}{2}}|\phi_1(tz)|(1-z)^{-\nu}dz$$
$$+|\phi_1(t)|+|\phi_1(t/2)|+\int_{1/2}^{1}|\phi_1'(tz)|(1-z)^{-\nu}dz\bigg).$$

Then using the conditions of the lemma for $\phi_1(t)$ and $\phi_1'(t)$ we obtain in the case $j=1$, $R(x,t)=O\big(\epsilon e^{-bt}t^{-\chi-\gamma}\big)$. Therefore making a change of the variables $\tau=tz$ and $pt^{1/\alpha}=y$ in the first summand of (4.10) we get (4.9). Since $\chi<\frac{1}{\alpha}$ making a change of the variable of integration $iy=p(t-\tau)^{\frac{1}{\alpha}}$ and using (4.8) we get

$$\bigg\|\int_{-i\infty}^{i\infty}dpe^{px}K(p)p^{-j}\int_0^t e^{-(K(p)+b)(t-\tau)}\tilde{u}_j(\tau)d\tau\bigg\|_{\mathbf{L}^2}$$
$$\le Ce^{-bt}\int_0^t e^{b\tau}|\tilde{u}_j(\tau)|d\tau\bigg(\int_{-i\infty}^{+i\infty}e^{-2\Re Ep^{\alpha}(t-\tau)}|p|^{2\alpha-2j}|dp|\bigg)^{\frac{1}{2}}$$
$$\le C\epsilon e^{-bt}\int_0^t\frac{d\tau}{\tau^{\chi+\frac{j-1}{\alpha}}(t-\tau)^{1-\frac{j}{\alpha}+1/2\alpha}}\bigg(\int_0^{+\infty}e^{-\Theta y^{\alpha}}y^{2\alpha-2j}dy\bigg)^{\frac{1}{2}}$$
$$<\frac{C\epsilon e^{-bt}}{(1+t)^{\chi-1/2\alpha}}.$$

From Lemmas 4.1 - 4.2 and asymptotics (4.9) we easily obtain

$$\left\| \int_{-i\infty}^{i\infty} dp e^{px} K(p) p^{-j} \int_0^t e^{-(K(p)+b)(t-\tau)} \tilde{u}_j(\tau) d\tau \right\|_{\mathbf{L}^\infty} < C\epsilon e^{-bt}(1+t)^{-\chi}.$$

Therefore Lemma 4.5 is proved. □

**Lemma 4.6.** *Let estimate (4.2) be valid. Then the following estimates*

$$\left\| \int_{-i\infty}^{i\infty} e^{px} K(p) p^{-n-j} dp \int_0^t e^{-(K(p)+b)(t-\tau)} v_j(\tau) d\tau \right\|_{\mathbf{L}^s}$$
$$= O\left(e^{-bt}(\epsilon + \epsilon_1^{\rho+1}) t^{-\chi-\gamma+\frac{1}{s\alpha}}\right)$$

*are true for all $j = 1, ..., m$, where $s = 2$ or $\infty$, $\gamma > 0$ is small.*

*Proof.* Via the estimate of Lemma 3.2 we have $t^{\frac{j-1}{\alpha}}|v_j(t)| \le \epsilon + \epsilon_1^{\rho+1}$ for $t \in [0,1], j = 1, ..., m$. Therefore by virtue of (4.2) making a change of the variables $tz = \tau$ and $y = pt^{\frac{1}{\alpha}}$ we obtain

$$\left\| \int_{-i\infty}^{i\infty} e^{px} K(p) p^{-n-j} dp \int_0^t e^{-(K(p)+b)(t-\tau)} v_j(\tau) d\tau \right\|_{\mathbf{L}^\infty}$$
$$\le Ce^{-bt} t^{\frac{n+j-1}{\alpha}} \left| \int_{-i\infty}^{i\infty} e^{xy/t^{1/\alpha}} y^{\alpha-n-j} dy \int_0^1 e^{-Ey^\alpha(1-z)+btz} v_j(tz) dz \right|$$
$$\le C(\epsilon + \epsilon_1^{\rho+1}) e^{-bt} \left( t^{-\frac{1}{\alpha}} \int_0^{+\infty} y^{\alpha-n-j} dy \int_{1/t}^1 e^{-\Theta y^\alpha(1-z)} z^{-\frac{n+j}{\alpha}} dz \right.$$
$$\left. + t^{\frac{n}{\alpha}} \left| \int_0^{1/t} \frac{dz}{z^{\frac{j-1}{\alpha}}} \right| \right) = O\big(e^{-bt}(\epsilon + \epsilon_1^{\rho+1}) t^{-\frac{1}{\alpha}} \log(2+t)\big)$$
$$= O\big(e^{-bt}(\epsilon + \epsilon_1^{\rho+1}) t^{-\chi-\gamma}\big)$$

with $\Theta = \min(\Re(i)^\alpha E, \Re(-i)^\alpha E) > 0$, where $\gamma > 0$ is small enough. Similarly making a change of the variables $p(t-\tau)^{\frac{1}{\alpha}} = y$ we get

$$\left\| \int_{-i\infty}^{i\infty} e^{px} K(p) p^{-n-j} dp \int_0^t e^{-(K(p)+b)(t-\tau)} v_j(\tau) d\tau \right\|_{\mathbf{L}^2}$$
$$= C \left\| K(p) p^{-n-j} dp \int_0^t e^{-(K(p)+b)(t-\tau)} v_j(\tau) d\tau \right\|_{\mathbf{L}^2}$$
$$\le Ce^{-bt} \left( \int_0^{+\infty} dy e^{-\Theta y^\alpha} y^{2\alpha-2n-2j} \right)^{\frac{1}{2}} \int_0^t \frac{e^{b\tau}|v_j(\tau)| d\tau}{|t-\tau|^{1+\frac{1}{2\alpha}-\frac{n+j}{\alpha}}}$$
$$\le Ce^{-bt}(\epsilon + \epsilon_1^{\rho+1}) \left( \int_0^1 \frac{d\tau}{\tau^{\frac{j-1}{\alpha}} (t-\tau)^{1+1/2\alpha-\frac{n+j}{\alpha}}} + \int_1^t \frac{d\tau}{\tau^{\frac{n+j}{\alpha}} (t-\tau)^{1+\frac{1}{2\alpha}-\frac{n+j}{\alpha}}} \right)$$
$$\le \frac{Ce^{-bt}(\epsilon + \epsilon_1^{\rho+1})}{(1+t)^{\frac{1}{2\alpha}}} \le \frac{Ce^{-bt}(\epsilon + \epsilon_1^{\rho+1})}{(1+t)^{\chi+\gamma-\frac{1}{2\alpha}}}$$

for all $j=1,...,m$. Lemma 4.6 is proved. □

**Lemma 4.7.** *Let estimates (4.1) be valid. Then we have*

$$\left\| \int_{-i\infty}^{i\infty} e^{px} dp \int_0^t e^{-(K(p)+b)(t-\tau)} \widehat{\mathbb{N}(u)}(p,\tau) d\tau \right\|_{\mathbf{L}^s} = O\left(\epsilon_1^{\rho+1} e^{-bt} t^{\frac{1}{s\alpha}-\chi-\gamma}\right)$$

*for all $t>1$, where $s=2$ or $\infty$, $\gamma>0$ is small.*

*Proof.* Via (4.1) we have

$$\left\| \widehat{\mathbb{N}(u)}(p,t) \right\|_{\mathbf{L}^\infty} = \sup_{p:\Re p=0} \left| \int_0^{+\infty} e^{-px} a(t)|u|^\rho u dx \right| \le |a(t)| \int_0^{+\infty} |u|^{\rho+1} dx$$
$$\le |a(t)| \|u\|_{\mathbf{L}^\infty}^{\rho-1} \|u\|_{\mathbf{L}^2}^2 \le \epsilon_1^{\rho+1} e^{-bt(\rho+1)} (1+t)^{-\beta},$$

where $\beta=\chi(\rho+1)-1/\alpha+\eta>1$ if $b=0$ by virtue of the condition of Theorem 1.2. Therefore making a change of the variables $p(t-\tau)^{\frac{1}{\alpha}}=iy$ we obtain

$$\left\| \int_{-i\infty}^{i\infty} e^{px} dp \int_0^t e^{-(K(p)+b)(t-\tau)} \widehat{\mathbb{N}(u)}(p,\tau) d\tau \right\|_{\mathbf{L}^\infty}$$
$$\le C\epsilon_1^{\rho+1} e^{-bt} \int_0^t \frac{e^{-b\rho\tau} d\tau}{(1+\tau)^\beta (t-\tau)^{1/\alpha}} \int_0^{+\infty} e^{-\Theta y^\alpha} dy = O(\epsilon_1^{\rho+1} e^{-bt} t^{-\frac{1}{\alpha}})$$

for $t>1$, where $\Theta=\min(\Re E i^\alpha, \Re E(-i)^\alpha)>0$. Similarly

$$\left\| \int_{-i\infty}^{i\infty} e^{px} dp \int_0^t e^{-(K(p)+b)(t-\tau)} \widehat{\mathbb{N}(u)}(p,\tau) d\tau \right\|_{\mathbf{L}^2}$$
$$\le Ce^{-bt} \int_0^t e^{b\tau} \frac{\|\widehat{\mathbb{N}(u)}(p,t)\|_{\mathbf{L}^\infty}}{(t-\tau)^{\frac{1}{2\alpha}}} \left( \int_0^\infty e^{-2\Theta y^\alpha} dy \right)^{\frac{1}{2}}$$
$$\le C\epsilon_1^{\rho+1} e^{-bt} \int_0^t \frac{e^{-b\rho\tau} d\tau}{(1+\tau)^\beta (t-\tau)^{\frac{1}{2\alpha}}}$$
$$\le C\epsilon_1^{\rho+1} e^{-bt} (1+t)^{-1/2\alpha} < Ce^{-bt} \epsilon_1^{\rho+1} (1+t)^{-\chi+\frac{1}{2\alpha}-\gamma}.$$

Lemma 4.7 is proved. □

*Proof of Theorem 1.2.* Let us prove the following estimate

$$e^{bt}(1+t)^\chi \left( \|u\|_{\mathbf{L}^\infty} + (1+t)^{-\frac{1}{2\alpha}} \|u\|_{\mathbf{L}^2} \right) < \epsilon_1 \tag{4.11}$$

for all $t>0$. By the contrary we suppose that the estimate (4.11) is violated for some time. By Theorem 1.1 the norms $\|u\|_{\mathbf{L}^2}$ and $\|u\|_{\mathbf{L}^\infty}$ are continuous.

Therefore there exist a maximal time $T > 0$ such that the nonstrict estimate (4.11) is valid on $[0, T]$. We have by formula (3.2)

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp e^{px}\Big(\hat{\bar{u}}(p)e^{-(K(p)+b)t} \\ &+ \sum_{j=1}^{n}\frac{K(p)}{p^j}\int_0^t e^{-(K(p)+b)(t-\tau)}\tilde{u}_j(\tau)d\tau \\ &+ \sum_{j=1}^{m}K(p)p^{-n-j}\int_0^t e^{-(K(p)+b)(t-\tau)}v_j(\tau)d\tau \\ &+ \int_0^t e^{-(K(p)+b)(t-\tau)}\widehat{\mathbb{N}(u)}(p,\tau)d\tau\Big). \end{aligned} \tag{4.12}$$

Whence by Lemmas 4.3 - 4.7 we get $e^{bt}(1+t)^{\chi}\left(\|u\|_{\mathbf{L}^\infty} + (1+t)^{-\frac{1}{2\alpha}}\|u\|_{\mathbf{L}^2}\right) < \epsilon_1$ for all $t \in [0,T]$ if $\epsilon > 0$ is sufficiently small. The contradiction obtained proves estimate (4.11). Then from (4.12) by virtue of Remark 3.1 we see that the solution $u(x,t) \in \mathbf{C}\left([0,\infty);\mathbf{L}^2 \cap \mathbf{L}^\infty\right) \cap \mathbf{C}\left(\mathbf{R}^+;\mathbf{W}_2^{[\alpha]-1} \cap \mathbf{C}^{[\alpha]-1}\right)$ and by virtue of Lemmas 4.1-4.7 it has asymptotics (1.4) for $t > 1$ uniformly with respect to $x \geq 0$. Theorem 1.2 is proved. $\square$

## §5 Asymptotics determined by the nonlinearity and initial data

This section is devoted to the proof of Theorem 1.3. We consider the case, when $\alpha > 1$ is not equal to an odd integer number. Before proving Theorem 1.3 we prepare some estimates in Lemmas 5.1 - 5.5. We suppose that we already have the following estimates for the solution

$$\sup_{t\in[0,T]} e^{bt}(1+t)^{\frac{1}{\alpha}}\left((1+t)^{-1/2\alpha}\|u\|_{\mathbf{L}^2} + t^{\gamma}(1+t)^{-\gamma}\|u\|_{\mathbf{L}^\infty}\right) < \epsilon_1, \tag{5.1}$$

where $\epsilon_1 > 0$ is some small constant, $T > 0$. Below everywhere the sum of the form $\sum_{j=1}^{0}$ we suppose to be identically zero (It appears for the case $\alpha \in (1,2)$ so $n = 0$ and the boundary data are absent). Also any condition in which $j$ varies from 1 to $n = 0$ we assume to be absent.

**Lemma 5.1.** *Let the initial data $\bar{u} \in \mathbf{L}^\infty$ and $x^\delta\bar{u} \in \mathbf{L}^1$, $\delta \in [0,\frac{1}{2}]$ and the boundary data $\tilde{u}_j \in \mathbf{Y}$, $j = 1,...,n$ and satisfy condition (1.3) with $\chi = \frac{1}{\alpha}$ as follows $\tilde{u}_j(t) = A_j e^{-bt}t^{-\frac{j}{\alpha}} + e^{-bt}\phi_j(t)$, where $|\phi_j(t)| \leq C\epsilon t^{-\frac{j}{\alpha}-\gamma}$ and $\phi_1'(t) = O\left(\epsilon t^{-\frac{j}{\alpha}-\gamma}\right)$. Suppose that the estimate (5.1) is valid and $\frac{\rho}{\alpha} + \eta > 1$, if $b = 0$. Then we have the following estimate $\|x^\delta u\|_{\mathbf{L}^2} \leq Ce^{-bt}$ for all $t > 1$.*

*Proof.* By formula (3.2) for the solution we get

$$
\begin{aligned}
\|x^\delta u\|_{\mathbf{L}^2} = C\left\|\partial_p^\delta \hat{u}(p,t)\right\|_{\mathbf{L}^2} \le Ce^{-bt}\Bigg(\left\|\partial_p^\delta\left(\hat{\bar{u}}(p)e^{-K(p)t}\right)\right\|_{\mathbf{L}^2} \\
+\sum_{j=1}^{n}\int_0^t e^{b\tau}|\tilde{u}_j(\tau)|\left\|\partial_p^\delta\left(e^{-K(p)(t-\tau)}p^{\alpha-j}\right)\right\|_{\mathbf{L}^2}d\tau \\
+\sum_{j=1}^{m}\int_0^t e^{b\tau}|v_j(\tau)|\left\|\partial_p^\delta\left(e^{-K(p)(t-\tau)}p^{\alpha-n-j}\right)\right\|_{\mathbf{L}^2}d\tau \\
+\int_0^t e^{b\tau}\left\|\partial_p^\delta\left(e^{-K(p)(t-\tau)}\widehat{\mathbb{N}(u)}(p,\tau)\right)\right\|_{\mathbf{L}^2}d\tau\Bigg) \equiv Ce^{-bt}\sum_{j=1}^{4}J_j,
\end{aligned}
\tag{5.2}
$$

where $\partial_p^\delta\phi = \mathcal{L}^{-1}(p^\delta\mathcal{L}\phi)$, by $\mathcal{L}$ we denote the Laplace transformation. We have the Sobolev embedding inequality (see, e.g., [1])

$$
\|\partial_p^\delta\phi\|_{\mathbf{L}^2} \le \|\phi\|_{\mathbf{L}^2}^{1-2\delta}\|\partial_p\phi\|_{\mathbf{L}^1}^{2\delta}. \tag{5.3}
$$

Consider the function $G(x,t) = \int_{-i\infty}^{i\infty} e^{px-Ep^\alpha t}dp$. Note that the Laplace transform $\hat{G}(p,t) = 2\pi i e^{-Ep^\alpha t}$. Then making a change of the variable of integration $y = pt^{1/\alpha}$ we easily obtain

$$
\|G(x,t)\|_{\mathbf{L}^2} = \|\hat{G}(p,t)\|_{\mathbf{L}^2} \le Ct^{-\frac{1}{2\alpha}} \tag{5.4}
$$

and

$$
\|x^\delta G(x,t)\|_{\mathbf{L}^2} = C\|\partial_p^\delta\hat{G}(p,t)\|_{\mathbf{L}^2} \le C\|\hat{G}(p,t)\|_{\mathbf{L}^2}^{1-2\delta}\|\partial_p\hat{G}(p,t)\|_{\mathbf{L}^1}^{2\delta} \le Ct^{\frac{2\delta-1}{2\alpha}} \tag{5.5}
$$

for $t>0$. Changing the order of integration in the first summand in (5.2) we get

$$
\begin{aligned}
J_1 &= \left\|x^\delta\int_{-i\infty}^{i\infty}dpe^{px-Ep^\alpha t}\int_0^{+\infty}e^{-py}\bar{u}(y)dy\right\|_{\mathbf{L}^2} \\
&= \left\|x^\delta\int_0^\infty \bar{u}(y)G(x-y,t)dy\right\|_{\mathbf{L}^2}.
\end{aligned}
$$

Since $\|\int_0^\infty\phi(x-y)\psi(y)dy\|_{\mathbf{L}^2} \le \|\phi\|_{\mathbf{L}^2}\|\psi\|_{\mathbf{L}^1}$ by virtue of (5.4) and (5.5) we have

$$
\begin{aligned}
J_1 &\le \left\|\int_0^\infty y^\delta\bar{u}(y)G(x-y,t)dy\right\|_{\mathbf{L}^2} + \left\|\int_0^\infty |x-y|^\delta\bar{u}(y)G(x-y,t)dy\right\|_{\mathbf{L}^2} \\
&\le \|x^\delta\bar{u}(x)\|_{\mathbf{L}^1}\|G(x,t)\|_{\mathbf{L}^2} + \|\bar{u}(x)\|_{\mathbf{L}^1}\|x^\delta G(x,t)\|_{\mathbf{L}^2} \le C
\end{aligned}
\tag{5.6}
$$

for all $t > 1$. Similarly we can estimate $J_4$. Indeed, by virtue of (5.1) we have

$$\|x^\delta \mathbb{N}(u)\|_{\mathbf{L}^1} \le C|a(t)| \|x^\delta u\|_{\mathbf{L}^2} \|u\|_{\mathbf{L}^2} \|u\|_{\mathbf{L}^\infty}^{\rho-1}$$
$$\le Ce^{-bt(\rho+1)} t^{-\gamma} (1+t)^{-\frac{2\rho-1}{2\alpha}-\eta+\gamma} \|x^\delta u\|_{\mathbf{L}^2}$$

and $\|\mathbb{N}(u)\|_{\mathbf{L}^1} \le C|a(t)| \|u\|_{\mathbf{L}^2}^2 \|u\|_{\mathbf{L}^\infty}^\rho \le Ce^{-bt(\rho+1)} t^{-\gamma}(1+t)^{-\frac{\rho}{\alpha}-\eta+\gamma}$. Therefore using (5.4) and (5.5) we get

$$J_4 \le \int_0^t \left( \|x^\delta \mathbb{N}(u)\|_{\mathbf{L}^1} \|G(x, t-\tau)\|_{\mathbf{L}^2} + \|\mathbb{N}(u)\|_{\mathbf{L}^1} \|x^\delta G(x, t-\tau)\|_{\mathbf{L}^2} \right) d\tau$$
$$\le C + \int_0^t e^{-b\tau(\rho+1)} \|x^\delta u\|_{\mathbf{L}^2} (t-\tau)^{-\frac{1}{2\alpha}} \tau^{-\gamma} (1+\tau)^{-\frac{2\rho-1}{2\alpha}-\eta+\gamma} d\tau \tag{5.7}$$

since $\frac{\rho}{\alpha} + \eta > 1$, if $b = 0$. Using the condition $e^{bt} t^{\frac{j}{\alpha}} |\tilde{u}_j(t)| \le C$ of the lemma, making a change of the variable of integration $y = p(t-\tau)^{1/\alpha}$ via (5.3) we obtain

$$J_2 \le C \sum_{j=1}^n \int_0^t e^{b\tau} |\tilde{u}_j(\tau)| \|e^{-Ep^\alpha (t-\tau)} p^{\alpha-j}\|_{\mathbf{L}^2}^{1-2\delta} \Big( \|e^{-Ep^\alpha(t-\tau)} p^{\alpha-j-1}\|_{\mathbf{L}^1}^{2\delta}$$
$$+ (t-\tau)^{2\delta} \|e^{-Ep^\alpha(t-\tau)} p^{2\alpha-j-1}\|_{\mathbf{L}^1}^{2\delta} \Big)$$
$$\le C \sum_{j=1}^n \int_0^t \tau^{-\frac{j}{\alpha}} (t-\tau)^{-1+\frac{j}{\alpha}+\frac{2\delta-1}{2\alpha}} d\tau \le C \tag{5.8}$$

for all $t \ge 1$. By Lemma 3.3 and Lemma 4.3 we have $e^{bt} t^{\frac{j-1}{\alpha}} (1+t)^{\frac{n+1}{\alpha}} |v_j(t)| \le C$. Then in the same way we obtain the following estimate for $J_3$

$$J_3 \le C \sum_{j=1}^m \int_0^t \tau^{\frac{1-j}{\alpha}} (1+\tau)^{-\frac{n+1}{\alpha}} (t-\tau)^{-1+\frac{j+n}{\alpha}-\frac{1-2\delta}{2\alpha}} d\tau \le C \tag{5.9}$$

for all $t \ge 1$. Substitution of (5.6) - (5.9) into (5.2) yields

$$\|\partial_p^\delta \hat{u}(p,t)\|_{\mathbf{L}^2} \le Ce^{-bt}$$
$$+ e^{-bt} \int_0^t e^{-b\tau(\rho+1)} \tau^{-\gamma} (1+\tau)^{-\frac{2\rho-1}{2\alpha}-\eta+\gamma} (t-\tau)^{-\frac{1}{2\alpha}} \|\partial_p^\delta \hat{u}(p,\tau)\|_{\mathbf{L}^2} d\tau.$$

Therefore via the Gronwall inequality we have

$\|\partial_p^\delta \hat{u}(p,t)\|_{\mathbf{L}^2} = \|x^\delta u(x,t)\|_{\mathbf{L}^2} \le Ce^{-bt}$ since $\frac{\rho}{\alpha} + \eta > 1$ if $b = 0$. Lemma 5.1 is proved. □

**Lemma 5.2.** *Let the initial data $\bar{u} \in \mathbf{X}$, $x\bar{u} \in \mathbf{L}^1$, and boundary data $\tilde{u}_j \in \mathbf{Y}$, $j = 1, ..., n$ be small $\|\bar{u}\|_{\mathbf{X}} + \|x\bar{u}\|_{\mathbf{L}^1} + \sum_{j=1}^{n} \|\tilde{u}_j\|_{\mathbf{Y}} \leq \epsilon$. Moreover let the following estimate for the Laplace transform of the boundary data $\hat{\tilde{u}}_j' = O(\frac{\epsilon}{|\xi|^2})$ be valid for $|\xi| > 1$, $j = 1, ..., n$. When $\alpha > 1$ is integer we also suppose that $\hat{\tilde{u}}_j'' = O(\frac{\epsilon}{|\xi|^2})$ for $|\xi| > 1$, $j = 1, ..., n$. Let estimate (5.1) be valid and $\frac{\rho}{\alpha} + \eta > 1$, if $b = 0$.*

*Then the following asymptotics*

$$v_j(t) = B_j e^{-bt} t^{-\frac{n+j}{\alpha}} + O\left(e^{-bt}(\epsilon + \epsilon_1^{\rho+1}) t^{-\frac{n+j}{\alpha} - \gamma}\right) \tag{5.10}$$

*takes place for the solutions $v_j(t)$ of system (2.7) (see (3.3)), where*

$$B_j = h\Gamma\left(1 - \frac{n+j}{\alpha}\right) \sum_{l=1}^{m} M_{j,l} r_l^{[\alpha]} \left(\hat{\bar{u}}(0) + \int_0^\infty \int_0^\infty e^{b\tau} \mathbb{N}(u)(x,\tau) dx d\tau\right),$$

*for $j = 1, ..., m$, $h = (\det \tilde{W})^{-1}$, $M_{j,l}$ are the algebraic minors of matrix $\tilde{W}$ (see 2.8), $\gamma > 0$ is some small constant. When $\alpha > 1$ is integer we denote $B_m = 0$. $\Gamma$ is the Euler Gamma-function.*

*Proof.* According to (3.3) changing the variable of integration $\xi' = \xi + b$ (prime we omit), we write the solutions $v_j(t)$ of the system (2.7) in the form

$$v_j(t) = he^{-bt} \sum_{l=1}^{m} r_l^{[\alpha]} M_{j,l} (I_1 + I_2 + I_3), \tag{5.11}$$

where $h = (\det \tilde{W})^{-1}$, $M_{j,l}$ are the algebraic minors of matrix $\tilde{W}$ (see 2.8),

$$I_1 = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\xi e^{\xi t} \xi^{-1+\frac{n+j}{\alpha}} \int_0^{+\infty} e^{-r_l \xi^{\frac{1}{\alpha}} x} \bar{u}(x) dx,$$

$$I_2 = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{-i\infty}^{i\infty} e^{\xi t} \xi^{\frac{n+j-k}{\alpha}} r_l^{-k} \hat{\tilde{u}}_k(\xi - b) d\xi$$

and

$$I_3 = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\xi e^{\xi t} \xi^{-1+\frac{n+j}{\alpha}} \int_0^{+\infty} e^{b\tau} d\tau \int_0^{+\infty} dx \mathbb{N}(u)(x,\tau) e^{-\xi\tau - r_l \xi^{\frac{1}{\alpha}} x}.$$

Now we estimate each summand in representation (5.11). We rewrite the first integral in (5.11) as follows

$$I_1 = \frac{\hat{\bar{u}}(0)}{2\pi i} \int_{-i\infty}^{i\infty} e^{\xi t} \xi^{\frac{n+j}{\alpha} - 1} d\xi + R = \hat{\bar{u}}(0) t^{-\frac{n+j}{\alpha}} \Gamma\left(1 - \frac{n+j}{\alpha}\right) + R \tag{5.12}$$

for $j=1,...,m$, if $\alpha\neq[\alpha]$ and $j=1,...,m-1$, if $\alpha=[\alpha]$, where

$$\begin{aligned} R = C\Bigg( & \left|\int_{|\xi|\le 1/t,\Re\xi=0} e^{\xi t}(\hat{u}(r_l\xi^{\frac{1}{\alpha}})-\hat{u}(0))\xi^{\frac{n+j}{\alpha}-1}d\xi\right| + \\ & \left|\int_{1\ge|\xi|\ge 1/t,\Re\xi=0} e^{\xi t}(\hat{u}(r_l\xi^{\frac{1}{\alpha}})-\hat{u}(0))\xi^{\frac{n+j}{\alpha}-1}d\xi\right| \\ & + \left|\int_{|\xi|>1,\Re\xi=0} e^{\xi t}(\hat{u}(r_l\xi^{\frac{1}{\alpha}})+\hat{u}(0))\xi^{\frac{n+j}{\alpha}-1}d\xi\right|\Bigg). \end{aligned} \tag{5.13}$$

Since $x\bar{u}(x)\in\mathbf{L}^1$ we have $|\hat{u}(r_l\xi^{\frac{1}{\alpha}})-\hat{u}(0)|\le C\epsilon|\xi|^{1/\alpha}$. Then for the first summand in (5.13) we get

$$\begin{aligned} & \left|\int_{|\xi|\le 1/t,\Re\xi=0} e^{\xi t}(\hat{u}(r_l\xi^{\frac{1}{\alpha}})-\hat{u}(0))\xi^{\frac{n+j}{\alpha}-1}d\xi\right| \\ & \le C\epsilon\int_0^{1/t}\xi^{-1+\frac{n+j+1}{\alpha}}d\xi = O(\epsilon t^{-\frac{n+j}{\alpha}-\gamma}) \end{aligned} \tag{5.14}$$

and since $|\hat{u}'|\le\epsilon$ integrating by parts we have for the second summand in (5.13)

$$\begin{aligned} & \left|\int_{i/t}^{i} e^{\xi t}(\hat{u}(r_l\xi^{\frac{1}{\alpha}})-\hat{u}(0))\xi^{\frac{n+j}{\alpha}-1}d\xi\right| \le \frac{1}{t}\Bigg(\left|(\hat{u}(r_l\xi^{\frac{1}{\alpha}})-\hat{u}(0))\xi^{\frac{n+j}{\alpha}-1}\Big|_{i/t}^{i}\right| \\ & + \left|\int_{i/t}^{i} e^{\xi t}(\hat{u}(r_l\xi^{\frac{1}{\alpha}})-\hat{u}(0))\xi^{\frac{n+j}{\alpha}-2}d\xi\right| + \left|\int_{i/t}^{i} e^{\xi t}\hat{u}'(r_l\xi^{\frac{1}{\alpha}})\xi^{\frac{n+1+j}{\alpha}-2}d\xi\right|\Bigg) \\ & = O\left(\epsilon t^{-\frac{n+j}{\alpha}-\gamma}\right), \end{aligned} \tag{5.15}$$

where $\gamma=\frac{\alpha-[\alpha]}{\alpha}$. Since $\|\bar{u}\|_{\mathbf{L}^1}\le\epsilon$ and $e^{-\Re\phi_l(\xi)x}\le C(\phi_l(\xi)x)^{-1}$ we obtain the estimate

$$\left|\frac{d}{d\xi}\hat{u}(\phi_l(\xi))\right| \le \left|\int_{-i\infty}^{i\infty} e^{-\Re\phi_l(\xi)x}\phi_l'(\xi)x\bar{u}(x)dx\right| \le C\epsilon|\xi|^{-1}.$$

Therefore integrating by parts we get for the third summand in (5.13)

$$\begin{aligned} & \left|\int_{|\xi|>1} e^{\xi t}(\hat{u}(r_l\xi^{\frac{1}{\alpha}})+\hat{u}(0))\xi^{\frac{n+j}{\alpha}-1}d\xi\right| \le \frac{1}{t}\Bigg(\left|e^{\xi t}(\hat{u}(r_l\xi^{1/\alpha})+\hat{u}(0))\xi^{\frac{n+j}{\alpha}-1}\Big|_{i}^{i\infty}\right| \\ & + \left|\int_{i}^{i\infty} e^{\xi t}(\hat{u}(r_l\xi^{\frac{1}{\alpha}})+\hat{u}(0))\xi^{\frac{n+j}{\alpha}-2}d\xi\right| + \left|\int_{i}^{i\infty} e^{\xi t}\xi^{\frac{n+j}{\alpha}-1}\frac{d}{d\xi}\hat{u}(r_l\xi^{\frac{1}{\alpha}})d\xi\right|\Bigg) \\ & = O\left(\epsilon t^{-\frac{n+j}{\alpha}-\gamma}\right). \end{aligned} \tag{5.16}$$

Substituting (5.14)-(5.16) into (5.13) we obtain for the first summand in (5.11)

$$I_1 = \hat{\bar{u}}(0)\Gamma\left(1 - \frac{n+j}{\alpha}\right) t^{-\frac{n+j}{\alpha}} + O(\epsilon t^{-\frac{n+j}{\alpha}-\gamma}) \tag{5.17}$$

for all $j = 1, ..., m$, if $\alpha \neq [\alpha]$ and $j = 1, ..., m-1$, if $\alpha = [\alpha]$. In the case $\alpha = [\alpha]$ for $j = m$ using Lemma 4.2 with $\delta = \gamma\alpha \leq 1$ we get

$$I_1 = Ct^{-1-\gamma} \int_0^{+\infty} x^{\alpha\gamma} |\bar{u}(x)| dx = O(\epsilon t^{-1-\gamma}). \tag{5.18}$$

By estimate (4.5) from Lemma 4.3 we have

$$I_2 = O(\epsilon e^{-bt} t^{-1}). \tag{5.19}$$

In the case $\alpha = [\alpha]$, using the condition $\hat{u}''(\xi) = O(\epsilon|\xi|^{-2})$ for $|\xi| > 1$ we integrate in (4.5) one more time by parts with respect to $\xi$ to obtain $I_2 = O(\epsilon t^{-2})$. Now let us estimate the third integral $I_3$ in the representation (5.11). Changing the order of integration we obtain

$$I_3 = -\Gamma\left(1 - \frac{n+j}{\alpha}\right) t^{-\frac{n+j}{\alpha}} \int_0^{+\infty} d\tau e^{b\tau} \int_0^{+\infty} \mathbb{N}(u)(x,\tau) dx + R \tag{5.20}$$

for $j = 1, ..., m$, if $\alpha$ is not integer and $j = 1, ..., m-1$, if $\alpha$ is integer, where

$$\begin{aligned} R = \int_0^t d\tau e^{b\tau} \int_0^{+\infty} dx \mathbb{N}(x,\tau) \big(\, &(F_l(0, t-\tau) - F_l(0,t)) \\ &+ (F_l(x, t-\tau) - F_l(0, t-\tau)) \big), \end{aligned}$$

here we denote $F_l(x,t) = \int_{-i\infty}^{i\infty} e^{\xi t - r_l \xi^{\frac{1}{\alpha}} x} \xi^{\frac{n+j}{\alpha}-1} d\xi$. From (5.1) and Theorem 1.1 we have

$$\int_0^{+\infty} (1 + x^{-\delta}) |\mathbb{N}(u)(x,\tau)| dx \leq \epsilon_1^{\rho+1} e^{-bt(\rho+1)} t^{-\gamma} (1+t)^{-\frac{\rho}{\alpha}-\eta+\gamma}, \tag{5.21}$$

where $0 < \delta < 1$, and by the Hölder inequality with $p = \frac{1}{\gamma\alpha}$, $q = \frac{1}{1-\gamma\alpha}$ we get

$$\int_0^{+\infty} |x^{\gamma\alpha} \mathbb{N}(u)(x,t)| dx \leq \epsilon_1^{\rho+1} e^{-bt(\rho+1)} t^{-\gamma} (1+t)^{-\frac{\rho}{\alpha}-\eta+2\gamma}. \tag{5.22}$$

Then using Lemma 3.1 with $\beta = 1 - \frac{n+j}{\alpha}$, $\mu = \frac{1}{\alpha}$, $\omega = \frac{1}{2} - \gamma\alpha$ and $\nu = \gamma > 0$ is small (we can chose $2\gamma < 1 - \eta - \frac{\rho}{\alpha}$ since $\frac{\rho}{\alpha} + \eta > 1$ if $b = 0$) we have

$$\begin{aligned} R \leq C \int_0^{+\infty} d\tau e^{b\tau} \tau^{\gamma} \left( |\tau|^{-\frac{n+j}{\alpha}-\gamma} + |t-\tau|^{-\frac{n+j}{\alpha}-\gamma} \right) \int_0^{+\infty} |\mathbb{N}(u)(x,\tau)| dx \\ + C \int_0^{+\infty} d\tau e^{b\tau} |t-\tau|^{-\frac{n+j}{\alpha}-\gamma} \int_0^{+\infty} x^{\gamma\alpha} |\mathbb{N}(u)(x,\tau)| dx = O(\epsilon_1^{\rho+1} t^{-\frac{n+j}{\alpha}-\gamma}). \end{aligned}$$

In the case $\alpha = [\alpha]$ interchanging the order of integration and using Lemma 4.2 with $\delta = -\gamma\alpha$ in the domain $|t-\tau| \le 1$ and $\delta = \gamma\alpha$ in the domain $|t-\tau| > 1$ by virtue of (5.21) and (5.22) we have

$$\begin{aligned} I_3 &= Ce^{-bt}\int_0^{+\infty} e^{b\tau}d\tau \int_0^{+\infty} \mathbb{N}(u)(x,\tau)dx \int_{-i\infty}^{i\infty} e^{\xi(t-\tau)-r_l\xi^{\frac{1}{\alpha}}x}d\xi \\ &\le Ce^{-bt}\int_{|t-\tau|\le 1} d\tau e^{b\tau}|t-\tau|^{\gamma-1}\int_0^{+\infty} |x^{-\gamma\alpha}\mathbb{N}(u)(x,\tau)|dx \\ +Ce^{-bt}&\int_{|t-\tau|\ge 1} d\tau e^{b\tau}|t-\tau|^{-1-\gamma}\int_0^{+\infty} |x^{\gamma\alpha}\mathbb{N}(u)(x,\tau)|dx = O\left(\epsilon_1^{\rho+1}t^{-1-\gamma}\right). \end{aligned} \tag{5.23}$$

From (5.11), (5.17) - (5.20) and (5.23) we get (5.10). Lemma 5.2 is proved. □

**Lemma 5.3.** *Let $\|(1+x^\delta)\bar{u}\|_{\mathbf{L}^1} \le \epsilon$, where $0 < \delta < \frac{1}{2}$. Then the estimate*

$$\left\|\int_{-i\infty}^{i\infty} e^{px-K(p)t}\hat{\bar{u}}(p)dp\right\|_{\mathbf{L}^2} = O\left(\epsilon t^{-\frac{1}{2\alpha}}\right)$$

*is valid for $t > 1$ and the following asymptotics as $t \to \infty$ uniformly with respect to $x > 0$*

$$\int_{-i\infty}^{i\infty} e^{px-K(p)t}\hat{\bar{u}}(p)dp = G_0(xt^{-\frac{1}{\alpha}})\hat{\bar{u}}(0)t^{-\frac{1}{\alpha}} + O\left(\epsilon t^{-\frac{1}{\alpha}-\gamma}\right),$$

*is true, where $G_0(q) = \int_{-i\infty}^{+i\infty} e^{yq-Ey^\alpha}dy$.*

*Proof.* Making a change of the variable of integration $y = ipt^{1/\alpha}$ and using the conditions of the lemma we get

$$\begin{aligned} \left\|\int_{-i\infty}^{i\infty} e^{px-K(p)t}\hat{\bar{u}}(p)dp\right\|_{\mathbf{L}^2} &\le \left(\int_{-i\infty}^{i\infty} e^{-2\Re Ep^\alpha t}|\hat{\bar{u}}|^2|dp|\right)^{\frac{1}{2}} \\ &\le C\epsilon t^{-\frac{1}{2\alpha}}\left(\int_0^{\infty} e^{-2\Theta y^\alpha}dy\right)^{\frac{1}{2}} = O\left(\epsilon t^{-\frac{1}{2\alpha}}\right) \end{aligned}$$

for $t > 1$, where $\Theta = \min(\Re E(i)^\alpha, \Re E(-i)^\alpha) > 0$. We write the representation

$$\begin{aligned} \int_{-i\infty}^{i\infty} e^{px-K(p)t}\hat{\bar{u}}(p)dp &= \hat{\bar{u}}(0)\int_{-i}^{i} e^{px-K(p)t}dp \\ +\int_{-i}^{i} e^{px-K(p)t}(\hat{\bar{u}}(p)-\hat{\bar{u}}(0))dp &+ \int_{|p|\ge 1,\Re p=0} e^{px-K(p)t}\hat{\bar{u}}(p)dp = J_1+J_2+J_3. \end{aligned}$$

Changing the variable of integration $y = pt^{1/\alpha}$ in the first integral $J_1$ we get

$$J_1 = \hat{\bar{u}}(0)G_0(xt^{-\frac{1}{\alpha}})t^{-\frac{1}{\alpha}} + O\left(\epsilon t^{-\frac{1}{\alpha}-\gamma}\right).$$

Since $|\hat{\bar{u}}(p) - \hat{\bar{u}}(0)| = |\int_0^\infty (e^{px} - 1)\bar{u}(x)dx| \leq |p|^{\gamma\alpha}\int_0^\infty x^{\gamma\alpha}|\bar{u}|dx \leq \epsilon|p|^{\gamma\alpha}$, making a change of the variable of integration $y = pt^{1/\alpha}$ we easily obtain for the second integral

$$|J_2| \leq C\int_{-i}^{i} e^{-\Theta|p|^\alpha t}|p|^{\gamma\alpha}|dp| = O\left(\epsilon t^{-\frac{1}{\alpha}-\gamma}\right),$$

where $\Theta = \min(E(i)^\alpha, E(-i)^\alpha) > 0$. Finally since $\|\hat{\bar{u}}\|_{\mathbf{L}^\infty} \leq \epsilon$ we have

$$|J_3| \leq Ce^{-\frac{\Theta t}{2}}\int_{|p|\geq 1, \Re p = 0} e^{-\frac{\Theta|p|^\alpha t}{2}}|\hat{\bar{u}}(p)||dp| = O\left(\epsilon t^{-\frac{1}{\alpha}-\gamma}\right).$$

Lemma 5.3 is proved. □

**Lemma 5.4.** *Let the functions $v_j(t)$, $j = 1, ..., m$ have asymptotics (5.14) as $t \to \infty$. Then the estimate*

$$\left\|\int_{-i\infty}^{i\infty} e^{px}dp\int_0^t e^{-K(p)(t-\tau)}K(p)\frac{e^{b\tau}v_j(\tau)}{p^{n+j}}d\tau\right\|_{\mathbf{L}^2} = O\left((\epsilon + \epsilon_1^{\rho+1})t^{-\frac{1}{2\alpha}}\right)$$

*is valid for $t > 1$ and the following asymptotics*

$$\int_{-i\infty}^{i\infty} e^{px}dp\int_0^t e^{-K(p)(t-\tau)}K(p)\frac{e^{b\tau}v_j(\tau)}{p^{n+j}}d\tau$$
$$= EB_jt^{-\frac{1}{\alpha}}G_j(x/t^{\frac{1}{\alpha}}) + O\left((\epsilon + \epsilon_1^{\rho+1})t^{-\frac{1}{\alpha}-\gamma}\right)$$

*is true as $t \to \infty$ uniformly with respect to $x > 0$, where*

$$G_j(q) = \int_{-i\infty}^{+i\infty} dye^{yq}y^{\alpha-n-j}dy\int_0^1 e^{-Ey^\alpha(1-z)}z^{-\frac{n+j}{\alpha}}dz$$

*for $j = 1, ..., m$ in the case $\alpha$ is not integer and $j = 1, ..., m-1$ if $\alpha$ is integer. In the case $\alpha$ is integer we denote*

$$G_m(q) = G_0(q)\int_0^{+\infty} v_m(\tau)d\tau, G_0(q) = \int_{-i\infty}^{i\infty} e^{qy-Ey^\alpha}dy,$$

*and $v_m$ is the boundary value of the last derivative of the solution defined by (5.11).*

*Proof.* By virtue of (5.10) and Lemma 3.2 we have

$$v_j = O\left(e^{-bt}(\epsilon + \epsilon_1^{\rho+1})t^{-\frac{j-1}{\alpha}}(1+t)^{-\frac{n+1}{\alpha}}\right)$$

for $j = 1, ..., m$. Then making a change of the variable of integration $y = p(t-\tau)^{1/\alpha}$ we get

$$\left\| \int_{-i\infty}^{i\infty} e^{px} dp \int_0^t e^{-K(p)(t-\tau)} K(p) \frac{e^{b\tau} v_j(\tau)}{p^{n+j}} d\tau \right\|_{\mathbf{L}^2}$$
$$\leq C(\epsilon + \epsilon_1^{\rho+1}) \left( \int_0^1 (t-\tau)^{-\frac{1}{2\alpha} - 1 + \frac{n+j}{\alpha}} \tau^{-\frac{j-1}{\alpha}} d\tau \right.$$
$$\left. + \int_1^t (t-\tau)^{-\frac{1}{2\alpha} - 1 + \frac{n+j}{\alpha}} \tau^{-\frac{n+j}{\alpha}} d\tau \right) = O\left( (\epsilon + \epsilon_1^{\rho+1}) t^{-\frac{1}{2\alpha}} \right).$$

Making changes of the variables of integration $\tau = tz$ and $y = pt^{1/\alpha}$ we get

$$\begin{aligned} I &= E \int_{-i\infty}^{i\infty} e^{px} p^{\alpha-n-j} dp \int_0^t e^{-K(p)(t-\tau)+b\tau} v_j(\tau) d\tau \\ &= Et^{\frac{n-1+j}{\alpha}} \int_{-i\infty}^{+i\infty} e^{yxt^{-\frac{1}{\alpha}}} y^{\alpha-n-j} dy \int_0^1 e^{-Ey^\alpha(1-z)} e^{btz} v_j(tz) dz \end{aligned} \tag{5.24}$$

for $j = 1, 2..., m$, if $\alpha \neq [\alpha]$ and for $j = 1, ..., m-1$, if $\alpha = [\alpha]$. Substitution of (5.10) into (5.24) yields

$$I = EB_j t^{-\frac{1}{\alpha}} \int_{-i\infty}^{+i\infty} e^{yxt^{-\frac{1}{\alpha}}} y^{\alpha-n-j} dy \int_0^1 e^{-Ey^\alpha(1-z)} z^{-\frac{n+j}{\alpha}} dz + R, \tag{5.25}$$

where

$$R = Ct^{\frac{n-1+j}{\alpha}} \int_{-i\infty}^{+i\infty} e^{yxt^{-\frac{1}{\alpha}}} y^{\alpha-n-j} dy \int_0^1 e^{-Ey^\alpha(1-z)} O\left( \frac{\epsilon + \epsilon_1^{\rho+1}}{(tz)^{\frac{n+j}{\alpha}+\gamma}} \right) dz.$$

Similarly to the proof of Lemma 4.1 we easily see that $R = O\left(\frac{\epsilon+\epsilon_1^{\rho+1}}{t^{\frac{1}{\alpha}+\gamma}}\right)$. In the case $\alpha = [\alpha]$ we have

$$I = E \int_{-i\infty}^{i\infty} dp e^{px} \int_0^t e^{-Ep^\alpha(t-\tau)+b\tau} v_m(\tau) d\tau.$$

By virtue of Theorem 1.1 and (5.12) the following estimate

$$|v_m(t)| \le C(\epsilon + \epsilon_1^{\rho+1}) t^{-\frac{m-1}{\alpha}} (1+t)^{-1+\frac{m-1}{\alpha}-\gamma}$$

is true. Then interchanging the order of integration and making the change of the variable of integration $y = pt^{1/\alpha}$ we get

$$I = Et^{-\frac{1}{\alpha}} \left( F(q,0) \int_0^{+\infty} v_m(\tau) d\tau + R \right),$$

where

$$R = \int_0^{t/2} e^{b\tau} v_m(\tau)(F(q,z) - F(q,0)) d\tau - F(q,0) \int_{t/2}^{\infty} e^{b\tau} v_m(\tau) d\tau + \int_{t/2}^{t} e^{b\tau} v_m(\tau) F(q,z) d\tau$$

and $F(q,z) = \int_{-i\infty}^{i\infty} e^{yq - Ey^\alpha (1-z)} dy$, $q = xt^{-\frac{1}{\alpha}}$, $z = \tau t^{-1}$. We have $|F(q,z)| \le \frac{C}{(1-z)^{\frac{1}{\alpha}}}$ and $|F_z(q,z)| = |E \int_{-i\infty}^{i\infty} e^{yq - Ey^\alpha(1-z)} y^\alpha dy| \le \frac{C}{(1-z)^{1+\frac{1}{\alpha}}}$. Therefore we obtain $R = O\left((\epsilon + \epsilon_1^\rho) t^{-\gamma}\right)$. Lemma 5.4 is proved. □

**Lemma 5.5.** *Let estimate (5.1) be true. Then the estimate*

$$\left\| \int_{-i\infty}^{i\infty} e^{px} dp \int_0^t e^{-K(p)(t-\tau)} e^{b\tau} \widehat{\mathbb{N}(u)}(p,\tau) d\tau \right\|_{\mathbf{L}^2} = O\left( \epsilon_1^{\rho+1} t^{-\frac{1}{2\alpha}} \right).$$

*is valid for $t > 1$ and the following asymptotics*

$$\begin{aligned} &\int_{-i\infty}^{i\infty} e^{px} dp \int_0^t e^{-K(p)(t-\tau)} e^{b\tau} \widehat{\mathbb{N}(u)}(p,\tau) d\tau \\ &= t^{-\frac{1}{\alpha}} G_0(xt^{-\frac{1}{\alpha}}) \int_0^{+\infty} e^{b\tau} \widehat{\mathbb{N}(u)}(0,\tau) d\tau + O\left( \epsilon_1^\rho t^{-\frac{1}{\alpha}-\gamma} \right) \end{aligned} \tag{5.26}$$

*is true as $t \to \infty$ uniformly with respect to $x > 0$, where $\widehat{\mathbb{N}(u)}(0,t) = \int_0^\infty \mathbb{N}(u) dx$, and $G_0(q) = \int_{-i\infty}^{+i\infty} e^{yq - Ey^\alpha} dy$.*

*Proof.* By virtue of (5.1) we have $\|\widehat{\mathbb{N}(u)}\|_{\mathbf{L}^\infty} \le C\epsilon_1^{\rho+1} e^{-bt(\rho+1)} t^{-\gamma} (1+t)^{-\frac{\rho}{\alpha}-\eta+\gamma}$. Therefore since $\frac{\rho}{\alpha} + \eta > 1$, if $b = 0$ we get

$$\begin{aligned} &\left\| \int_{-i\infty}^{i\infty} e^{px} dp \int_0^t e^{-K(p)(t-\tau)} e^{b\tau} \widehat{\mathbb{N}(u)}(p,\tau) d\tau \right\|_{\mathbf{L}^2} \\ &\le \int_0^t e^{b\tau} \|\widehat{\mathbb{N}(u)}\|_{\mathbf{L}^\infty} \left( \int_{-i\infty}^{i\infty} e^{-2\Theta |p|^\alpha (t-\tau)} |dp| \right)^{\frac{1}{2}} d\tau \\ &\le C\epsilon_1^{\rho+1} \int_0^t \frac{\tau^{-\gamma} d\tau}{(1+\tau)^{\frac{\rho}{\alpha}+\eta-\gamma} (t-\tau)^{\frac{1}{2\alpha}}} = O\left( \epsilon_1^{\rho+1} t^{-\frac{1}{2\alpha}} \right), \end{aligned}$$

where $\Theta = \min(\Re E i^\alpha, \Re E(-i)^\alpha) > 0$. We write the representation

$$\begin{aligned}&\int_{-i\infty}^{i\infty} e^{px} dp \int_0^t e^{-K(p)(t-\tau)} e^{b\tau} \widehat{\mathbb{N}(u)}(p,\tau) d\tau \\ &= \int_{-i}^{i} e^{px-K(p)t} dp \int_0^{\frac{t}{2}} e^{b\tau} \widehat{\mathbb{N}(u)}(0,\tau) d\tau + R,\end{aligned} \tag{5.27}$$

where

$$\begin{aligned}R = &\int_{-i}^{i} e^{px} dp \int_0^{\frac{t}{2}} (e^{-K(p)(t-\tau)} - e^{-K(p)t}) e^{b\tau} \widehat{\mathbb{N}(u)}(0,\tau) d\tau \\ &+ \int_{-i}^{i} e^{px} dp \int_0^{\frac{t}{2}} e^{-K(p)(t-\tau)} e^{b\tau} (\widehat{\mathbb{N}(u)}(p,\tau) - \widehat{\mathbb{N}(u)}(0,\tau)) d\tau \\ &+ \int_{-i}^{i} e^{px} dp \int_{\frac{t}{2}}^{t} e^{-K(p)(t-\tau)} e^{b\tau} \widehat{\mathbb{N}(u)}(p,\tau) d\tau \\ &+ \int_{|p|>1, \Re p = 0} e^{px} dp \int_0^t e^{-K(p)(t-\tau)} e^{b\tau} \widehat{\mathbb{N}(u)}(p,\tau) d\tau\Bigg) = \sum_1^4 J_j.\end{aligned}$$

Making a change of the variable of integration $y = pt^{1/\alpha}$ in the first summand of representation (5.27) we get

$$\begin{aligned}&\int_{-i}^{i} e^{px-K(p)t} dp \int_0^{\frac{t}{2}} e^{b\tau} \widehat{\mathbb{N}(u)}(0,\tau) d\tau \\ &= t^{-\frac{1}{\alpha}} G_0(x/t^{\frac{1}{\alpha}}) \int_0^\infty e^{b\tau} \widehat{\mathbb{N}(u)}(0,\tau) d\tau + O\left(\epsilon_1^{\rho+1} t^{-\frac{1}{\alpha}-\gamma}\right).\end{aligned} \tag{5.28}$$

We have $|e^{-Ep^\alpha(t-\tau)} - e^{-Ep^\alpha t}| \le Ce^{-\Theta|p|^\alpha(t-\tau)}|p|^{\alpha\gamma}\tau^\gamma$ for all $p \in [-i,i]$, since $\Theta = \min(\Re E i^\alpha, \Re E(-i)^\alpha) > 0$. Therefore making a change of the variable of integration $y = p(t-\tau)^{1/\alpha}$ we get via (5.1)

$$\begin{aligned}J_1 &\le C\int_0^{\frac{t}{2}} e^{b\tau} |\widehat{N(u)}(0,\tau)| d\tau \int_{-i}^{i} e^{-\Theta|p|^\alpha(t-\tau)} |p|^{\alpha\gamma}\tau^\gamma |dp| \\ &\le C\epsilon_1^{\rho+1} \int_0^{\frac{t}{2}} \frac{e^{-b\rho\tau} d\tau}{(1+\tau)^{\frac{\rho}{\alpha}+\eta-\gamma}(t-\tau)^{\frac{1}{\alpha}+\gamma}} = O\left(\epsilon_1^{\rho+1} t^{-\frac{1}{\alpha}-\gamma}\right).\end{aligned} \tag{5.29}$$

By estimate of Lemma 5.1 $\|x^\delta u\|_{\mathbf{L}^2} \le C$ for $0 < \delta \le \frac{1}{2}$ we get

$$\begin{aligned}|\widehat{\mathbb{N}(u)}(p,\tau) - \widehat{\mathbb{N}(u)}(0,\tau)| &= \left|\int_0^\infty (e^{-px} - 1)\mathbb{N}(u)(x,\tau) dx\right| \\ &\le \sqrt{|p|} \int_0^\infty \sqrt{x} |\mathbb{N}(u)(x,\tau)| dx \\ &\le Ct^{-\eta}\sqrt{|p|}\|u\|_{\mathbf{L}^\infty}^{\rho-1}\|u\|_{\mathbf{L}^2}\|\sqrt{x}u\|_{\mathbf{L}^2} \le C\epsilon_1^{\rho+1}\sqrt{|p|}(1+t)^{-\frac{2\rho-1}{2\alpha}-\eta}\end{aligned}$$

for all $p \in [-i, i]$. Making a change of the variable of integration $y = p(t-\tau)^{1/\alpha}$ we obtain

$$J_2 \le C\epsilon_1^{\rho+1} \int_0^{\frac{t}{2}} \frac{e^{-b\tau(\rho-1)}\tau^{-\gamma} d\tau}{(1+\tau)^{\frac{2\rho-1}{2\alpha}+\eta-\gamma}(t-\tau)^{\frac{1}{\alpha}+\frac{1}{2\alpha}}} \int_0^\infty e^{-\Theta y^\alpha} \sqrt{y} dy = O\left(\frac{\epsilon_1^{\rho+1}}{t^{\frac{1}{\alpha}+\gamma}}\right). \tag{5.30}$$

Via (5.1) we get $\|\widehat{\mathbb{N}(u)}\|_{\mathbf{L}^\infty} \le \epsilon_1^{\rho+1}|a(\tau)|\|u\|_{\mathbf{L}^\infty}^{\rho-1}\|u\|_{\mathbf{L}^2}^2 \le C\epsilon_1^{\rho+1}e^{-b\tau(\rho+1)}(1+\tau)^{-\frac{\rho}{\alpha}-\eta}$, whence making a change of the variable of integration $y = p(t-\tau)^{1/\alpha}$ we have

$$\begin{aligned} |J_3| &\le \int_{\frac{t}{2}}^t \|\widehat{\mathbb{N}(u)}(p,\tau)\|_{\mathbf{L}^\infty} e^{b\tau} d\tau \int_{-i}^i e^{-\Theta|p|^\alpha(t-\tau)}|dp| \\ &\le C\epsilon_1^{\rho+1} \int_{\frac{t}{2}}^t \frac{e^{-b\tau\rho} d\tau}{(1+\tau)^{\frac{\rho}{\alpha}+\eta}(t-\tau)^{\frac{1}{\alpha}}} \int_0^{+\infty} e^{-\Theta y^\alpha} dy = O\left(\epsilon_1^{\rho+1} t^{-\frac{1}{\alpha}-\gamma}\right). \end{aligned} \tag{5.31}$$

For the last integral $J_4$ we easily obtain

$$\begin{aligned} |J_4| &\le \int_0^t e^{-\frac{\Theta(t-\tau)}{2}} \|\widehat{\mathbb{N}(u)}(p,\tau)\|_{\mathbf{L}^\infty} e^{b\tau} d\tau \int_{|p|\ge 1, \Re p=0} e^{-\Theta|p|^\alpha(t-\tau)/2}|dp| \\ &\le \int_0^t \frac{e^{-\frac{\Theta(t-\tau)}{2}-b\tau\rho}}{(1+\tau)^{\frac{\rho}{\alpha}+\eta}(t-\tau)^{\frac{1}{\alpha}}} d\tau \int_0^\infty e^{-\Theta y^\alpha/2} dy = O\left(\epsilon_1^{\rho+1} t^{-\frac{1}{\alpha}-\gamma}\right). \end{aligned} \tag{5.32}$$

From estimates (5.28) - (5.32) the result of Lemma 5.5 follows. □

*Proof of Theorem 1.3.* Let us prove the following estimate

$$e^{bt}(1+t)^{\frac{1}{\alpha}}\left((1+t)^{-\frac{1}{2\alpha}}\|u\|_{\mathbf{L}^2} + t^\gamma(1+t)^{-\gamma}\|u\|_{\mathbf{L}^\infty}\right) < \epsilon_1 \tag{5.33}$$

for all $t > 0$. By the contrary we suppose that estimate (5.33) is violated for some time. By Theorem 1.1 the left hand side of (5.33) is continuous. Therefore there exists a maximal time $T > 1$ such that the nonstrict estimate (5.33) is valid on $[0, T]$. Thus the supposition (5.1) is valid on the time interval $[0, T]$ and we can apply Lemmas 5.3 - 5.5 to representation (3.2) of the solution. Hence we get estimate (5.33) for all$[0, T]$. The contradiction obtained proves estimate (5.33) for all $t > 0$. Moreover by virtue of Remark 3.1, Lemmas 5.3 - 5.5 and Lemma 4.5 with $\chi = \frac{1}{\alpha}$ we see that the solution $u(x,t) \in \mathbf{C}([0,\infty); \mathbf{L}^2) \cap \mathbf{C}\left(\mathbf{R}^+; \mathbf{W}_2^{[\alpha]-1} \cap \mathbf{C}^{[\alpha]-1}\right)$ has asymptotics (1.5) with the coefficient

$$B_0 = \frac{1}{2\pi i}\left(\hat{\hat{u}}(0) + \int_0^{+\infty} d\tau e^{b\tau} \int_0^\infty \mathbb{N}(u)(x,\tau) dx + R\right),$$

where $R = \int_0^{+\infty} v_m(\tau)d\tau$, if $\alpha$ is integer and $R = 0$ if $\alpha$ is not integer ($v_m$ is the boundary value of the derivative of the solution of order $n+m$ and is defined by (5.11)) the coefficients $B_j = \frac{A_j}{2\pi i}$, for $j = 1, ..., n$ and

$$B_j = \Gamma\left(1 - \frac{n+j}{\alpha}\right) \sum_{l=1}^{m} \frac{hr_l^{[\alpha]} M_{j,l}}{2\pi i} \left(\hat{\hat{u}}(0) - \int_0^{+\infty} d\tau e^{b\tau} \int_0^{+\infty} \mathbb{N}(u)(x,\tau)dx\right)$$

for $j = n+1, ..., n+m$, where $h = (\det \tilde{W})^{-1}$, $M_{j,l}$ are the algebraic minors of matrix $\tilde{W}$ (see 2.8). Integrals $B_j$ converge for any $b \geq 0$ in view of estimate (5.33) (see estimate (5.21) of Lemma 5.2 for details). Theorem 1.3 is proved. □

**Acknowledgments.** This paper was started, when one of the authors (E.I. Kaikina) visited the Department of Applied Mathematics of Science University of Tokyo. She would like to thank Science University of Tokyo for kind hospitality.

## References

1. A. Friedman, *Partial Differential Equations*, Krieger, 1983.
2. J. Ginibre and G. Velo, *On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case; II Scattering theory, general case*, J. Funct. Anal. **32** (1979), 1-71.
3. N. Hayashi, K. Nakamitsu and M. Tsutsumi, *On solutions of the initial value problem for nonlinear Schrödinger equations*, J. Funct. Anal. **71** (1987), 218-245.
4. E.I. Kaikina, *The Cauchy problem for the nonlocal Schrödinger equation. I, II*, Mat. Modelirovanie **3 (11)** (1991), 83-108.
5. P.I. Naumkin and I.A. Shishmarev, *Nonlinear Nonlocal Equations in the Theory of Waves*, Translations of Monographs, vol. 133, A.M.S., Providence, R.I., 1994.
6. I.A. Shishmarev, M. Tsutsumi and E.I. Kaikina, *Asymptotics in time for the nonlinear nonlocal Schrödinger equations with a source*, J. Math. Soc. Japan **51** (1999), 463-484.
7. V.S. Vladimirov, *Equations of Mathematical Physics*, Moscow, Nauka, 1976.

Elena I. Kaikina
Departamento de Ciencias Básicas, Instituto Tecnologico de Morelia
CP 58120, Morelia, Michoacan, MEXICO
*E-mail*: `kaikina@ifm1.ifm.umich.mx`

Pavel I. Naumkin
Instituto de Física y Matemática, Universidad Michoacana
AP 2-82, CP 58040, Morelia, Michoacan, MEXICO
*E-mail*: `pavelni@zeus.ccu.umich.mx`

Ilya A. Shishmarev
Department of Computational Mathematics and Cybernetics, Moscow State University,
119 899 Moscow, RUSSIA
*E-mail*: `Ilia@shishmarev.msk.su`