SUT Journal of Mathematics
Vol. 34, No. 2 (1998), 91–103

ON CERTAIN INEQUALITIES FOR FOURIER COEFFICIENTS

Amir Kamaly

(Received March 31, 1998)

Abstract. A new proof of the one-dimensional Carlson inequality of discrete and integral type is introduced. Also, a new proof of the multidimensional Carlson inequality of integral type is presented. Additionally, the duality between the discrete and integral type of Carlson's inequality are explored.

AMS 1991 *Mathematics Subject Classification.* 26D15, 42AXX, 42BXX, 47A30, 47A63.

Key words and phrases. Discrete and integral inequalities, Hilbert's inequalities, Carlson's inequalities, multidimensional inequalities, the Hausdorff-Young inequality.

0. Introduction

Inequalities are basic tools in the study of Fourier analysis. A classical result relating to L^p-estimates for a function and its Fourier transform is the Hausdorff-Young inequality (1912-1923) which states that, for any complex-valued function g in the Banach space $L^p(\mathbb{T})$

$$\|\widehat{g}\|_{p'} \leq \|g\|_p \tag{0.1}$$

holds for $1 \leq p \leq 2$. This inequality was proved by Young [19, 20] for even p' and by Hausdorff [9] in the general case. It is easy to see that for functions $g(x) = Ae^{-2\pi imx}$ the inequality (0.1) turns to be an equality. With the other words, the oscillating exponent maximized the Hausdorff-Young inequality. Moreover, Hardy and Littlewood showed that the converse result was also true, i.e. every maximizer of (0.1) must be the functions $Ae^{-2\pi imx}$ with some constant A and integers m (see e.g. [21, p. 105], Theorem (2.25)). Their proof is based on the one-functional relation for maximizers of the Riesz-Thorin inequality (see (1.24) in [21, p. 98]), the properties of entire functions and on the Riemann-Lebesgue theorem for Fourier coefficients.

In 1924 Titchmarsh proved (0.1) for the space $L^p(\mathbb{R})$. Thus, it was natural to consider the question about maximizers of the Hausdorff-Young inequality for the real-line group, i.e. to find the real-line analogue of the above Hardy-Littlewood result. Of course, an oscillating exponent does not have to be a maximizer (it is even non-summable on the real line). On the other hand, since the decomposition of an oscillating exponent into Fourier series coincides with the initial function, thus, in the real-line case, it will be natural to consider a maximizer as a function which is invariant under the action of the Fourier transform, i.e. the Gaussian function $\exp(-\pi x^2)$. Moreover, if we consider the n-dimensional version, then one may expect that if an extremal function exists, it should be rotationally invariant. The n-dimensional norm will be a power of the one-dimensional one, so we need an extremal function for which a product of functions is radial in separate variables as well as in the variables jointly. This is possible only for a Gaussian function. But if we admit that the maximizer is Gaussian, we get the improved constant in the Hausdorff-Young inequality. That is

$$\|\widehat{f}\|_{p'} \le B_p^n \|f\|_p, \qquad \text{with} \qquad B_p = \sqrt{\frac{p^{\frac{1}{p}}}{p'^{\frac{1}{p'}}}}. \tag{0.2}$$

This inequality for the space $L^p(\mathbb{R}^n)$ and for even integer p' was established by Babenko [2] in 1961. The first problem in the proof of (0.2) is to prove the existence of extremal functions. The difficulty is that the Fourier transform is not a compact operator from $L^p(\mathbb{R}^n)$ to $L^{p'}(\widehat{\mathbb{R}^n})$. So we can regularize this operator. A very natural way to regularize Fourier series is to generate the Abel means. This idea, in some sense, was realized by Babenko, who introduced an integral operator $K_t f$ with the classical Mehler kernel

$$K(x,y,t) = \sum_{n=0}^{\infty} t^n \psi_n(x)\psi_n(y)$$

where ψ_n are the Hermite orthogonal functions. Obviously, the operator $K_t f$ forms the Abel means of the Hermite expansions of the function f and $K_t f = \widehat{f}$ for $t = 1$.

The idea of the proof of this moment is similar, in some sense, to the one of Hardy and Littlewood. Namely, since K_t is a good (compact) operator, then by a weak compactness argument, a solution g_0, $\|g_0\|_p \le 1$ of the extremal problem

$$\|K_t g_0\|_{p'} = \sup_{\|f\|_p \le 1} \|K_t f\|_{p'} \equiv \mu_{p,t}$$

will exist and satisfy an integral identity, whose right-hand side contains $|f|^{p'-2}$ (see (25) in [2]). But the application of the method of entire functions (in fact the property that bounded entire functions are only constants) requires to avoid operations with absolute values. Since $|f|^{p'-2} = (f\overline{f})^{p'/2-1}$, thus to save uniqueness we can require that the value of p' will be even integers. Then using the Phragmen-Lindelöf method and rearranging contour integrals in the complex plane, one can calculate the value $\mu_{p,t}$ and as a consequence, get the inequality

$$\|K_t g\|_{p'} \leq \mu_{p,t}\|g\|_p.$$

A limiting argument gives the inequality (0.2) since

$$\lim_{t\to 0^+} \mu_{p,t} = B_p.$$

Babenko mentioned that the equality in (0.2) can be realized for the functions $f(x) = \exp(-ax^2 + ibx)$, $a > 0$ and introduced the hypothesis that the inequality (0.2) was still true for all values $1 < p \leq 2$ and that extremal functions must be the Gaussians.

The first part of Babenko's hypothesis was proved by Beckner [4] in 1975, i.e. he established the inequality (0.2) for all $1 < p \leq 2$. Now B_p is called the Babenko-Beckner constant. He also showed that for all $1 < p \leq 2$ the sharp constant (found by Babenko) in (0.2) is given by the Gaussians. However, Beckner's method did leave open the question whether the Gaussians were the only maximizers or not. The positive answer was given by Lieb [15] in 1990.

Further, for the even integers p', Andersson (1992) [1] and for all p, Sjölin (1994) [18] proved a Babenko-Beckner type of the classical Hausdorff-Young inequality (0.1) for functions in the space $L^p(\mathbb{T})$, with small supports. One defined

$$H_{p,a} := \sup\left\{\frac{\|\widehat{g}\|_{p'}}{\|g\|_p} : g \in L^p(\mathbb{T}^n),\ \mathrm{supp}\ g \subset \overline{B}(0,a),\ \|g\|_p \neq 0\right\}$$

and let $H_p := \lim_{a\to 0^+} H_{p,a}$ (see [11, p. 3]). In these terms, the one-dimensional result of Andersson and Sjölin states that

$$H_p = B_p.$$

The author proved the identity $H_p = B_p^n$ for $p \in [1,2]$, using other reasons. Namely, we proved Carlson's inequality on n-dimensional torus and applied it to find some upper bounds for $H_{p,a}$. One of the possible estimates is the following

$$H_{p,a} \leq (1 + C_0 a)B_p^n, \qquad 1 \leq p \leq 2. \tag{0.3}$$

Moreover, the author mentioned some applications of the Babenko-Beckner constant to related problems of Fourier Analysis [12]. □

1. On Carlson's Inequality of discrete and integral type

For an even, periodic, real-valued function f on $\mathbb{R}$ such that $f \in \mathcal{A}([-\pi,\pi])$ and $\widehat{f}(n) = a_n \geq 0$ for all $n \in \mathbb{N}$ with $a_0 = 0$, Fritz Carlson (1934) proved that

$$\sum_{n=1}^{\infty} a_n < \sqrt{\pi}\left(\sum_{n=1}^{\infty} a_n^2\right)^{\frac{1}{4}}\left(\sum_{n=1}^{\infty} n^2 a_n^2\right)^{\frac{1}{4}} \tag{1.1}$$

holds and $\sqrt{\pi}$ is the best possible constant. By the mean value theorem for integral together with partial integration he [6] got

$$f^2(0) = -2\int_0^{\xi} f(x)f'(x)dx \tag{1.2}$$

for some $\xi \in (-\pi,\pi)$. Acting the Schwarz-Cauchy inequality on (1.2) and using Parseval's identity were other parts of Carlson's proof. He also noted that (1.1) does not follow from Hölder's inequality [14] in the following way

$$\sum_{n=1}^{\infty} a_n < \left(\sum_{n=1}^{\infty} n^{-h}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} a_n^2\right)^{\frac{1}{4}}\left(\sum_{n=1}^{\infty} n^{2h} a_n^2\right)^{\frac{1}{4}}$$

because $\sum_{n=1}^{\infty} n^{-h} \to \infty$ as $h \to 1^+$. However, in 1936 Hardy [7] presented a simple proof of (1.1) and observed that (1.1) in fact followed even from the Schwarz-Cauchy inequality $\sum_{n=1}^{\infty} x_n y_n < \left(\sum_{n=1}^{\infty} x_n^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} y_n^2\right)^{\frac{1}{2}}$ applied to the sequences $x_n := a_n\left(\alpha + \beta n^2\right)^{\frac{1}{2}}$ and $y_n := \left(\alpha + \beta n^2\right)^{-\frac{1}{2}}$ and he even got the best possible constant $\sqrt{\pi}$ by making a suitable choice of α and β and invoking Parseval's identity.

Hardy's technique can be used for any periodic complex-valued function $f \in \mathcal{A}(\mathbb{T})$ with $\widehat{f}(0) = 0$. Moreover, another expression of (1.1) is

$$\|f\|_{\mathcal{A}(\mathbb{T})} \leq C\biggl(\|\widehat{f}\|_2\|\widehat{f'}\|_2\biggr)^{\frac{1}{2}} \tag{1.3}$$

which is equivalent to

$$\|f\|_{\mathcal{A}(\mathbb{T})} \leq C\biggl(\|f\|_2\|f'\|_2\biggr)^{\frac{1}{2}}. \tag{1.4}$$

Thus, the best possible constant C (which depends on the definition of the Fourier series of f) will be $\sqrt{2\pi}$ and 1 if $\widehat{f}(n) := \frac{1}{2\pi}\int_{|x|\leq\pi} f(x)e^{-inx}dx$ and $\widehat{f}(n) := \int_{|x|\leq\frac{1}{2}} f(x)e^{-2\pi inx}dx$ respectively.

Here $\|f\|_{\mathcal{A}(\mathbb{T})} := \sum_{m\in\mathbb{Z}} |\widehat{f}(m)|$ and $\mathcal{A}(\mathbb{T})$ is the space of continuous functions on $\mathbb{T}$ having an absolutely convergent Fourier series. Note that all the sums are supposed to be finite. Furthermore, $\|\widehat{f}\|_2 = \|\widehat{f}\|_{\ell^2(\mathbb{Z})}$ and $\|f\|_2 = \|f\|_{L^2(\mathbb{T})}$.

The author proved (1996) the multidimensional discrete inequality of types (1.3) and (1.4) by applying Hardy's technique and invoking the classical Hausdorff-Young inequality [11].

$$\begin{aligned} \|f\|_{\mathcal{A}(\mathbb{T}^n)} &\le K_{n,q}^{(\alpha)} \|\widehat{f}\|_{q'}^{1-\frac{n}{q\alpha}} \left(\sum_{|\beta|=\alpha} \|\widehat{D^\beta f}\|_{q'} \right)^{\frac{n}{q\alpha}} \\ &\le K_{n,q}^{(\alpha)} \|f\|_{q}^{1-\frac{n}{q\alpha}} \left(\sum_{|\beta|=\alpha} \|D^\beta f\|_{q} \right)^{\frac{n}{q\alpha}} . \end{aligned} \tag{1.5}$$

Here the absolute value of the multi-index β is equal to the positive integer $\alpha \ge 1$ such that $\alpha > \frac{n}{q}$ with $q > 1$ and $1 < q \le 2$ for the first respectively the second part of (1.5) where $q' = \frac{q}{q-1}$ is the dual exponent of q. The positive constant $K_{n,q}^{(\alpha)}$ depends only on n, α and q. Furthermore, $\|\widehat{f}\|_{q'} = \|\widehat{f}\|_{\ell^{q'}(\mathbb{Z}^n)}$ and $\|f\|_q = \|f\|_{L^q(\mathbb{T}^n)}$.

In the case $\widehat{f}(0) \ne 0$, we obtain

$$\|f\|_{\mathcal{A}(\mathbb{T}^n)} \le \|f\|_1 + K_{n,q}^{(\alpha)} \|f\|_{q}^{1-\frac{n}{q\alpha}} \left(\sum_{|\beta|=\alpha} \|D^\beta f\|_{q} \right)^{\frac{n}{q\alpha}} .$$

The other versions of (1.5) for $1 < q \le 2$ are

$$\|f\|_{\mathcal{A}(\mathbb{T}^n)} \le K_{n,q}^{(\alpha)} \|\widehat{f}\|_{q'}^{1-\frac{n}{q\alpha}} \left(\sum_{|\beta|=\alpha} \|D^\beta f\|_{q} \right)^{\frac{n}{q\alpha}} \quad \text{and}$$

$$\|f\|_{\mathcal{A}(\mathbb{T}^n)} \le K_{n,q}^{(\alpha)} \|f\|_{q}^{1-\frac{n}{q\alpha}} \left(\sum_{|\beta|=\alpha} \|\widehat{D^\beta f}\|_{q'} \right)^{\frac{n}{q\alpha}} .$$

The corresponding integral inequalities [3, 6] of (1.1) and (1.3) are

$$\int_0^\infty f(x)dx \le \sqrt{\pi} \left(\int_0^\infty f^2(x)dx \right)^{\frac{1}{4}} \left(\int_0^\infty x^2 f^2(x)dx \right)^{\frac{1}{4}} \tag{1.6}$$

respectively

$$\int_{-\infty}^{\infty} |f(x)|dx \leq C \left(\int_{-\infty}^{\infty} |f(x)|^2 dx \right)^{\frac{1}{4}} \left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \right)^{\frac{1}{4}}. \tag{1.7}$$

There is equality in (1.6) when $f(x) := \frac{1}{r+sx^2}$, for any positive number r and s.

The inequality (1.6) is proved by Carlson [6] and Hardy [7]. The version (1.7) was proved by A. Beurling [5]. The inequality (1.6) is generalized in many directions [5, 8, 14]. B. Kjellberg [13] and D. Müller (see [16], Lemma 3.1) proved a multidimensional extension of Carlson's inequality of the integral type. The purpose of the next section is to present new proofs of (1.1) and (1.6). These proofs are established by application of Hilbert's double series theorem and its corresponding theorem for integrals.

2. One-dimensional cases

Hilbert's inequality [8] states that for nonnegative sequences $a := (a_n)$ and $b := (b_n)$, $n \in \mathbb{N}$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{n+m} \leq \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\sum_{n=1}^{\infty} a_n^p \right)^{\frac{1}{p}} \left(\sum_{m=1}^{\infty} b_n^{p'} \right)^{\frac{1}{p'}} \tag{2.1}$$

holds for $p > 1$. There is equality here if a or b is null.

The corresponding Hilbert's inequality for integrals [8] is

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x+y} dxdy \leq \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\int_0^{\infty} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^{\infty} g^{p'}(y)dy \right)^{\frac{1}{p'}} \tag{2.2}$$

for nonnegative functions $f \in L^p$ and $g \in L^{p'}$ and for $p > 1$. Here the equality occurs if $f \equiv 0$ or $g \equiv 0$. In both (2.1) and (2.2) $p' = \frac{p}{p-1}$ and the positive constant $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible one.

Now, we give a very short proof of (1.1) *and* (1.6) *but with a worse constant.*

Let $p = 2$ then by (2.1) we get

$$\begin{aligned}\left(\sum_{n=1}^{\infty} a_n\right)^2 &= \sum_{n\ge 1}\sum_{m\ge 1}\frac{(n+m)a_n a_m}{n+m}\\ &= \sum_{n\ge 1}\sum_{m\ge 1}\frac{n a_n a_m}{n+m} + \sum_{n\ge 1}\sum_{m\ge 1}\frac{m a_n a_m}{n+m}\\ &= 2\sum_{n\ge 1}\sum_{m\ge 1}\frac{(n a_n) a_m}{n+m} \le 2\pi\left(\sum_{n=1}^{\infty} n^2 a_n^2\right)^{\frac12}\left(\sum_{n=1}^{\infty} a_n^2\right)^{\frac12}.\end{aligned} \tag{2.3}$$

Similarly, for $p = 2$ and by (2.2) we obtain

$$\begin{aligned}\left(\int_0^{\infty} f(x)dx\right)^2 &= \int_0^{\infty}\int_0^{\infty}\frac{f(x)f(y)(x+y)}{x+y}dxdy\\ &\le 2\pi\left(\int_0^{\infty} x^2 f^2(x)dx\right)^{\frac12}\left(\int_0^{\infty} f^2(y)dy\right)^{\frac12}.\end{aligned} \tag{2.4}$$

As we mentioned before, we derive (1.1) and (1.6) with the worse constant $\sqrt{2\pi}$. □

3. Fritz Carlson's inequality of integral type

Kjellberg and Müller proved a multidimensional extension of Carlson's inequality of the integral type, as mentioned before. Using the same technique in the proof of (1.5) in [11], we get the recent result:

3.1. Theorem (Generalization of Carlson's inequality of integral type). *Let absolute value of the multi-index β be equal to the positive integer $\alpha \ge 1$ and $\alpha > \frac{n}{q'}$ with $q' = \frac{q}{q-1}$ where $q > 1$. Let f be a real-valued function on $\mathbb{R}^n$ such that $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ and $x^\beta f \in L^q(\mathbb{R}^n)$. Then*

$$\|f\|_{L^1(\mathbb{R}^n)} \le C_{n,q}^{(\alpha)} \|f\|_{L^q(\mathbb{R}^n)}^{1-\frac{n}{q'\alpha}} \left(\sum_{|\beta|=\alpha} \|x^\beta f\|_{L^q(\mathbb{R}^n)}\right)^{\frac{n}{q'\alpha}}. \tag{3.1}$$

The positive constant $C_{n,q}^{\alpha}$ does only depend on n, q and α.

Proof of Theorem 3.1. Let

$$S := \|f\|_{L^q(\mathbb{R}^n)}^q \quad \text{and} \quad T := \sum_{|\beta|=\alpha} \|x^\beta f\|_{L^q(\mathbb{R}^n)}^q.$$

For $t > 0$, define

$$\Theta := \sum_{|\beta|=\alpha} \left(1 + t|x^\beta|^q\right).$$

Then

$$T \leq \left(\sum_{|\beta|=\alpha} \|x^\beta f\|_{L^q(\mathbb{R}^n)} \right)^q. \tag{3.2}$$

Here and everywhere in this paper $x^\beta := \prod_{k=1}^n x_k^{\beta_k}$.

By Hölder's inequality we get

$$\begin{aligned} \|f\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |f(x)|\Theta^{\frac{1}{q}}\Theta^{-\frac{1}{q}}\, dx \\ &\leq \left(\int_{\mathbb{R}^n} |f(x)|^q \Theta(x)\, dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} \Theta^{-\frac{q'}{q}}\, dx \right)^{\frac{1}{q'}} \\ &\leq \left(c_{n,\alpha} S + tT \right)^{\frac{1}{q}} \left\{ \int_{\mathbb{R}^n} \left[\sum_{|\beta|=\alpha} \left(1 + t|x^\beta|^q\right) \right]^{\frac{-q'}{q}} dx \right\}^{\frac{1}{q'}} \\ &\leq \left(c_{n,\alpha} S + tT \right)^{\frac{1}{q}} \left[\int_{\mathbb{R}^n} \frac{dx}{(1 + tC_{n,\alpha}|x|^{q\alpha})^{\frac{q'}{q}}} \right]^{\frac{1}{q'}}. \end{aligned} \tag{3.3}$$

Because

$$\sum_{|\beta|=\alpha} \left(1 + t|x^\beta|^q\right) = c_{n,\alpha} + t \sum_{|\beta|=\alpha} |x^\beta|^q \geq 1 + tC_{n,\alpha}|x|^{q\alpha}. \tag{3.4}$$

Here $c_{n,\alpha} := \sum_{|\beta|=\alpha} 1$ and the positive constant $C_{n,\alpha}$ does only depend on n and α.

It is not hard to see that $\left[\int_{\mathbb{R}^n} \frac{dx}{(1 + |x|^{q\alpha})^{\frac{q'}{q}}} \right]^{\frac{1}{q'}}$ is finite for $\alpha > \frac{n}{q'}$ and

$$\int_0^\infty \frac{dx}{(1 + x^{\frac{q\alpha}{n}})^{\frac{q'}{q}}} = \frac{q\alpha \Gamma\left(\frac{n}{q\alpha}\right) \Gamma\left(\frac{q\alpha + n(1-q)}{q(q-1)\alpha}\right)}{n\Gamma\left(\frac{1}{q-1}\right)}. \tag{3.5}$$

Now, by (3.3) and (3.5) we obtain

$$
\begin{aligned}
\|f\|_{L^1(\mathbb{R}^n)} &\leq (tC_{n,\alpha})^{-\frac{n}{qq'\alpha}} \left(c_{n,\alpha}S + tT\right)^{\frac{1}{q}} \left[\int_{\mathbb{R}^n} \frac{dx}{(1+|x|^{q\alpha})^{\frac{q'}{q}}}\right]^{\frac{1}{q'}} \\
&= (tC_{n,\alpha})^{\frac{-n}{qq'\alpha}} \left(c_{n,\alpha}S + tT\right)^{\frac{1}{q}} \left(\int_0^\infty \int_{\{x\in\mathbb{R}^{n-1}:|x|=1\}} \frac{r^{n-1}drdx}{(1+r^{q\alpha})^{\frac{q'}{q}}}\right)^{\frac{1}{q'}} \\
&= \left(\frac{w_{n-1}}{n}\right)^{\frac{1}{q'}} (tC_{n,\alpha})^{\frac{-n}{qq'\alpha}} (c_{n,\alpha}S + tT)^{\frac{1}{q}} \left[\int_0^\infty \frac{dr}{\left(1+r^{\frac{q\alpha}{n}}\right)^{\frac{q'}{q}}}\right]^{\frac{1}{q'}} \\
&= B_{n,q}^{(\alpha)} \left(t^{\frac{-n}{q'\alpha}} c_{n,\alpha}S + t^{1-\frac{n}{q'\alpha}} T\right)^{\frac{1}{q}}.
\end{aligned}
$$

Here

$$
B_{n,q}^{(\alpha)} := \left[\frac{q\alpha w_{n-1}\Gamma\left(\frac{n}{q\alpha}\right)\Gamma\left(\frac{q\alpha+n(1-q)}{q(q-1)\alpha}\right)}{n^2\Gamma\left(\frac{1}{q-1}\right)(C_{n,\alpha})^{\frac{n}{q\alpha}}}\right]^{\frac{q-1}{q}},
$$

and w_{n-1} is the surface area of the unit sphere in $\mathbb{R}^{n-1}$.

Choose $t = \frac{S}{T}$, then by (3.2)

$$
\begin{aligned}
\|f\|_{L^1(\mathbb{R}^n)} &\leq B_{n,q}^{(\alpha)} (c_{n,\alpha}+1)^{\frac{1}{q}} S^{\frac{1-\frac{n}{q'\alpha}}{q}} T^{\frac{n}{qq'\alpha}} \\
&= C_{n,q}^{(\alpha)} \|f\|_{L^q(\mathbb{R}^n)}^{1-\frac{n}{q'\alpha}} \left(\sum_{|\beta|=\alpha} \|x^\beta f\|_{L^q(\mathbb{R}^n)}\right)^{\frac{n}{q'\alpha}}.
\end{aligned}
$$

Remarks.

1. For $q = 2$, $n = \alpha = 1$ and for a real-valued function f, we have the Beurling result (1.7) and for a positive function f, we obtain the classical Carlson inequality (1.6) of integral type.
2. For $q > 1$ and $f \in L^1(\mathbb{R}^n) \cap L^{q'}(\mathbb{R}^n)$ and $x^\beta f \in L^{q'}(\mathbb{R}^n)$ we obtain

$$
\|f\|_{L^1(\mathbb{R}^n)} \leq c_0^{-1} K_{n,q}^{(\alpha)} \|f\|_{L^{q'}(\mathbb{R}^n)}^{1-\frac{n}{q\alpha}} \left(\sum_{|\beta|=\alpha} \|x^\beta f\|_{L^{q'}(\mathbb{R}^n)}\right)^{\frac{n}{q\alpha}},
$$

if we define $\Theta := \sum_{|\beta|=\alpha}\left(1+t|x^\beta|^{q'}\right)$. The positive constant $K_{n,q}^{(\alpha)}$ is found in (1.5) and for the positive constant c_0, please see [12, pp. 5-6]. $\square$

4. A duality between the discrete and integral type of Carlson's inequality

In [7] Hardy noted that (1.1) implies (1.6). Applying this observation on the second part of (1.5) we obtain

4.1. Theorem. *Let the absolute value of the multi-index β be equal to the positive integer $\alpha \geq 1$ and $\alpha > \frac{n}{q}$ with $1 < q \leq 2$ where $q' = \frac{q}{q-1}$. Let g be a positive function on $\mathbb{R}^n$ such that $\widehat{g}$ is a smooth function (for instance $\widehat{g} \in C_0^\infty(\widehat{\mathbb{R}^n})$) and its support is localized on $\mathbb{T}^n$. Further, $\widehat{g}$ and $\widehat{x^\beta g} \in L^q(\widehat{\mathbb{R}^n})$. Then*

$$\|g\|_{L^1(\mathbb{R}^n)} \leq K_{n,q}^{(\alpha)} \|\widehat{g}\|_{L^q(\widehat{\mathbb{R}^n})}^{1-\frac{n}{q\alpha}} \left(\sum_{|\beta|=\alpha} \|\widehat{x^\beta g}\|_{L^q(\widehat{\mathbb{R}^n})} \right)^{\frac{n}{q\alpha}}. \tag{4.1}$$

Proof of Theorem 4.1. Let $\mathcal{R}$ denote the reflexion operator that is $\mathcal{R}g := \widetilde{g} = g(-x)$ then it is obvious that $\widetilde{\widetilde{g}} = g$, $\widetilde{g} = \widehat{\widehat{g}}$ and $\widetilde{\widehat{g}} = \widehat{\widetilde{g}}$. Define $f := \widehat{g}$, then $g := \mathcal{R}\widehat{f}$ such that $\widehat{f}(m) \geq 0$ for all $m \in \mathbb{Z}^n$ and $f \in L^q(\widehat{\mathbb{R}^n}) \cap C_0^\infty(\widehat{\mathbb{R}^n})$ and its support is localized on $\mathbb{T}^n$. Also, $\mathcal{R}\widehat{f} \geq 0$, $D^\beta f = D^\beta \widehat{g} = \widehat{(2\pi i x)^\beta g}$ and $f(0) = \widehat{g}(0)$. So, f satisfies the second part of (1.5). Moreover, $f(x) = \sum_{m\in\mathbb{Z}^n} \widehat{f}(m) e^{2\pi i <m,x>}$ with $<m,x> := \sum_{k=1}^n m_k x_k$. Furthermore,

$$f(0) = \|f\|_{A(\mathbb{T}^n)}, \quad \widehat{g}(0) = \|g\|_{L^1(\mathbb{R}^n)} \quad \text{and} \tag{4.2}$$

$$\|f\|_{L^q(\mathbb{T}^n)} = \|\widehat{g}\|_{L^q(\widehat{\mathbb{R}^n})}, \quad \|D^\beta f\|_{L^q(\mathbb{T}^n)} = \|\widehat{x^\beta g}\|_{L^q(\widehat{\mathbb{R}^n})}. \tag{4.3}$$

Now by the second part of (1.5) together with (4.2) and (4.3) we obtain (4.1) for smooth functions $\widehat{g}$.

Remark. For $q = 2$ and invoking Parseval's identity we get

$$\int_{\mathbb{R}^n} g(x)dx \leq K_{n,2}^{(\alpha)} \left(\int_{\mathbb{R}^n} g(x)^2 dx \right)^{\frac{1}{2}[1-\frac{n}{2\alpha}]} \left[\sum_{|\beta|=\alpha} \left(\int_{\mathbb{R}^n} \left(x^\beta g\right)^2 dx \right)^{\frac{1}{2}} \right]^{\frac{n}{2\alpha}}$$

which gives (1.6) for $n = \alpha = 1$.

4.2. Theorem. *Suppose f is a function on $\mathbb{T}^n := \{x \in \mathbb{R}^n : |x_k| \le \frac{1}{2};\ 1 \le k \le n\}$ such that $f(e^{2\pi i x}) = 0$ on $\mathbb{T}^n \setminus \overline{B_\delta}$ where $\overline{B_\delta} := \overline{B_\delta}(0) = \{x \in \mathbb{R}^n : |x| \le \delta;\ \delta := \frac{1}{2} - a\}$ for $0 < a < \frac{1}{2}$. Let g be defined on $\mathbb{R}^n$ by*

$$g(x) := \begin{cases} f(e^{2\pi i x}) & \text{if } x \in \mathbb{T}^n \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in \mathcal{A}(\mathbb{T}^n)$ if and only if $g \in \mathcal{A}(\mathbb{R}^n)$. Moreover, there are positive constants $C_1,\ C_2$ (depending on a) such that

$$C_1 \|g\|_{\mathcal{A}(\mathbb{R}^n)} \le \|f\|_{\mathcal{A}(\mathbb{T}^n)} \le C_2 \|g\|_{\mathcal{A}(\mathbb{R}^n)}.$$

Proof of Theorem 4.2. The technique is analogous to the case $n = 1$, due to Rudin (see [17, pp. 56-57], Lemma 2.7.6). Let h be a smooth function on $\mathbb{R}^n$, for instance $h \in C^\infty(\mathbb{R}^n)$, such that $\overline{B_\delta} \subseteq \operatorname{supp} h \subset \mathbb{T}^n$ and $h \equiv 1$ on $\overline{B_\delta}$ and $h \equiv 0$ outside of the torus. The Fourier transform of $D^\beta h$ is $(2\pi i x)^\beta \widehat{h}$; it follows that

$$|\widehat{h}| \le \frac{A}{1 + C_{n,\alpha}|x|^{2\alpha}} \quad x \in \widehat{\mathbb{R}^n} \tag{4.4}$$

for some positive constants A and $C_{n,\alpha}$ (see (3.4)). Because

$$\sum_{|\beta|=2\alpha} |\widehat{D^\beta h}| = |\widehat{h}| \sum_{|\beta|=2\alpha} |(2\pi i x)^\beta| \quad \text{and}$$

$$|\widehat{h}| = \frac{\sum_{|\beta|=2\alpha} |\widehat{D^\beta h}|}{\sum_{|\beta|=2\alpha} |(2\pi i x)^\beta|} \le \frac{A}{1 + C_{n,\alpha}|x|^{2\alpha}}; \quad \text{by (3.4).}$$

Here α satisfies the conditions on (1.5) and $A := \sum_{|\beta|=2\alpha} |\widehat{D^\beta h}|$ is finite . So, $h \in \mathcal{A}(\mathbb{R}^n)$ by the inversion theorem. If $f \in \mathcal{A}(\mathbb{T}^n)$ and $F(x) := f(e^{2\pi i x})$ for all $x \in \mathbb{R}^n$ then F is bounded, uniformly continuous and translation invariant (see [17, p. 15], 1.3.3). Moreover, $\|F\|_{\mathcal{B}(\mathbb{R}^n)} = \|f\|_{\mathcal{A}(\mathbb{T}^n)}$, $g = hF$ and hence $g \in \mathcal{A}(\mathbb{R}^n)$ and $\|g\|_{\mathcal{A}(\mathbb{R}^n)} \le \|h\|_{\mathcal{A}(\mathbb{R}^n)} \|f\|_{\mathcal{A}(\mathbb{T}^n)}$. If $g \in \mathcal{A}(\mathbb{R}^n)$, then $g = gh$, and so

$$\widehat{f}(m) = \int_{\mathbb{T}^n} g(x) e^{-2\pi i <x,m>}\, dx = \int_{\mathbb{R}^n} g(x) h(x) e^{-2\pi i <x,m>}\, dx \tag{4.5}$$

for $m \in \mathbb{Z}^n$. The inversion theorem holds for h by (4.4); substitution into (4.5) yields

$$\widehat{f}(m) = \int_{\widehat{\mathbb{R}^n}} \widehat{g}(x) \widehat{h}(m - x)\, dx. \tag{4.6}$$

By (4.4), there is a positive constant C_2 such that $\sum_{m\in\mathbb{Z}^n} |\widehat{h}(m-x)| < C_2$ for all $x \in \widehat{\mathbb{R}^n}$. Hence

$$\|f\|_{\mathcal{A}(\mathbb{T}^n)} = \sum_{m\in\mathbb{Z}^n} |\widehat{f}(m)| \leq C_2 \int_{\widehat{\mathbb{R}^n}} |\widehat{g}(x)|\, dx = C_2 \|g\|_{\mathcal{A}(\mathbb{R}^n)}$$

by (4.6) and the proof is complete.

As a consequence of Theorem 4.2. we obtain

4.3. Corollary. *Let absolute value of the multi-index β be equal to the positive integer $\alpha \geq 1$ and $\alpha > \frac{n}{q}$ where $1 < q \leq 2$. Then*

$$\|f\|_{L^1(\mathbb{R}^n)} \leq K_{n,q}^{(\alpha)} \|f\|_{L^q(\mathbb{R}^n)}^{1-\frac{n}{q\alpha}} \left(\sum_{|\beta|=\alpha} \|x^\beta f\|_{L^q(\mathbb{R}^n)} \right)^{\frac{n}{q\alpha}} \quad \text{and} \tag{4.7}$$

$$\|f\|_{\mathcal{A}(\mathbb{T}^n)} \leq K_{n,q}^{(\alpha)} \|f\|_{L^q(\mathbb{T}^n)}^{1-\frac{n}{q\alpha}} \left(\sum_{|\beta|=\alpha} \|D^\beta f\|_{L^q(\mathbb{T}^n)} \right)^{\frac{n}{q\alpha}} \tag{1.5}$$

are equivalent. Here $f \in \mathcal{A}(\mathbb{T}^n)$ and $\widehat{f}(0) = 0$ for the discrete case (1.5) *and the real-valued function $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ and $x^\beta f \in L^q(\mathbb{R}^n)$ for the integral version* (4.7). □

References

1. M. E. Andersson, *Local variants of the Hausdorff-Young inequality*, Analysis, algebra, and computers in mathematical research (Luleå, 1992), 25-34 (1994), Lecture Notes in Pure and Appl. Math., 156, Dekker, New York.
2. K.I. Babenko, *An inequality in the theory of Fourier integrals*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 25 (1961), 531-542.
3. S. Barza, V. Burenkov, J. Pećarić and L.E. Persson, *Sharp multidimensional multiplicative inequalities for weighted L^p-spaces with homogeneous weights*, To appear in Journal of Math. Ineq. Appl.
4. W. Beckner, *Inequalities in Fourier analysis*, Ann. of Math. (2) 102, no.1 (1975), 159-182.
5. A. Beurling, *Sur les intégrales de Fourier absolutement convergences et leur application á une transformation fonctionelle*, Proc. from the ninth Nordic congress of mathematicians, Helsinki, (1938).
6. F. Carlson, *Une inégalité*, Arkiv för matematik, astronomi och fysik **25 B, No. 1** (1934).

7. G.H. Hardy, *A note on two inequalities*, J. London Math.Soc. **11** (1936), 167-170.
8. ——— J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, London (1934).
9. F. Hausdorff, *Eine Ausdehnung des Parsevalschen Stazes über Fourierreihen*, Math. Z. 16 (1923), 163-172.
10. L. Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 256 (1983), ix+391 pp, Springer-Verlag, Berlin-New York.
11. A. Kamaly, *Fritz Carlson's inequality and its application*, To appear in Math. Scandinavica **84, No. 2** (1999).
12. ———, *A new local variant of the Hausdorff-Young inequality*, TRITA-MAT-1998-20, ISSN 1401-2278, Royal Inst. of Tech., Stockholm (Apr. 1998), Presented at the International Symposium on Complex Analysis and Related Topics, Cuernavaca, Mexico, Nov. 18-22, 1996.
13. B. Kjellberg, *Ein Momentenproblem*, Arkiv för matematik, astronomi och fysik **29 A, No. 2** (1942).
14. N.Y. Krugljak, L. Maligranda and L.E. Persson, *A Carlson type inequality with blocks and interpolation*, Studia Math. **104, No. 2** (1993), 161-180.
15. E.H. Lieb, *Gaussian kernels have only Gaussian maximizers*, Invent. Math., 102 (1990), 179-208.
16. D. Müller, *On the Spectral Synthesis Problem for Hypersurfaces of* $\mathbb{R}^n$, J. of Functional Analysis **47** (1982), 247-280.
17. W. Rudin, *Fourier analysis on groups*, Interscience Publishers (1962), 56-57.
18. P. Sjölin, *A remark on the Haussdorf-Young onequality*, Proc. Amer. Math.Soc., 123 (1995), 3085-3088.
19. W.H. Young, *On the multiplication of successions of Fourier constant*, Proc. Roy. Soc. A., 87 (1912), 331-340.
20. ———, *On the determination of summability of functions by means of its Fourier constants*, Proc. London Math. Soc. 12 (1913), 71-88.
21. A. Zygmund, *Trigonometric Series*.

Amir Kamaly
Department of Electromagnetic Theory, Royal Institute of Technology,
S-100 44 Stockholm, Sweden.

E-mail: amir@math.kth.se