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GEODESICS REFLECTING ON A PSEUDO-RIEMANNIAN SUBMANIFOLD

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Abstract. In a pseudo-Riemannian manifold, we consider curves through a fixed pseudo-Riemannian submanifold. The first variation formula and the second variation formula of a reflecting geodesic are obtained. Moreover, we study the index form and conjugate points for a reflecting geodesic. Variation formulae for energy are also considered.

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§0. Introduction

In the paper [2], Innami considered a geodesic reflecting at a boundary point of a Riemannian manifold with boundary. Let M be a Riemannian manifold with boundary $\partial M =: B \neq \emptyset$ which is a union of smooth hypersurfaces. A broken geodesic on M is said to be a reflecting geodesic if it satisfies the reflection law. As usual, a variation of a reflecting geodesic γ through reflecting geodesics yields a Jacobi vector field Y along γ which satisfies the Jacobi equation. In the case of a reflection, such a Jacobi vector field is discontinuous at the boundary in general, but certain conditions hold at the boundary. In this case, he defined and studied the index form, conjugate points and so on, as in the case of a usual geodesic. We note that Hasegawa studied special cases in [1] and [2].

In this paper, we consider the case where M is a pseudo-Riemannian manifold and B is a pseudo-Riemannian submanifold. We generalize the notion of a reflecting geodesic and generalize some of Innami's results in a sense.

In Section 1, for a piecewise smooth curve on M through a point of B, we define a variation of such a curve. The details will be described in Definition 1.1. In Section 2, we prove the first variation formula of arclength for the variation above. In Section 3, we provide the second variation formula. In Section 4, we formalize the index form for our case. In Section 5, we consider the variation of a reflecting geodesic through reflecting geodesics and give definitions of an admissible Jacobi field and a conjugate point. In Section 6, we study a reflecting geodesic whose tangent vector at a point of B is normal to B. In Section 7, we consider the first and second variation formulas of energy.

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§1. Preliminaries

Let M be a pseudo-Riemannian manifold with a metric $< \cdot, \cdot >$ and D the Levi-Civita connection. A tangent vector v to M is said to be

spacelike if $< v, v >> 0$ or $v = 0$,
null if $< v, v >= 0$ and $v \neq 0$,
timelike if $< v, v >< 0$.

The category into which a given tangent vector falls is called its *causal character*. The *norm* $|v|$ of a tangent vector is $| < v, v > |^{\frac{1}{2}}$. A curve α in M is *spacelike* if all of its velocity vectors $\alpha'(t)$ are spacelike; similary for *timelike* and *null*. An arbitrary curve need not have one of these *causal characters*, but a geodesic always does. The class of curves α with $|\alpha'| > 0$ consists of all spacelike regular curves and all timelike (hence regular) curves, the two cases distinguished by the *sign of* α; that is, $\varepsilon := sgn(< \alpha', \alpha' >) = \pm 1$.

Let B be a pseudo-Riemannian submanifold in M and $\tilde{\Omega}$ the set of all piecewise smooth curves $\alpha : [a, b] \to M$ through B.

Definition 1.1. Let $\alpha : [a, b] \to M$ be a piecewise smooth curve such that $\alpha(t_0) \in B$ $(t_0 \in [a, b])$. A *piecewise smooth variation of* α *in* $\tilde{\Omega}$ (or, simply, a *variation of* α *in* $\tilde{\Omega}$) is a map

$$\varphi : [a, b] \times (-\delta, \delta) \to M,$$

for some $\delta > 0$, such that

$$\varphi_s(\cdot) := \varphi(\cdot, s) \in \tilde{\Omega}, \tag{1.1}$$

$$\varphi_0(t) = \alpha(t) \qquad \text{for all } a \leq t \leq b, \tag{1.2}$$

$$\varphi(t_0(s), s) \in B, \tag{1.3}$$

where $a = a_0(s) < a_1(s) < \cdots < t_0(s) = a_j(s) < \cdots < a_k(s) < a_{k+1}(s) = b$ are the breaks of φ_s ($a_i(0) = a_i$ ($i = 1, \cdots, k$) and $t_0(0) = t_0 = a_j$). We assume that $a_i(s)$'s are smooth with respect to s.

A *fixed endpoint* variation φ of α is a variation such that

$$\varphi(a, s) = \alpha(a) \quad \text{and} \quad \varphi(b, s) = \alpha(b). \tag{1.4}$$

There is no loss of generality in assuming that $a_1(s) < \cdots < t_0(s) = a_j(s) < \cdots < a_k(s)$ are the breaks of φ_s, since we can always add *trivial breaks* at which α or φ is smooth. The vector fields Y and A on α given by $Y(t) := \dfrac{\partial \varphi}{\partial s}(t,0)$ and $A(t) := \dfrac{D}{\partial s}\dfrac{\partial \varphi}{\partial s}(t, 0)$ are called *variation vector field* and *transverse acceleration vector field* of φ respectively, where $\dfrac{D}{\partial s} := D_{\frac{\partial}{\partial s}}$ and $\dfrac{D}{\partial t} := D_{\frac{\partial}{\partial t}}$. Unusually, in our case, Y and A are not piecewise smooth vector fields, for they are, possibly, discontinuous at breaks. We write $X(t, s) = \dfrac{\partial \varphi}{\partial t}(t, s)$ ($X(t)= X(t, 0) = \alpha'(t)$), $Y(t, s) = \dfrac{\partial \varphi}{\partial s}(t, s)$ ($Y(t) = Y(t, 0)$) and $A(t, s) = \dfrac{D}{\partial s}\dfrac{\partial \varphi}{\partial s}(t, s)$ ($A(t)= A(t, 0)$). For a function or vector field f on $[a, b]$, we put $\Delta_t f = f(t - 0) - f(t + 0)$ ($t \in (a, b)$), $\Delta_a f = -f(a+0)$ and $\Delta_b f = f(b-0)$, where $f(t \pm 0) = \lim_{t \to t \pm 0} f(t)$.

Let I be an interval in the real line $\boldsymbol{R}$. A *geodesic* in M is a curve $\gamma : I \to M$ whose vector field γ' is parallel, that is, $\gamma'' = D_{\gamma'}\gamma' = 0$. Furthermore a piecewise smooth curve α such that $|\alpha'| > 0$ is said to have *constant speed* and *constant sign* if $|\alpha'| = constant$ and $sgn(<\alpha', \alpha'>) = constant$, respectively. We note that geodesics have constant speed and sign.

Definition 1.2. A piecewise smooth curve α such that $\alpha(t_0) \in B$ is a *reflecting geodesic* if α satisfies the following conditions:

$$\alpha \text{ is a geodesic on } [a, t_0] \text{ and } [t_0, b], \tag{1.5}$$

$$\Delta_{t_0} X \text{ is normal to } B, \tag{1.6}$$

$$\Delta_{t_0} < X, X >= 0, \tag{1.7}$$

where we ignore this condition in the case of $t_0 = a$ or $t_0 = b$,

$$\Delta_{t_0} X \neq 0. \tag{1.8}$$

From (1.5) and (1.7), a reflecting geodesic have constant speed and sign. If $\Delta_{t_0}X = 0$ instead of (1.8), then α is a usual geodesic.

For each $s \in (-\delta, \delta)$, let $L(s)$ be the length of the *longitudinal curve* $\varphi_s : t \mapsto \varphi(t, s)$. We shall find formulas for the *first* and *second variation of arclength on* φ, that is, for

$$L'(0) = \frac{dL}{ds}|_{s=0} \quad \text{and} \quad L''(0) = \frac{d^2L}{ds^2}|_{s=0},$$

where the latter is considered when $L'(0) = 0$.

§2. First variation

For a variation φ, we define a curve $\beta_i : (-\delta, \delta) \to M$ by $\beta_i(s) = \varphi(a_i(s), s)$ (i $=0, 1, \cdots, k+1$). In particular, we put $\beta(s) := \beta_j(s) = \varphi(t_0(s), s)$. Hence β is a curve on B. First we show that variation vector fields have the following properties.

Lemma 2.1. *Let* $\alpha : [a, b] \to M$ *be a piecewise smooth curve such that* $\alpha(t_0) \in B$. *If* φ *is a variation of* α *in* $\tilde{\Omega}$ *with the variation vector field* Y, *then*

$$a_i'(0)X(a_i - 0) + Y(a_i - 0) = a_i'(0)X(a_i + 0) + Y(a_i + 0). \tag{2.1}$$

In particular,

$$\begin{aligned} &t_0'(0)X(t_0 - 0) + Y(t_0 - 0) \\ &= t_0'(0)X(t_0 + 0) + Y(t_0 + 0) \in T_{\alpha(t_0)}B. \end{aligned} \tag{2.2}$$

Proof. Since a curve β_i satisfies that $\beta_i'(s) = \dfrac{d}{ds}\varphi(a_i(s) - 0, s) = \dfrac{d}{ds}\varphi(a_i(s) + 0, s)$, it follows that

$$\begin{aligned} &a_i'(s)\frac{\partial\varphi}{\partial t}(a_i(s) - 0, s) + \frac{\partial\varphi}{\partial s}(a_i(s) - 0, s) \\ &= a_i'(s)\frac{\partial\varphi}{\partial t}(a_i(s) + 0, s) + \frac{\partial\varphi}{\partial s}(a_i(s) + 0, s). \end{aligned} \tag{2.3}$$

In particular, since β is a curve on B, we have $\beta'(0) \in T_{\alpha(t_0)}B$. □

This lemma shows that variation vector fields are element of the set $T_\alpha\tilde{\Omega}$ defined as below:

Definition 2.3. If $\alpha \in \tilde{\Omega}$, the set $T_\alpha\tilde{\Omega}$ consists of all piecewise smooth vector fields Y on α, which possibly may be discontinuous at t_0, such that for $i = 1, \cdots, k$

(2.4) there is a real number d_i such that

$$d_i X(a_i - 0) + Y(a_i - 0) = d_i X(a_i + 0) + Y(a_i + 0),$$

and, in particular,

$$d_j X(t_0 - 0) + Y(t_0 - 0) = d_j X(t_0 + 0) + Y(t_0 + 0) \in T_{\alpha(t_0)}B. \tag{2.5}$$

For example, piecewise smooth vector fields Y on α such that $Y(t_0 - 0) = Y(t_0 + 0) \in T_{\alpha(t_0)}B$ are elements of $T_\alpha\tilde{\Omega}$.

Conversely, given $Y \in T_\alpha\tilde{\Omega}$ we can choose a variation φ whose vector field is Y. In fact, we can know this claims from the following lemmas.

Lemma 2.4. *Let $\alpha \in \tilde{\Omega}$ and Y be a piecewise smooth vector field on α such that $Y(t_0 - 0) = Y(t_0 + 0) \in T_{\alpha(t_0)}B$. Then there is a variation of α in $\tilde{\Omega}$ whose variation vector field is Y.*

proof. We take t_1 and t_2 ($t_1 < t_0 < t_2$) such that $\alpha|[t_1, t_0]$, $\alpha|[t_0, t_2]$, $Y|[t_1, t_0]$ and $Y|[t_0, t_2]$ are smooth and $\alpha|[t_1, t_2]$ lies within one coordinate neighborhood. Choosing $\delta > 0$ sufficiently small, we can construct a variation as follows. Let

$$\varphi(t, s) = \exp_{\alpha(t)}(sY(t)) \text{ on } [a, t_1] \times (-\delta, \delta) \text{ and } [t_2, b] \times (-\delta, \delta).$$

Then we have $\varphi(t, 0) = \alpha(t)$ and $\dfrac{\partial\varphi}{\partial s}(t, 0) = Y(t)$.

Next we take a curve $\beta : (-\delta, \delta) \to B$ such that $\beta(0) = \alpha(t_0)$ and $\beta'(0) = Y(t_0)$. And we extend $Y|[t_1, t_0]$ and $Y|[t_0, t_2]$ to a smooth vector fields Z^- and Z^+ on a neighborhood of $\alpha|[t_1, t_0]$ and $\alpha|[t_0, t_2]$ respectively which satisfy the following conditions:

$$Z^\pm_{\beta(s)} = \beta'(s) \quad \text{and} \quad Z^\pm_{\varphi(t_l, s)} = \frac{\partial\varphi}{\partial s}(t_l, s),$$

for $l = 1, 2$. Let $\varphi^\pm_s$ be a local 1-parameter group of transformations which induce $Z^\pm$ and

$$\varphi(t, s) = \begin{cases} \varphi^-_s(\alpha(t)) & \text{on } [t_1, t_0] \times (-\delta, \delta) \\ \varphi^+_s(\alpha(t)) & \text{on } [t_0, t_2] \times (-\delta, \delta) \end{cases}.$$

Then we get a desired variation. □

Lemma 2.5. *If $\alpha \in \tilde{\Omega}$ and $Y \in T_\alpha \tilde{\Omega}$, then there is a variation of α whose variation vector field is Y.*

proof. There is a real number d_i such that, for $i = 1, \cdots, k$,

$$Y(a_i - 0) + d_i X(a_i - 0) = Y(a_i + 0) + d_i X(a_i + 0).$$

We define a function f by, for $i = 0, \cdots, k$,

$$f(t) = \frac{(t - a_i)d_{i+1} - (t - a_{i+1})d_i}{a_{i+1} - a_i} \quad \text{on } [a_i, a_{i+1}],$$

where we put $d_0 = d_{k+1} = 0$. Then, let $\bar{Y}(t) = Y(t) + f(t)X(t)$. Since $\bar{Y}(a_i \pm 0) = Y(a_i \pm 0) + d_i X(a_i \pm 0)$, by Lemma 2.4, there is a variation ψ of α whose variation vector field is $\bar{Y}$. Let $\varphi : [a, b] \times (-\delta, \delta) \to M$ such that $\varphi(t, s) = \psi(\hat{t}(t, s), s)$, where

$$\hat{t}(t, s) = t - f(t)s.$$

It follows that

$$\frac{\partial \varphi}{\partial s}(t, 0) = -f(t)X(t) + \bar{Y}(t).$$

Hence φ is a desired variation. □

We compute the first variation formula.

Proposition 2.6. (*First Variation Formula*) *Let $\alpha : [a, b] \to M$ be a piecewise smooth curve with constant speed $c > 0$ and sign ε such that $\alpha(t_0) \in B$. If φ is a variation of α in $\tilde{\Omega}$ with the variation vector field Y, then*

$$L'(0) = -\frac{\varepsilon}{c}\int_a^b < Y, \alpha'' > dt + \frac{\varepsilon}{c}\sum_{i=1}^{k} \Delta_{a_i} < Y, \alpha' > + \frac{\varepsilon}{c} < Y, \alpha' > |_a^b,$$

where $a_1 < \cdots < t_0 = a_j < \cdots < a_k$ are the breaks of α.

proof. If the s-interval $(-\delta, \delta)$ is small enough, $|X(t, s)|$ is positive, hence differentiable. Differentiating both sides of

$$L(s) = \sum_{i=1}^{k+1} \int_{a_{i-1}(s)}^{a_i(s)} |X(t, s)| dt,$$

we have

$$L'(s) = \sum_{i=1}^{k+1} \{ \int_{a_{i-1}(s)}^{a_i(s)} \frac{\partial}{\partial s} |X(t, s)| dt \tag{2.6}$$

$$+a_i'(s)|X(a_i(s)-0,s)| - a_{i-1}'(s)|X(a_{i-1}(s)+0,s)|\}$$

$$= \int_a^b \frac{\partial}{\partial s}|X(t,s)|dt + \sum_{i=1}^{k} a_i'(s)\{|X(a_i(s)-0,s)| - |X(a_i(s)+0,s)|\}.$$

Since the causal character of longitudinal curves is preserved for small $|s|$, we can compute

$$\frac{\partial}{\partial s}|X(t,s)| \tag{2.7}$$

$$= \frac{1}{2}(\varepsilon < X(t,s), X(t,s) >)^{-\frac{1}{2}} 2\varepsilon < \frac{DX}{\partial s}(t,s), X(t,s) >$$

$$= \varepsilon < \frac{DY}{\partial t}(t,s), X(t,s) > /|X(t,s)|$$

and

$$< \frac{DY}{\partial t}(t,s), X(t,s) >$$

$$= \frac{\partial}{\partial t} < Y(t,s), X(t,s) > - < Y(t,s), \frac{DX}{\partial t}(t,s) > .$$

Hence we have

$$L'(0) = \frac{\varepsilon}{c}\sum_{i=1}^{k+1} < Y(t), X(t) > |_{a_{i-1}}^{a_i} - \frac{\varepsilon}{c}\int_a^b < Y(t), \alpha''(t) > dt$$

$$= \frac{\varepsilon}{c}\sum_{i=1}^{k} \Delta_{a_i} < Y, \alpha' > + \frac{\varepsilon}{c} < Y, \alpha' > |_a^b - \frac{\varepsilon}{c}\int_a^b < Y(t), \alpha''(t) > dt.$$

□

In the case of $t_0 = a$ or b, we ignore the condition $\Delta_{t_0} < X, X >= 0$ from now on.

Lemma 2.7. *Let $\alpha : [a,b] \to M$ be a piecewise smooth curve with $\Delta_{t_0} < X, X >= 0$ such that $\alpha(t_0) \in B$. Then the followings are equivalent:*

$$\Delta_{t_0} X \textit{ is normal to } B. \tag{2.8}$$

$$< Y(t_0 - 0) + Y(t_0 + 0), \Delta_{t_0} X >= 0 \quad \textit{for any } Y \in T_\alpha \tilde{\Omega}. \tag{2.9}$$

$$\Delta_{t_0} < Y, X >= 0 \quad \textit{for any } Y \in T_\alpha \tilde{\Omega}. \tag{2.10}$$

proof. For simplicity, we put $X_\pm := X(t_0 \pm 0)$, $Y_\pm := Y(t_0 \pm 0)$, $d := t_0'(0)$ and $\Delta X := \Delta_{t_0} X$.

(2.8)⇒(2.9)⇒(2.10): If ΔX is normal to B, then, from (2.2),

$$< (dX_- + Y_-) + (dX_+ + Y_+), \Delta X >= 0.$$

Since $< X_- + X_+, \Delta X >= 0$, it holds that

$$< Y_- + Y_+, \Delta X >= 0.$$

Hence, by (2.2), we have

$$\begin{aligned} F :=< Y_-, X_- > - < Y_+, X_+ > &=< Y_-, X_+ > - < Y_+, X_- > \\ &=< Y_+ - d\Delta X, X_+ > - < Y_- + d\Delta X, X_- > \\ &=< Y_+, X_+ > - < Y_-, X_- > -d < \Delta X, X_+ + X_- >= -F. \end{aligned}$$

It follows that $F = 0$.

(2.10)⇒(2.9)⇒(2.8): Suppose $F = 0$. Then, from (2.2), we get

$$\begin{aligned} 2 < dX_- + Y_-, \Delta X > &=< (dX_- + Y_-) + (dX_+ + Y_+), \Delta X > \\ &=< Y_- + Y_+, \Delta X > \\ &=< Y_-, X_- > - < Y_+, X_+ > - < Y_-, X_+ > + < Y_+, X_- > \\ &= - < Y_+ - d\Delta X, X_+ > + < Y_- + d\Delta X, X_- > \\ &= - < Y_+, X_+ > + < Y_-, X_- > +d < \Delta X, X_+ + X_- > \\ &= 0. \end{aligned}$$

It follows that $< dX_- + Y_-, \Delta X >= 0$. This means that $< y, \Delta X >= 0$ for any $y \in T_{\alpha(t_0)} B$ from Lemma 2.4. Hence ΔX is normal to B. □

For a fixed endpoint variation φ, the first and last transverse curves are constant, so all longitudinal curves run from $\alpha(a)$ to $\alpha(b)$. In particular, the variation vector field Y vanishes at a and b, and so does the last term in the first variation formula. Given any neighborhood $\mathcal{U}$ of a point $t \in I$ there is a smooth real-valued function f on an interval I, called a *bump function* at t, such that $0 \le f \le 1$ on I, $f = 1$ on some neighborhood of t and $\mathrm{supp} f \subset \mathcal{U}$.

Corollary 2.8. *A piecewise smooth curve α with constant speed $c > 0$ and sign ε such that $\alpha(t_0) \in B$ is a reflecting geodesic or a geodesic if and only if* *the first variation of arc length is zero for every fixed endpoint variation of* α *in* $\tilde{\Omega}$.

proof. We assume that α is a reflecting geodesic. Then $\alpha'' = 0$ and $\Delta_{a_i}\alpha' = 0$ $(i \neq j)$. Hence, for $i \neq j$, we get

$$\Delta_{a_i} < Y, \alpha' > = < \Delta_{a_i} Y, \alpha'(a_i) > = 0,$$

since (2.1). For fixed endpoint variations, $Y(a)$ and $Y(b)$ are zero. Moreover using Lemma 2.7, we have $L'(0) = 0$.

Conversely, suppose $L'(0) = 0$ for every fixed endpoint variation φ. First we show that each segment $\alpha|I_i$ is geodesic, where

$$I_i = [a_{i-1}, a_i] \ (i = 1, \cdots, k+1). \tag{2.11}$$

It suffices to show that $\alpha''(t) = 0$ for $t \in I_i^\circ$, where $I_i^\circ := (a_{i-1}, a_i)$. Let y be any tangent vector to M at $\alpha(t)$, and let f be a bump function at t on $[a, b]$ with supp$f \subset [t - \zeta, t + \zeta] \subset I_i$. Let V be the vector field on α obtained by parallel translation of y, and let $Y = fV$. Since $Y(a)$ and $Y(b)$ are both zero, exponential formula $\varphi(t, s) = \exp_{\alpha(t)}(sY(t))$ produces a fixed endpoint variation of α whose variation vector field is Y. Since $L'(0) = 0$, the formula in Proposition 2.6 reduces to

$$0 = -\int_a^b < Y, \alpha'' > dt = \int_{t-\zeta}^{t+\zeta} < fV, \alpha'' > dt.$$

This holds for all y and $\zeta > 0$. Hence $< y, \alpha''(t) > = 0$ for all $y \in T_{\alpha(t_0)}M$. Thus we have $\alpha'' = 0$.

As before, let y be an arbitrary tangent vector at $\alpha(a_i)$ $(i \neq j)$, and let f be a bump function at a_i with supp$f \subset I_i \cup I_{i+1}$ $(i \neq j)$. For a fixed endpoint variation with vector field fV the first variation formula now reduces to

$$0 = L'(0) = \frac{\varepsilon}{c}\Delta_{a_i} < Y, \alpha' > = \frac{\varepsilon}{c} < y, \Delta_{a_i}\alpha' > \quad for\ all\ y.$$

Hence $\Delta_{a_i}\alpha' = 0$ $(i \neq j)$. This shows that (1.5) is true and $\Delta_{a_i} < Y, \alpha' > = 0$ $(i \neq j)$.

Finally Lemma2.7 implies (1.6). □

§3. Second variation

For a variation φ of a curve α, our aim is to compare $L(s)$, $|s|$ small, with the length $L(0)$ of α. Thus $L''(0)$ is needed only when $L'(0) = 0$. By Corollary

2.8, it suffices to find a formula for $L''(0)$ in the case where α is a reflecting geodesic. Let R be the Riemannian curvature tensor defined by

$$R(X,Y)W := D_X D_Y W - D_Y D_X W - D_{[X,Y]}W,$$

for any vector field X, Y and W on M, and S the shape operator defined by

$$S_Z(V) := -tan D_V Z,$$

for any vector field V tangent to B and Z normal to B. A vector field Y on a piecewise smooth curve $\alpha : [a,b] \to M$ is a *tangent to* α if $Y = f\alpha'$ for some function f on $[a,b]$ and *perpendicular to* α if $< Y, \alpha' >= 0$. If $|\alpha'| > 0$, then each tangent space $T_{\alpha(t)}M$ has a direct sum decomposition $\boldsymbol{R}\alpha' + \alpha'^{\perp}$. Hence each vector field Y on α has a unique expression $Y = Y^T + Y^{\perp}$, where Y^T is tangent to α and $Y^{\perp}$ is perpendicular to α, that is,

$$Y^{\perp} = Y - \frac{< Y, \alpha' >}{< \alpha', \alpha' >}\alpha'.$$

If γ is a nonnull reflecting geodesic, then $(Y')^T = (Y^T)'$ and $(Y')^{\perp} = (Y^{\perp})'$.

Definition 3.1. Let $\gamma : [a,b] \to M$ be a reflecting geodesic such that $\gamma(t_0) \in B$ and $\Delta_{t_0}X$ is nonnull. A linear operator $P : T_\gamma\tilde{\Omega} \to T_{\gamma(t_0)}B$ is defined by

$$\begin{aligned} P(Y) &:= Y(t_0+0) - \frac{< \Delta_{t_0}Y, \Delta_{t_0}X >}{< \Delta_{t_0}X, \Delta_{t_0}X >}X(t_0+0) \\ &= Y(t_0+0) - \frac{< Y(t_0+0), \Delta_{t_0}X >}{< X(t_0+0), \Delta_{t_0}X >}X(t_0+0). \end{aligned} \tag{3.1}$$

It follows from (2.2) that

$$\begin{aligned} P(Y) &= Y(t_0-0) - \frac{< \Delta_{t_0}Y, \Delta_{t_0}X >}{< \Delta_{t_0}X, \Delta_{t_0}X >}X(t_0-0) \\ &= Y(t_0-0) - \frac{< Y(t_0-0), \Delta_{t_0}X >}{< X(t_0-0), \Delta_{t_0}X >}X(t_0-0) \end{aligned} \tag{3.2}$$

If $Y \in T_\gamma\tilde{\Omega}$ is tangent to γ, then $P(Y) = 0$. For a continuous vector field Y such that $Y(t_0) \in T_{\gamma(t_0)}B$, $P(Y) = Y(t_0)$ holds.

We prepare the following lemma for the proof of the second variation formula.

Lemma 3.2. *Let* $\alpha : [a,b] \to M$ *be a piecewise smooth curve such that* $\alpha(t_0) \in B$. *If* φ *is a variation of* α *in* $\tilde{\Omega}$ *with the variation vector field* Y, *then*

$$\begin{aligned}(3.3)\quad & a_i''(0)X(a_i-0)+2a_i'(0)Y'(a_i-0)+A(a_i-0)+(a_i'(0))^2X'(a_i-0)\\ &= a_i''(0)X(a_i+0)+2a_i'(0)Y'(a_i+0)+A(a_i+0)+(a_i'(0))^2X'(a_i+0).\end{aligned}$$

In particular, if α *is a reflecting geodesic, then*

$$(3.4)\qquad 2a_i'(0)Y'(a_i-0)+A(a_i-0)=2a_i'(0)Y'(a_i+0)+A(a_i+0),$$

for $i \neq j$, *and*

$$\begin{aligned}(3.5)\qquad & t_0''(0)X(t_0-0)+2t_0'(0)Y'(t_0-0)+A(t_0-0)\\ &= t_0''(0)X(t_0+0)+2t_0'(0)Y'(t_0+0)+A(t_0+0).\end{aligned}$$

proof. We use a curve $\beta_i(s)=\varphi(a_i(s),s)$ as in §.2. Then we have

$$\begin{aligned}\beta_i''(s) &= D_{\beta_i'(s)}\beta_i'(s)\\ &= a_i''(s)X(a_i(s)-0,s)+2a_i'(s)\frac{DY}{\partial t}(a_i(s)-0,s)\\ &\quad +A(a_i(s)-0,s)+(a_i'(0))^2X'(a_i-0)\\ &= a_i''(s)X(a_i(s)+0,s)+2a_i'(s)\frac{DY}{\partial t}(a_i(s)+0,s)\\ &\quad +A(a_i(s)+0,s)+(a_i'(0))^2X'(a_i+0).\end{aligned}$$

□

Theorem 3.3. (*Second Variation Formula*) *Let* $\gamma : [a,b] \to M$ *be a reflecting geodesic with constant speed* $c > 0$ *and sign* ε *such that* $\gamma(t_0) \in B$ *and* $\Delta X := \Delta_{t_0}X$ *is nonnull. If* φ *is a variation of* γ *in* $\tilde{\Omega}$, *then*

$$\begin{aligned}L''(0) &= \frac{\varepsilon}{c}\int_a^b \{< Y^{\perp'}, Y^{\perp'} > - < R(Y,\gamma')\gamma', Y >\}dt\\ &\quad + \frac{\varepsilon}{c} < A, \gamma' > |_a^b + \frac{\varepsilon}{c} < S_{\Delta X}(P(Y)), P(Y) > ,\end{aligned}$$

where Y *is the variation vector field and* A *is the transverse acceleration vector field of* φ.

proof. Let $h = h(t,s) = |\frac{\partial\varphi}{\partial t}(t,s)|$, so $L(s)=\int_a^b hdt$. From (2.6), we have

$$\frac{\partial h}{\partial s} = \frac{\varepsilon}{h} < \frac{\partial\varphi}{\partial t}, \frac{D}{\partial s}\frac{\partial\varphi}{\partial t} > .$$

Thus we get

$$\frac{\partial^2 h}{\partial s^2} = \frac{\varepsilon}{h^2}\{h\frac{\partial}{\partial s} < \frac{\partial\varphi}{\partial t}, \frac{D}{\partial s}\frac{\partial\varphi}{\partial t} > - < \frac{\partial\varphi}{\partial t}, \frac{D}{\partial s}\frac{\partial\varphi}{\partial t} > \frac{\partial h}{\partial s}\}$$

$$= \frac{\varepsilon}{h}\{< \frac{D}{\partial s}\frac{\partial\varphi}{\partial t}, \frac{D}{\partial s}\frac{\partial\varphi}{\partial t} > + < \frac{\partial\varphi}{\partial t}, \frac{D}{\partial s}\frac{D}{\partial s}\frac{\partial\varphi}{\partial t} > - \frac{\varepsilon}{h^2} < \frac{\partial\varphi}{\partial t}, \frac{D}{\partial s}\frac{\partial\varphi}{\partial t} >^2\}.$$

Since $\frac{D}{\partial s}\frac{\partial\varphi}{\partial t} = \frac{D}{\partial t}\frac{\partial\varphi}{\partial s}$ and

$$\frac{D}{\partial s}\frac{D}{\partial s}\frac{\partial\varphi}{\partial t} = \frac{D}{\partial s}\frac{D}{\partial t}\frac{\partial\varphi}{\partial s} = R(\frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial t})\frac{\partial\varphi}{\partial s} + \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial\varphi}{\partial s}$$

hold, hence we have

$$\frac{\partial^2 h}{\partial s^2} = \frac{\varepsilon}{h}\{< \frac{D}{\partial s}\frac{\partial\varphi}{\partial t}, \frac{D}{\partial s}\frac{\partial\varphi}{\partial t} > + < \frac{\partial\varphi}{\partial t}, R(\frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial t})\frac{\partial\varphi}{\partial s} >$$
$$+ < \frac{\partial\varphi}{\partial t}, \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial\varphi}{\partial s} > - \frac{\varepsilon}{h^2} < \frac{\partial\varphi}{\partial t}, \frac{D}{\partial s}\frac{\partial\varphi}{\partial t} >^2\}.$$

Setting $s = 0$ in this equation produces the following changes: $h \to c$, $\frac{\partial\varphi}{\partial t} \to \gamma'$, $\frac{\partial\varphi}{\partial s} \to Y$, $\frac{D}{\partial s}\frac{\partial\varphi}{\partial t} \to Y'$ and $\frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial\varphi}{\partial s} \to A'$. Thus, rearranging the curvature term, we find

$$\frac{\partial^2 h}{\partial s^2}|_{s=0} = \frac{\varepsilon}{c}\{< Y', Y' > - < Y, R(Y, \gamma')\gamma' > + < \gamma', A' > - \frac{\varepsilon}{c^2} < \gamma', Y' >^2\}.$$

Since γ is a reflecting geodesic, it follows that $< \gamma', A' >= \frac{d}{dt} < \gamma', A >$ and

$$Y' = \frac{\varepsilon}{c^2} < Y', \gamma' > \gamma' + Y'^{\perp};$$

hence

$$< Y', Y' >= \frac{\varepsilon}{c^2} < Y', \gamma' >^2 + < Y'^{\perp}, Y'^{\perp} > .$$

Substitution then gives

$$\frac{\partial^2 h}{\partial s^2}|_{s=0} = \frac{\varepsilon}{c}\{< Y'^{\perp}, Y'^{\perp} > - < Y, R(Y, \gamma')\gamma' > + \frac{d}{dt} < \gamma', A >\}.$$

Now, by (2.6), we have

$$L''(s) = \sum_{i=1}^{k+1}\{\int_{a_{i-1}(s)}^{a_i(s)} \frac{\partial^2}{\partial s^2}|X(t,s)|dt \tag{3.6}$$

$$+a_i'(s)\frac{\partial}{\partial s}|X(t,s)||_{t=a_i(s)-0} \; - a_{i-1}'(s)\frac{\partial}{\partial s}|X(t,s)||_{t=a_{i-1}(s)+0}\}$$

$$+\sum_{i=1}^{k}[a_i''(s)\{|X(a_i(s)-0,s)|-|X(a_i(s)+0,s)|\}$$

$$+a_i'(s)\{\frac{d}{ds}|X(a_i(s)-0,s)|-\frac{d}{ds}|X(a_i(s)+0,s)|\}].$$

Setting $s=0$ in (3.6) produces the following changes:

$$|X(a_i(s)-0,s)|-|X(a_i(s)+0,s)|\to 0,$$

$$\frac{\partial}{\partial s}|X(t,s)||_{t=a_i(s)\pm 0}\to\frac{\varepsilon}{c}< Y'(a_i\pm 0),X(a_i\pm 0) > ,$$

and

$$\frac{d}{ds}|X(a_i(s)\pm 0,s)|\to\frac{\varepsilon}{c}< Y'(a_i\pm 0),X(a_i\pm 0) > .$$

Thus we get

$$L''(0)=\frac{\varepsilon}{c}\int_a^b\{< Y^{\perp\prime},Y^{\perp\prime} > - < R(Y,\gamma')\gamma',Y >\}dt \tag{3.7}$$

$$+\frac{\varepsilon}{c}\{\sum_{i=1}^{k+1} < A,X > |_{a_{i-1}}^{a_i}+2\sum_{i=1}^{k}a_i'(0)\Delta_{a_i} < Y',X >$$

$$=\frac{\varepsilon}{c}\int_a^b\{< Y^{\perp\prime},Y^{\perp\prime} > - < R(Y,\gamma')\gamma',Y >\}dt$$

$$+\frac{\varepsilon}{c}\{< A,X > |_a^b+\sum_{i=1}^{k}\Delta_{a_i} < A,X > +2\sum_{i=1}^{k}a_i'(0)\Delta_{a_i} < Y',X >\}.$$

In the rest of proof, we use the notation simplified as in the proof of Lemma 2.7. We show the following facts:

$$\Delta_{t_0} < A,X > +2d\Delta_{t_0} < Y',X > = < S_{\Delta X}(dX_+ + Y_+),dX_+ + Y_+ > ,$$

and

$$\Delta_{a_i} < A,X > +2a_i'(0)\Delta_{a_i} < Y',X > = 0 \quad (i\neq j).$$

In fact, let $\beta:(-\delta,\delta)\to B$ be $\beta(s):=\varphi(t_0(s),s)$, then $\beta'(0)=dX_+ + Y_+ = dX_- + Y_-$ and $\beta''(0)=A_+ + 2dY_+' + eX_+ = A_- + 2dY_-' + eX_-$ by Lemma 3.2, where $Y_\pm' := Y'(t_0\pm 0)$, $A_\pm := A(t_0\pm 0)$ and $e=t_0''(0)$. Thus we have

$$< S_{\Delta X}(dX_+ + Y_+),dX_+ + Y_+ >$$

$$= < D_{\beta'(0)}\beta',\Delta X > = < A_+ + 2dY_+' + eX_+,\Delta X > .$$

Hence, from (3.5), we find

$$< S_{\Delta X}(dX_+ + Y_+), dX_+ + Y_+ >$$

$$=< A_- + 2dY'_- + eX_-, X_- > - < A_+ + 2dY'_+ + eX_+, X_+ >$$

$$=< A_- + 2dY'_-, X_- > - < A_+ + 2dY'_+, X_+ > + e\{< X_-, X_- > - < X_+, X_+ >\}$$

$$= \Delta_{t_0} < A + 2dY', X > .$$

By (3.4), we have

$$\Delta_{a_i} < A, X > + 2a'_i(0)\Delta_{a_i} < Y', X >$$

$$=< \Delta_{a_i} A, X(a_i) > + 2a'_i(0) < \Delta_{a_i} Y', X(a_i) >= 0 \quad (i \neq j).$$

It follows that

$$L''(0) = \frac{\varepsilon}{c}\int_a^b \{< Y^{\perp\prime}, Y^{\perp\prime} > - < R(Y,\gamma')\gamma', Y >\}dt \tag{3.8}$$

$$+ \frac{\varepsilon}{c} < A, X > |_a^b + \frac{\varepsilon}{c} < S_{\Delta X}(dX_+ + Y_+), dX_+ + Y_+ >\}.$$

From (2.2), we get

$$0 =< d\Delta X + \Delta Y, \Delta X >= d < \Delta X, \Delta X > + < \Delta Y, \Delta X > ,$$

$$(=< dX_+ + Y_+, \Delta X >= d < X_+, \Delta X > + < Y_+, \Delta X >)$$

where $\Delta Y := \Delta_{t_0} Y$. Thus we have

$$d = -\frac{< \Delta Y, \Delta X >}{< \Delta X, \Delta X >} = -\frac{< Y_+, \Delta X >}{< X_+, \Delta X >}.$$

This completes the proof. □

For a fixed endpoint variation, since $< A, \gamma' > |_a^b = 0$, $L''(0)$ depends only on the variation vector field Y.

§4. The index form

Let p and q be points of M. And let $\Omega = \Omega(p,q) \subset \tilde{\Omega}$ be the set of all piecewise smooth curves $\alpha : [a,b] \to M$ such that $\alpha(a) = p$ and $\alpha(b) = q$. A subspace $T_\alpha\Omega$ in $T_\alpha\tilde{\Omega}$ is defined by

$$T_\alpha\Omega := \{Y \in T_\alpha\tilde{\Omega} : \ Y(a) = 0, \ Y(b) = 0\}.$$

We assume that $\Delta_{t_0}X$ is nonnull and nonzero. If $Y \in T_\alpha\tilde{\Omega}$, then

$$d_Y := d_j = -\frac{< Y(t_0+0), \Delta_{t_0}X >}{< X(t_0+0), \Delta_{t_0}X >} = -\frac{< \Delta_{t_0}Y, \Delta_{t_0}X >}{< \Delta_{t_0}X, \Delta_{t_0}X >}.$$

Hence, if $Y, V \in T_\alpha\tilde{\Omega}$, then $d_{Y+V} = d_Y + d_V$.

When we assume that $\Delta_{t_0} < X, X >= 0$ and $\Delta_{t_0}X$ is normal to B, if $Y \in T_\alpha\tilde{\Omega}$, then, by Lemma 2.7, Y^T and $Y^\perp$ are elements of $T_\gamma\tilde{\Omega}$. Furthermore followings hold:

$$\Delta_{t_0} < Y, Y >= 0, \quad \text{for any } Y \in T_\alpha\tilde{\Omega}. \tag{4.1}$$

$$\begin{aligned} &< norY(t_0-0), norY(t_0-0) > \\ &=< norY(t_0+0), norY(t_0+0) >, \quad \text{for any } Y \in T_\alpha\tilde{\Omega}. \end{aligned} \tag{4.2}$$

$$\Delta_{t_0} < Y^T, Y^T >= 0, \quad \text{for any } Y \in T_\alpha\tilde{\Omega}, \tag{4.3}$$

hence

$$\Delta_{t_0} < Y^\perp, Y^\perp >= 0, \quad \text{for any } Y \in T_\alpha\tilde{\Omega}. \tag{4.4}$$

In fact, from (2.5) and (2.9),

$$< Y(t_0-0) + Y(t_0+0), \Delta_{t_0}Y >=< Y(t_0-0) + Y(t_0+0), -d_Y\Delta_{t_0}X >$$

$$= -d_Y < Y(t_0-0) + Y(t_0+0), \Delta_{t_0}X >= 0.$$

Hence (4.1) holds. Since $\Delta_{t_0}Y$ is normal to B, $tanY(t_0-0) = tanY(t_0+0)$. Thus, by (4.1), (4.2) is true. Finally we show (4.3). Here, we can put $Y^T(t_0-0) = cX(t_0-0)$ and $Y^T(t_0+0) = cX(t_0+0)$ for some constant c, since $\Delta_{t_0} < Y^T, X >= 0$ and $\Delta_{t_0} < X, X >= 0$. Thus we have

$$\Delta_{t_0} < Y^T, Y^T >= c^2\Delta_{t_0} < X, X >= 0.$$

Lemma 4.1. *Let P be a linear operator defined by definition 3.2. Then*

$$P(Y^\perp) = P(Y) \qquad \textit{for all} \quad Y \in T_\gamma\tilde{\Omega}, \tag{4.5}$$

and $P : T_\gamma\tilde{\Omega} \to T_{\gamma(t_0)}B$ is a surjection.

proof. The proof is a straightforward calculation. For simplicity, we use the notation as in the proofs of Lemma 2.7 and Theorem 3.3. If $\Delta X \neq 0$, then

$$P(Y^\perp) = Y_+^\perp - \frac{< \Delta Y^\perp, \Delta X >}{< \Delta X, \Delta X >}X_+$$

$$= Y_+ - \frac{< Y_+, X_+ >}{< X_+, X_+ >} X_+$$

$$-\frac{1}{< \Delta X, \Delta X >} < Y_- - \frac{< Y_-, X_- >}{< X_-, X_- >} X_- - Y_+ + \frac{< Y_+, X_+ >}{< X_+, X_+ >} X_+, \Delta X > X_+$$

$$= Y_+ - \frac{< Y_+, X_+ >}{< X_+, X_+ >} X_+ - \frac{1}{< \Delta X, \Delta X >} < \Delta Y, \Delta X > X_+$$

$$+\frac{1}{< \Delta X, \Delta X >} \frac{1}{< X_+, X_+ >} << Y_-, X_- > X_- - < Y_+, X_+ > X_+, \Delta X > X_+$$

$$= P(Y) - \frac{1}{< X_+, X_+ >} \{< Y_+, X_+ > X_+$$

$$-\frac{1}{< \Delta X, \Delta X >} < Y_+, X_+ >< \Delta X, \Delta X > X_+\}$$

$$= P(Y). \qquad \square$$

Definition 4.2. The *index form* I_γ of a nonnull reflecting geodesic $\gamma \in \Omega$ for which $\Delta_{t_0} X$ is nonnull is the unique symmetric bilinear form

$$I_\gamma : T_\gamma \Omega \times T_\gamma \Omega \to \boldsymbol{R},$$

such that

$$I_\gamma(Y, Y) = L''(0),$$

where L is the length function of a fixed endpoint variation of γ in Ω with variation vector field $Y \in T_\gamma \Omega$.

Corollary 4.3. *If $\gamma \in \Omega$ is a reflecting geodesic of constant speed $c > 0$ and sign ε such that $\gamma(t_0) \in B$ and $\Delta X := \Delta_{t_0} X$ is nonnull, then*

$$I_\gamma(Y, W) = \frac{\varepsilon}{c} \int_a^b \{< Y^{\perp\prime}, W^{\perp\prime} > - < R(Y, \gamma')\gamma', W >\} dt$$

$$+\frac{\varepsilon}{c} < S_{\Delta X}(P(Y)), P(W) > ,$$

for all $Y, W \in T_\gamma \Omega$.

From Lemma 4.1, it follows immediately that

$$I_\gamma(Y, W) = I_\gamma(Y^\perp, W^\perp) \quad \text{for all } Y, W \in T_\gamma \Omega.$$

Thus there is no loss of information in restricting the index form I_γ to

$$T_\gamma^\perp \Omega := \{Y \in T_\gamma \Omega : Y \perp \gamma'\}.$$

We write $I_\gamma^\perp$ for this restriction.

Integration by parts produces a new version of the formula above.

Corollary 4.4. *Let $\gamma \in \Omega$ be a reflecting geodesic of constant speed $c > 0$ and sgn ε such that $\gamma(t_0) \in B$ and $\Delta X := \Delta_{t_0} X$ is nonnull. If Y and $W \in T_\gamma \Omega$ have breaks $a_1 < \cdots < t_0 = a_j < \cdots < a_k$, then*

$$I_\gamma(Y, W) = -\frac{\varepsilon}{c} \int_a^b < Y^{\perp\prime\prime} + R(Y, \gamma')\gamma', W^\perp > dt$$

$$+\frac{\varepsilon}{c} < S_{\Delta X}(P(Y)) + \Delta_{t_0} Y^{\perp\prime}, P(W) > +\frac{\varepsilon}{c} \sum_{i \neq j} < \Delta_{a_i} Y^{\perp\prime}, W^\perp(a_i) > .$$

proof. In Corollary 4.3, we can rewrite

$$< Y^{\perp\prime}, W^{\perp\prime} >= \frac{d}{dt} < Y^{\perp\prime}, W^\perp > - < Y^{\perp\prime\prime}, W^\perp > .$$

Then we get

$$I_\gamma(Y, W) = \frac{\varepsilon}{c} \int_a^b \{ \frac{d}{dt} < Y^{\perp\prime}, W^\perp > - < Y^{\perp\prime\prime}, W^\perp > - < R(Y, \gamma')\gamma', W > \} dt$$

$$+\frac{\varepsilon}{c} < S_{\Delta X}(P(Y)), P(W) >$$

$$= -\frac{\varepsilon}{c} \int_a^b \{ < Y^{\perp\prime\prime}, W^\perp > + < R(Y^\perp, \gamma')\gamma', W^\perp > \} dt$$

$$+\frac{\varepsilon}{c} \sum_{i=1}^{k+1} < Y^{\perp\prime}, W^\perp > |_{a_{i-1}}^{a_i} + \frac{\varepsilon}{c} < S_{\Delta X}(P(Y)), P(W) >$$

$$= -\frac{\varepsilon}{c} \int_a^b \{ < Y^{\perp\prime\prime}, W^\perp > + < R(Y^\perp, \gamma')\gamma', W^\perp > \} dt$$

$$+\frac{\varepsilon}{c} \sum_{i \neq j} < \Delta_{a_i} Y^{\perp\prime}, W^\perp(a_i) > +\frac{\varepsilon}{c} \Delta_{t_0} < Y^{\perp\prime}, W^\perp >$$

$$+\frac{\varepsilon}{c} < S_{\Delta X}(P(Y)), P(W) > .$$

For simplicity, we use the notation as in the proofs of Lemma 2.7 and Theorem 3.3. Then we have

$$\Delta_{t_0} < Y^{\perp\prime}, W^\perp >=< {Y^{\perp\prime}}_-, W_-^\perp > - < {Y^{\perp\prime}}_+, W_+^\perp >$$

$$=< {Y^{\perp\prime}}_-, P(W^\perp) + \frac{1}{< \Delta X, \Delta X >} < \Delta_{t_0} W^\perp, \Delta X > X_- >$$

$$- < {Y^{\perp}}'_{+}, P(W^{\perp}) + \frac{1}{< \Delta X, \Delta X >} < \Delta_{t_0} W^{\perp}, \Delta X > X_{+} >$$

$$=< \Delta_{t_0} {Y^{\perp}}', P(W) > + \frac{1}{< \Delta X, \Delta X >} < \Delta_{t_0} W^{\perp}, \Delta X >$$

$$\times \{ < {Y^{\perp}}'_{-}, X_{-} > - < {Y^{\perp}}'_{+}, X_{+} > \}$$

$$=< \Delta_{t_0} {Y^{\perp}}', P(W) > ,$$

since $< {Y^{\perp}}'_{-}, X_{-} > - < {Y^{\perp}}'_{+}, X_{+} >= \Delta_{t_0} \frac{d}{dt} < Y^{\perp}, X >$. □

Corollary 4.8. *Let $\gamma \in \Omega$ be a reflecting geodesic of constant speed $c > 0$ and sgn ε such that $\gamma(t_0) \in B$ and $\Delta X := \Delta_{t_0} X$ is nonnull. Then $Y \in T^{\perp}_{\gamma}\Omega$ is an element of the nullspace of $I^{\perp}_{\gamma}$ if and only if Y satisfies following two properties*:

$$\text{(4.6)} \qquad Y \textit{ is a Jacobi field on } [a, t_0] \text{ and } [t_0, b],$$

$$\text{(4.7)} \qquad S_{\Delta X}(P(Y)) + \Delta_{t_0} Y' \text{ is normal to } B.$$

proof. Let Y be in the nullspace of $I^{\perp}_{\gamma}$ and have breaks $a_1 < \cdots < t_0 = a_j < \cdots < a_k$. First we show that each restriction $Y|I_i$ is a Jacobi field. For a fixed t inside the interval I_i, let y be an arbitrary tangent vector to M at $\gamma(t)$. Construct $W = fV$ as in the proof of Corollary 2.8. Then, since $Y \perp \gamma'$, we have

$$0 = I_{\gamma}(Y, W) = -\frac{\varepsilon}{c} \int_{t-\zeta}^{t+\zeta} < Y'' + R(Y, \gamma')\gamma', fV^{\perp} > dt.$$

It follows as before that $Y'' + R(Y, \gamma')\gamma'$ is zero at t, hence identically zero on I_i, and so Y is Jacobi there. The proof that Y is differentiable on $[a, t_0]$ and $[t_0, b]$ again follows the same pattern as for the proof of Corollary 2.8. Thus

$$0 = I_{\gamma}(Y, W) = \frac{\varepsilon}{c} < S_{\Delta X}(P(Y)) + \Delta_{t_0} Y', P(W) > .$$

Since P is a surjection, $S_{\Delta X}(P(Y)) + \Delta_{t_0} Y'$ is normal to B.

Conversely, if (4.6) and (4.7) hold, then Y is an element of the nullspace of $I^{\perp}_{\gamma}$. □

§5. Conjugate points

Let $\gamma : [a, b] \to M$ be a reflecting geodesic such that $\gamma(t_0) \in B$ and $\Delta X := \Delta_{t_0} X$ is nonnull. Consider a variation $\varphi : [a, b] \times (-\delta, \delta) \to M$ such that $\varphi(t, 0) = \gamma(t)$ and $\varphi_s = \varphi(\cdot, s)$ is a reflecting geodesic for each s and the parameters $t_0(s)$ at which the geodesics reflect is smooth for s. Let Y be the variation vector field. Then, we can prove the following.

Lemma 5.1.

$$Y'' + R(Y,X)X = 0 \qquad \textit{on } [a,t_0] \textit{ and } [t_0,b], \tag{5.1}$$

$$S_{\Delta X}(P(Y)) + \Delta_{t_0}Y' \textit{ is normal to } B, \tag{5.2}$$

$$< Y, X >= C_1t + C_2 \quad \textit{for some constant } C_1 \textit{ and } C_2. \tag{5.3}$$

Proof. (1): Since φ is a variation through reflecting geodesics, Y is a Jacobi field along γ on $[a,t_0]$ and $[t_0,b]$, hence, satisfies (5.1).

(2): Let $\beta : (-\delta,\delta) \to B$ be $\beta(s) = \varphi(t_0(s),s)$. And we put $Z(s) = X(t_0(s)-0,s) - X(t_0(s)+0,s)$. Then, we find

$$S_{\Delta X}(P(Y)) = S_{Z(0)}(\beta'(0)) = (S_Z(\beta'))(0) = -tan(D_{\beta'}Z)(0).$$

Further, it holds that

$$D_{\beta'}Z = t_0'(s)\frac{DX}{\partial t}(t_0(s)-0,s) + \frac{DX}{\partial s}(t_0(s)-0,s)$$

$$-(t_0'(s)\frac{DX}{\partial t}(t_0(s)+0,s) + \frac{DX}{\partial s}(t_0(s)+0,s))$$

$$= \frac{DY}{\partial t}(t_0(s)-0,s) - \frac{DY}{\partial t}(t_0(s)+0,s).$$

Hence, we have

$$S_{\Delta X}(P(Y)) = -tan\Delta Y'.$$

(3): We set $< X(t,s), X(t,s) >= c(s)$, then

$$< \frac{DY}{\partial t}(t,s), X(t,s) >=< \frac{DX}{\partial s}(t,s), X(t,s) >$$

$$= \frac{1}{2}\frac{\partial}{\partial s} < X(t,s), X(t,s) >= \frac{1}{2}c'(s).$$

Hence we get

$$< Y(t), X(t) >' =< Y'(t), X(t) >= \frac{1}{2}c'(0).$$

Thus, for some constant $C_i (i = 1, 2, 3)$, we have

$$< Y, X >= \begin{cases} C_1 t + C_2 & \text{on } [a, t_0] \\ C_1 t + C_3 & \text{on } [t_0, b] \end{cases} \quad (C_1 := \frac{1}{2} c'(0)).$$

The result follows from $\Delta_{t_0} < Y, X >= 0$. □

Lemma 5.2. *If φ is a variation through reflecting geodesic of constant speed $c > 0$, then*

$$< Y, X >= const. \tag{5.4}$$

Furthermore,

$$Y' = Y^{\perp'} \qquad \text{on } [a, t_0] \text{ and } [t_0, b]. \tag{5.5}$$

proof. Since $< X(t, s), X(t, s) >= const.$, we find

$$< \frac{DY}{\partial t}(t, s), X(t, s) >=< \frac{DX}{\partial s}(t, s), X(t, s) >$$
$$= \frac{1}{2} \frac{\partial}{\partial s} < X(t, s), X(t, s) >= 0.$$

Hence we get

$$\frac{\partial}{\partial t} < Y(t, s), X(t, s) >=< \frac{DY}{\partial t}(t, s), X(t, s) >= 0.$$

Since $\Delta_{t_0} < Y, X >= 0$, (5.4) holds. Furthermore, we have

$$Y' = D_X Y = D_X (Y^{\perp} + \frac{< Y, X >}{< X, X >} X) = D_X Y^{\perp} + \frac{< Y, X >}{< X, X >} D_X X = Y^{\perp'}. \quad \square$$

Definition 5.3. Let γ be a reflecting geodesic such that $\gamma(t_0) \in B$ and $\Delta_{t_0} X$ is nonnull. If $Y \in T_\gamma \tilde{\Omega}$ satisfies the conditions (5.1), (5.2) and (5.3), then Y is called an *admissible Jacobi field* along γ. Let $\mathcal{J}_\gamma$ be the set of all admissible Jacobi fields on γ. An admissible Jacobi field Y along γ is a *perpendicular admissible Jacobi field* if Y is normal to γ. Let $\mathcal{J}_\gamma^{\perp}$ be the set of all perpendicular admissible Jacobi fields on γ. An admissible Jacobi field Y along γ is a *continuous admissible Jacobi field* if $Y(t_0) \in T_{\gamma(t_0)} B$. Let $\mathcal{J}_\gamma^{con}$ be the set of all continuous admissible Jacobi fields on γ.

By Corollary 4.5 elements of the nullspace of $I_\gamma^{\perp}$ are perpendicular admissible Jacobi fields. If Y is an admissible Jacobi field, then $Y \perp \gamma \Leftrightarrow$ there exist $t_i \in [a,b]$ $(i=1,2)$ such that $Y(t_i) \perp \gamma$ $(i=1,2)$ $\Leftrightarrow$ there exist $t_i \in [a,b]$ $(i=1,2)$ such that $Y(t_1) \perp \gamma$ and $Y'(t_2) \perp \gamma$, since (5.3). Y is an admissible Jacobi field if and only if Y^T and $Y^{\perp}$ are admissible Jacobis. $\mathcal{J}_\gamma$, $\mathcal{J}_\gamma^{\perp}$ and $\mathcal{J}_\gamma^{con}$ forms real vector spaces.

Lemma 5.4. *Let Y be an admissible Jacobi field on a reflecting geodesic γ. Then Y is the variation vector field of a variation φ of γ through reflecting geodesics.*

proof. Let $\beta : (-\delta,\delta) \to B$ be a curve with $\beta(0) = \gamma(t_0)$ and $\beta'(0) = P(Y)$. Let $A^{tan}(s)$ and $B^{tan}(s)$ be the vector fields on β gotten by B parallel translation of $tanX(t_0-0)(= tanX(t_0+0))$ and $tanY'(t_0-0) + S_{norX(t_0-0)}(P(Y))$ $(=tanY'(t_0+0) + S_{norX(t_0+0)}(P(Y)))$ along β. And let $A^{nor}_{\pm}(s)$ and $B^{nor}_{\pm}(s)$ be the vector fields on β gotten by normal parallel translation of $norX(t_0 \pm 0)$ and $norY'(t_0 \pm 0) - II(P(Y), tanX(t_0 \pm 0))$ along β. Where the function II is the shape tensor defined by $II(V,W) = norD_V W$ for any tangent vector field V and W to B. Finally, we put $A_{\pm}(s) = A^{tan}(s) + A^{nor}_{\pm}(s)$ and $B_{\pm}(s) = B^{tan}(s) + B^{nor}_{\pm}(s)$. If $Z_{\pm}(s) = A_{\pm}(s) + sB_{\pm}(s)$ for all s, then $Z_{\pm}(0) = X(t_0 \pm 0)$. Furthermore,

$$Z'_{\pm}(0) = A'_{\pm}(0) + B_{\pm}(0) = Y'(t_0 - 0).$$

For $Z_{\pm}$ as above, we now define a required variation φ as follows. Let exp be the exponential map and $t_0(s) = d_Y s + t_0$. Then

$$\varphi(t,s) = \begin{cases} \exp_{\beta(s)}((t - t_0(s))Z_-(s)) \text{ on } t \in [a, t_0(s)] \\ \exp_{\beta(s)}((t - t_0(s))Z_+(s)) \text{ on } t \in [t_0(s), b] \end{cases} \tag{5.7}$$

defines a variation of γ. The longitudinal curves of φ satisfy $X(t_0(s) \pm 0, s) = Z_{\pm}(s)$. Consequently, we have

$$X(t_0(s)-0,s) - X(t_0(s)+0,s)$$

$$= A^{nor}_-(s) - A^{nor}_+ + s(B^{nor}_-(s) - B^{nor}_+(s)),$$

and this is normal to B.

If V is the variation vector field of φ, then $V(t_0 \pm 0) = Y(t_0 \pm 0)$ since $P(V) = \beta'(0) = P(Y)$. By construction, it follows that

$$V'(t_0 \pm 0) = \frac{DX}{\partial s}(t_0 \pm 0, 0) = (D_{\beta'} Z_{\pm})(0) = Z'_{\pm}(0).$$

Thus we get $V = Y$. $\square$

Definition 5.5. Let γ be a reflecting geodesic such that $\gamma(t_0) \in B$ and $\Delta_{t_0}X$ is nonnull. We say that $\gamma(t_2)$ is a *conjugate point to* $\gamma(t_1)$ $(t_1 \neq t_2)$ *with respect to* B if there exists a nontrivial admissible Jacobi field Y along γ with $Y(t_1) = 0$ and $Y(t_2) = 0$.

Example 1. Let $M = \boldsymbol{R}^2$ be the Euclidean plane and

$$B = \{(x,y) | \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\} \quad (0 < a \leq b).$$

And let $r = (a,0) \in B$ and $p = (0,be)$, $q = (0,-be) \in M$, where $e = \sqrt{b^2 - a^2}/b$. A curve $\gamma : [0,2b] \to M$ is defined by

$$\gamma(t) = \begin{cases} (\frac{at}{b}, be(1 - \frac{t}{b})) & \text{on } [0,b] \\ (\frac{a}{b}(2b - t), be(1 - \frac{t}{b})) & \text{on } [b,2b] \end{cases}.$$

It holds that $\gamma(0) = p$, $\gamma(b) = r$ and $\gamma(2b) = q$. If $U_1 = \partial/\partial x$ and $U_2 = \partial/\partial y$ are the natural frame field, then

$$\gamma'(t) = \begin{cases} \frac{a}{b}U_1 - eU_2 & \text{on } [0,b] \\ -\frac{a}{b}U_1 - eU_2 & \text{on } [b,2b] \end{cases}.$$

Thus γ is a unit-speed reflecting geodesic. We define a variation $\varphi : [0,2b] \times (-\delta,\delta) \to M$ of γ by

$$\varphi(t,\theta) = \begin{cases} (\frac{a\cos\theta}{t_0(\theta)}t, be + \frac{b(\sin\theta - e)}{t_0(\theta)}t) & \text{on } I_-(\theta) \\ (\frac{a(t-2b)\cos\theta}{t_0(\theta) - 2b}, -be + \frac{b(t-2b)(\sin\theta + e)}{t_0(\theta) - 2b}) & \text{on } I_+(\theta) \end{cases},$$

where

$$t_0(\theta) = \sqrt{(a\cos\theta)^2 + (b\sin\theta - be)^2},$$

$$I_-(\theta) = [0, t_0(\theta)] \times (-\delta,\delta)$$

$$I_+(\theta) = [t_0(\theta), 2b] \times (-\delta,\delta).$$

Then we have

$$\frac{\partial\varphi}{\partial\theta}(t,\theta) = -\frac{at}{t_0(\theta)^2}(t_0(\theta)\sin\theta + t_0'(\theta)\cos\theta)U_1$$

$$+\frac{bt}{t_0(\theta)^2}(t_0(\theta)\cos\theta - t_0'(\theta)(\sin\theta - e))U_2 \quad \text{on } I_-(\theta)$$

and

$$\frac{\partial \varphi}{\partial \theta}(t,\theta) = -\frac{a(t-2b)}{(t_0(\theta)-2b)^2}((t_0(\theta)-2b)\sin\theta + t_0'(\theta)\cos\theta)U_1$$

$$+\frac{b(t-2b)}{(t_0(\theta)-2b)^2}((t_0(\theta)-2b)\cos\theta - t_0'(\theta)(\sin\theta+e))U_2 \quad \text{on } I_+(\theta).$$

Since $t_0'(\theta) = -be$, the variation vector field Y is

$$Y(t) = \begin{cases} \dfrac{aet}{b}U_1 + \dfrac{a^2 t}{b^2}U_2 & \text{on } [0,b] \\ \dfrac{ae(t-2b)}{b}U_1 - \dfrac{a^2(t-2b)}{b^2}U_2 & \text{on } [b,2b] \end{cases}.$$

It follows that

$$Y(b-0) = aeU_1(r) + \frac{a^2}{b}U_2(r)$$

and

$$Y(b+0) = -aeU_1(r) + \frac{a^2}{b}U_2(r).$$

Thus an admissible Jacobi field Y is discontinuous for $a \neq b$. Furthermore $\gamma(0)= p$ is a conjugate point to $\gamma(2b) = q$ with respect to B since $Y(0) = 0$ and $Y(2b) = 0$. We note that Hasegawa mentioned this example in [2].

Example 2. Let $M = \boldsymbol{R}^3$ be the Euclidean space and B be a regular smooth curve on

$$\tilde{B} = \{(x,y,z) \in M | \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1\} \quad (0 < a \leq b).$$

Then $p = (0, be, 0)$ is a conjugate point to $q = (0, -be, 0)$ with respect to B and $\tilde{B}$.

Example 3. Let $M = \boldsymbol{R}^3_1$ be the Lorentzian space with the metric

$$< (x,y,z),(x,y,z) >= -x^2 + y^2 + z^2$$

and

$$B = \{(0,y,z) | y^2 + z^2 = 1\},$$

that is, a sphere $S^1(1)$ in the hyperplane $x = 0$. And take $r = (0,1,0) \in B$ and $p = (c,0,0)$, $q = (d,0,0) \in M$ with $(1-c^2)(1-d^2) > 0$. Let $\gamma : [0, \bar{c}+\bar{d}] \to M$ ($\bar{c} := \sqrt{|1-c^2|}$, $\bar{d} := \sqrt{|1-d^2|}$) be a curve defined by

$$\gamma(t) = \begin{cases} (\dfrac{c(\bar{c}-t)}{\bar{c}}, \dfrac{t}{\bar{c}}, 0) & \text{on } [0,\bar{c}] \\ (\dfrac{d(t-\bar{c})}{\bar{d}}, \dfrac{\bar{c}+\bar{d}-t}{\bar{d}}, 0) & \text{on } [\bar{c}, \bar{c}+\bar{d}] \end{cases}.$$

Then $\gamma(0) = p$, $\gamma(\bar{c}) = r$ and $\gamma(\bar{c}+\bar{d}) = q$ hold. If $U_0 = \partial/\partial x$, $U_1 = \partial/\partial y$, $U_2 = \partial/\partial z$ are the natural frame field of $\boldsymbol{R}^3_1$ such that $< U_0, U_0 >= -1$, then

$$\gamma'(t) = \begin{cases} -\frac{c}{\bar{c}}U_0 + \frac{1}{\bar{c}}U_1 & \text{on } [0,\bar{c}] \\ \frac{d}{\bar{d}}U_0 - \frac{1}{\bar{d}}U_1 & \text{on } [\bar{c},\bar{c}+\bar{d}] \end{cases}$$

and

$$< \gamma', \gamma' >= \begin{cases} \frac{1-c^2}{\bar{c}^2} & \text{on } [0,\bar{c}] \\ \frac{1-d^2}{\bar{d}^2} & \text{on } [\bar{c},\bar{c}+\bar{d}] \end{cases}.$$

Thus γ is a timelike or spacelike unit-speed geodesic. Since er have

$$\gamma'(\bar{c}-0) - \gamma'(\bar{c}+0) = -(\frac{c}{\bar{c}} + \frac{d}{\bar{d}})U_0(r) + (\frac{1}{\bar{c}} + \frac{1}{\bar{d}})U_1(r)$$

and

$$T_rB = Span\{U_2(r)\},$$

γ is a reflecting geodesic. We define a variation $\varphi : [0, \bar{c}+\bar{d}] \times (-\delta, \delta) \to M$ of γ by

$$\varphi(t,\theta) = \begin{cases} (\frac{c(\bar{c}-t)}{\bar{c}}, \frac{t\cos\theta}{\bar{c}}, \frac{t\sin\theta}{\bar{c}}) & \text{on } [0,\bar{c}] \times (-\delta,\delta) \\ (\frac{d(t-\bar{c})}{\bar{d}}, \frac{(\bar{c}+\bar{d}-t)\cos\theta}{\bar{d}}, \frac{(\bar{c}+\bar{d}-t)\sin\theta}{\bar{d}}) & \text{on } [\bar{c},\bar{c}+\bar{d}] \times (-\delta,\delta) \end{cases}.$$

It holds that

$$\frac{\partial\varphi}{\partial t}(t,\theta) = \begin{cases} -\frac{c}{\bar{c}}U_0 + \frac{\cos\theta}{\bar{c}}U_1 + \frac{\sin\theta}{\bar{c}}U_2 & \text{on } [0,\bar{c}] \times (-\delta,\delta) \\ \frac{d}{\bar{d}}U_0 - \frac{\cos\theta}{\bar{d}}U_1 - \frac{\sin\theta}{\bar{d}}U_2 & \text{on } [\bar{c},\bar{c}+\bar{d}] \times (-\delta,\delta) \end{cases}.$$

Let

$$v := -\sin\theta \cdot U_1(\varphi(\bar{c},\theta)) + \cos\theta \cdot U_2(\varphi(\bar{c},\theta))$$

and

$$w := \frac{\partial\varphi}{\partial t}(\bar{c}-0,\theta) - \frac{\partial\varphi}{\partial t}(\bar{c}+0,\theta).$$

Then it follows that $< v, w >= 0$ since $T_{\varphi(D,\theta)}B = Span\{v\}$ and

$$w = -(\frac{c}{\bar{c}} + \frac{d}{\bar{d}})U_0(\varphi(\bar{c},\theta)) + \cos\theta(\frac{1}{\bar{c}} + \frac{1}{\bar{d}})U_1(\varphi(\bar{c},\theta)) + \sin\theta(\frac{1}{\bar{c}} + \frac{1}{\bar{d}})U_2(\varphi(\bar{c},\theta)).$$

Hence φ is a variation through reflecting geodesics. Furthermore it holds that

$$\frac{\partial\varphi}{\partial\theta}(t,\theta) = \begin{cases} -\frac{t\sin\theta}{\bar{c}}U_1 + \frac{t\cos\theta}{\bar{c}}U_2 & \text{on } [0,\bar{c}] \times (-\delta,\delta) \\ -\frac{(\bar{c}+\bar{d}-t)\sin\theta}{\bar{d}}U_1 + \frac{(\bar{c}+\bar{d}-t)\cos\theta}{\bar{d}}U_2 & \text{on } [\bar{c},\bar{c}+\bar{d}] \times (-\delta,\delta) \end{cases}.$$

Thus the variation vector field Y is

$$Y(t)=\begin{cases}\dfrac{t}{\bar{c}}U_2 \quad \text{on } [0,\bar{c}]\\ \dfrac{\bar{c}+\bar{d}-t}{\bar{d}}U_2 \quad \text{on } [\bar{c},\bar{c}+\bar{d}]\end{cases}.$$

This shows that Y is a perpendicular and continuous admissible Jacobi field and $\gamma(0)$ is a conjugate point to $\gamma(\bar{c}+\bar{d})$ with respect to B.

§6. Normal reflecting geodesics

In this section we treat special cases of reflecting geodesics.

Definition 6.1. Let γ be a reflecting geodesic. If $X(t_0-0)$ is normal to B (thus so is $X(t_0+0)$), γ is called a *normal reflecting geodesic.*

For example, a reflecting geodesic with $\gamma(a)=\gamma(b)$ is a normal reflecting geodesic and so is a reflecting geodesic with $\gamma(a)$ or $\gamma(b)\in B$.

Proposition 6.2. *An admissible Jacobi field Y on a normal reflecting geodesic γ is the variation vector field of a variation φ of γ through normal reflecting geodesics if and only if*

$$S_{X(t_0\pm 0)}(P(Y))+Y'(t_0\pm 0) \text{ are normal to } B. \tag{6.1}$$

proof. Let $\varphi:[a,b]\times(-\delta,\delta)\to M$ be such a variation with the variation vector field Y and $\beta:(-\delta,\delta)\to B$ a curve defined to be $\beta(s)=\varphi(t_0(s),s)$. Then $\beta'(0)=P(Y)$ and we put

$$Z_\pm(s):=X(t_0(s)\pm 0,s).$$

These are normal to B and

$$D_{\beta'(s)}Z_\pm=t_0'(s)\frac{DX}{\partial t}(t_0(s)\pm 0,s)+\frac{DX}{\partial s}(t_0(s)\pm 0,s)=\frac{DY}{\partial t}(t_0(s)\pm 0,s).$$

Hence $Z'_\pm(0)=Y'(t_0\pm 0)$. Furthermore, we have

$$tan Z'_\pm=tan D_{\beta'}Z_\pm=-S_{Z_\pm}(\beta'),$$

hence

$$tan Y'(t_0\pm 0)=-S_{X(t_0\pm 0)}(P(Y)).$$

It follows that

$$S_{X(t_0\pm 0)}(P(Y))+Y'(t_0\pm 0)=nor Y'(t_0\pm 0).$$

Converse is the case of $A^{tan} = B^{tan} = 0$ in Lemma 5.4. $\square$

Corollary 6.3. *An admissible Jacobi field Y on a normal (reflecting) geodesic γ with $t_0 = a$ is the variation vector field of a variation φ of γ through normal (reflecting) geodesics if and only if*

$$S_{X(a)}(Y(a)) + Y'(a) \text{ is normal to } B.$$

This coincides with the well-known fact, see Proposition 10.28 in [4] , for example.

§7. Variation of energy

Let $\alpha : [a, b] \to M$ be a piecewise smooth curve. Then the integral

$$E = \frac{1}{2}\int_a^b < \alpha', \alpha' > dt$$

is called *energy*. Let $E(s)$ be the value of E on the longitudinal curve $t \mapsto \varphi(t, s)$, so

$$E(s) = \frac{1}{2}\int_a^b < \frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial t} > dt,$$

where φ is the variation of α in $\tilde{\Omega}$. By contrast with L, the function E is always smooth without restriction on φ. Formulas for the first and second variations of E are simpler analogues of those for L.

Lemma 7.1. *Let $\alpha : [a, b] \to M$ be a piecewise smooth curve such that $\alpha(t_0) \in B$. Let φ be a variation of α in $\tilde{\Omega}$, with Y and A the variation and transverse acceleration vector fields of φ. If $f = f(t, s) =< \frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial t} >$, then*

$$\frac{1}{2}\frac{\partial f}{\partial s}|_{s=0} =< Y', \alpha' >= - < Y, \alpha'' > + \frac{d}{dt} < Y, \alpha' > , \tag{7.1}$$

$$\frac{1}{2}\frac{\partial^2 f}{\partial s^2}|_{s=0} =< Y', Y' > - < R(Y, \alpha')\alpha', Y > + < A', \alpha' > \tag{7.2}$$

$$= - < Y'' + R(Y, \alpha')\alpha', Y > + < A', \alpha' > + \frac{d}{dt} < Y', Y > .$$

proof. We readily compute

$$\frac{1}{2}\frac{\partial f}{\partial s} =< \frac{D}{\partial s}\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial t} >=< \frac{D}{\partial t}\frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial t} > ,$$

$$\frac{1}{2}\frac{\partial^2 f}{\partial s^2} =< \frac{D}{\partial t}\frac{\partial\varphi}{\partial s}, \frac{D}{\partial s}\frac{\partial\varphi}{\partial t} > + < \frac{D}{\partial s}\frac{D}{\partial t}\frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial t} >$$

$$=< \frac{D}{\partial t}\frac{\partial\varphi}{\partial s}, \frac{D}{\partial t}\frac{\partial\varphi}{\partial s} > + < R(\frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial t})\frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial t} > + < \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial t} > .$$

Hence it holds that

$$\frac{1}{2}\frac{\partial f}{\partial s}|_{s=0} =< Y', \alpha' >= - < Y, \alpha'' > + \frac{d}{dt} < Y, \alpha' > ,$$

$$\frac{1}{2}\frac{\partial^2 f}{\partial s^2}|_{s=0} =< Y', Y' > - < R(Y, \alpha')\alpha', Y > + < A', \alpha' >$$

$$= - < Y'' + R(Y, \alpha')\alpha', Y > + < A', \alpha' > + \frac{d}{dt} < Y', Y > . \quad \square$$

Proposition 7.2. (*First Variation Formula*) *Let $\alpha : [a, b] \to M$ be a piecewise smooth curve such that $\alpha(t_0) \in B$. Let φ be a variation of α in $\tilde{\Omega}$ with the variation vector field Y. Then*

$$E'(0) = - \int_a^b < Y, \alpha'' > dt + < Y, \alpha' > |_a^b$$

$$+ \frac{1}{2}\sum_{i=1}^{k} < Y(a_i - 0) + Y(a_i + 0), \Delta_{a_i}\alpha' > ,$$

where $a_1 < \cdots < a_j = t_0 < \cdots < a_k$ are the breaks of α.

proof. As in the proof of Proposition 2.6, we get

$$E'(0) = \frac{d}{ds}(\frac{1}{2}\int_a^b f(t, s)dt)|_{s=0} = \frac{1}{2}(\frac{d}{ds}\sum_{i=1}^{k+1}\int_{a_{i-1}(s)}^{a_i(s)} f(t, s)dt)|_{s=0}$$

$$= \int_a^b \frac{1}{2}\frac{\partial f}{\partial s}|_{s=0}dt + \frac{1}{2}\sum_{i=1}^{k+1}\{a_i'(0)f(a_i - 0, 0) - a_{i-1}'(0)f(a_{i-1} + 0, 0)\}$$

$$= \int_a^b \frac{1}{2}\frac{\partial f}{\partial s}|_{s=0}dt + \frac{1}{2}\sum_{i=1}^{k} a_i'(0)\{f(a_i - 0, 0) - f(a_i + 0, 0)\}.$$

By Lemma 7.2, it holds that

$$E'(0) = - \int_a^b < Y, \alpha'' > dt + \sum_{i=1}^{k+1} < Y, \alpha' > |_{a_{i-1}}^{a_i} + \frac{1}{2}\sum_{i=1}^{k} a_i'(0)\{< \alpha'(a_i - 0), \alpha'(a_i - 0) > - < \alpha'(a_i + 0), \alpha'(a_i + 0) >\}$$

$$= - \int_a^b < Y, \alpha'' > dt + \sum_{i=1}^{k} \Delta_{a_i} < Y, \alpha' >$$

$$+ < Y, \alpha' > |_a^b + \frac{1}{2}\sum_{i=1}^{k} a_i'(0)\Delta_{a_i} < \alpha', \alpha' > .$$

Furthermore, we have

$$
\begin{aligned}
& a_i'(0)\Delta_{a_i} < \alpha', \alpha' > +2\Delta_{a_i} < Y, \alpha' > \\
&= < a_i'(0)\alpha'(a_i - 0) + 2Y(a_i - 0), \alpha'(a_i - 0) > \\
&- < a_i'(0)\alpha'(a_i + 0) + 2Y(a_i + 0), \alpha'(a_i + 0) > \\
&= < a_i'(0)\alpha'(a_i + 0) + Y(a_i + 0) + Y(a_i - 0), \alpha'(a_i - 0) > \\
&- < a_i'(0)\alpha'(a_i - 0) + Y(a_i - 0) + Y(a_i + 0), \alpha'(a_i + 0) > \\
&= < Y(a_i + 0) + Y(a_i - 0), \Delta_{a_i}\alpha' > \\
&+ < a_i'(0)\alpha'(a_i + 0), \alpha'(a_i - 0) > - < a_i'(0)\alpha'(a_i - 0), \alpha'(a_i + 0) > \\
&= < Y(a_i + 0) + Y(a_i - 0), \Delta_{a_i}\alpha' > ,
\end{aligned}
$$

since (2.1). □

Corollary 7.3. *Let $\alpha : [a, b] \to M$ be a piecewise smooth curve such that $\alpha(t_0) \in B$. The first variation of energy is zero for every fixed endpoint variation of α in $\tilde{\Omega}$ if and only if α is a reflecting geodesic or a geodesic.*

proof. Suppose $E'(0) = 0$ for every fixed endpoint variation φ. First we show that each segment $\alpha|I_i$ is geodesic. It suffices to show that $\alpha''(t) = 0$ for $t \in I_i^\circ$. Let y be any tangent vector to M at $\alpha(t)$, and let f be a bump function on $[a, b]$ with supp$f \subset [t - \zeta, t + \zeta] \subset I_i$. Let V be the vector field on α obtained by parallel translation of y, and finally let $Y = fV$.

Since $Y(a)$ and $Y(b)$ are both zero, exponential formula $\varphi(t, s) = \exp_{\alpha(t)} (sY(t))$ produces a fixed endpoint variation of α whose variation vector field is Y. Since $E'(0) = 0$, the formula in Proposition 7.2 reduces to

$$
0 = -\int_a^b < Y, \alpha'' > dt = \int_{t-\zeta}^{t+\zeta} < fV, \alpha'' > dt.
$$

This holds for all y and $\zeta > 0$. Hence $< y, \alpha''(t) >= 0$ for all y; hence $\alpha'' = 0$.

As before, let y be an arbitrary tangent vector at $\alpha(a_i)$ $(i \neq j)$, and let f be a bump function at a_i with supp$f \subset I_i \cup I_{i+1}$ $(i \neq j)$. For a fixed endpoint variation with vector field fV the first variation formula now reduces to

$$
\begin{aligned}
0 = E'(0) &= \frac{1}{2} < Y(a_i - 0) + Y(a_i + 0), \Delta_{a_i}\alpha' > \\
&= < y, \Delta_{a_i}\alpha' > \quad \text{for all } y.
\end{aligned}
$$

Hence $\Delta_{a_i}\alpha' = 0$ $(i \neq j)$. This shows that (1.5) is true and $0 =< Y(t_0-0)+Y(t_0+0),\Delta_{t_0}X >$. The latter means that $< y, \Delta_{t_0}X >= 0$ for any $y \in T_{\alpha(t_0)}B$. Furthermore, for a fixed endpoint variation of α with $t_0'(0) \neq 0$,

$$0 =< Y(t_0 - 0) + t_0'(0)X(t_0 - 0) + Y(t_0 + 0) + t_0'(0)X(t_0 + 0), \Delta_{t_0}X >$$

$$=< Y(t_0 - 0) + Y(t_0 + 0), \Delta_{t_0}X > + t_0'(0)\Delta_{t_0} < X, X > .$$

Consequently (1.7) is true.

Conversely we assume that α is a reflecting geodesic. For any fixed endpoint variation of α whose vector field is Y, by the first variation formula,

$$E'(0) = \frac{1}{2} < Y(t_0 - 0) + Y(t_0 + 0), \Delta_{t_0}X >= 0. \qquad \Box$$

Proposition 7.4. (*Second Variation Formula*) *Let $\gamma : [a,b] \to M$ be a reflecting geodesic such that $\gamma(t_0) \in B$ and $\Delta X := \Delta_{t_0}X$ is nonnull. If φ is a variation of γ in $\tilde{\Omega}$, then*

$$E''(0) = \int_a^b \{< Y', Y' > - < R(Y, \gamma')\gamma', Y >\} dt$$

$$+ < A, \gamma' > |_a^b + < S_{\Delta X}(P(Y)), P(Y) > .$$

proof. Using (7.2), we can prove as in Theorem 3.2. $\Box$

If γ is such a reflecting geodesic, then strictly analogous to the index form I_γ for L is the *Hessian* H_γ for E. Explicitly, H_γ is the unique $\boldsymbol{R}$-linear form on $T_\gamma\Omega$ such that $H_\gamma(Y,Y) = E''(0)$, where E is the energy function of a variation of γ in $\tilde{\Omega}$ whose variation vector field is Y. By the second variation formula above it follows as in Corollary 4.6,

$$H_\gamma(Y, W) = \int_a^b \{< Y', W' > - < R(Y, \gamma')\gamma', W >\} dt$$

$$+ < S_{\Delta X}(P(Y)), P(W) > ,$$

for $Y, W \in T_\gamma\Omega$.

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