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THE DEGREE THEORY OF A NEW CLASS OF OPERATORS AND ITS APPLICATION

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Abstract. This paper defines a concept of a semi-k-set-contraction operator, and establishes a degree theory for it. As its application, we discuss the existence for the solution of two-point boundary value problems for nonlinear second order integro-differential equations in Banach spaces.

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§1. INTRODUCTION

It is well known that the degree theory for the strict-set-contraction operator and the condensing operator has many applications to the existence of the solutions of some equations(see [1], [2], [3], [4]). However, some important operators are not strict-set-contraction operators or condensing operators. Now we give an example.

Let E be a Banach space, $C([0,\ 1],\ E) = \{x,\ x$ is a mapping from $[0,\ 1]$ into E and $x(t)$ is continuous at every $t \in [0,\ 1]\}$. Obviously $C([0,\ 1], E)$ is a Banach space with norm $\|x\| = \max\{\|x(t)\|,\ t \in [0,1]\}$. For $x \in C([0,\ 1],\ E)$, let

$$(Ax)(t) = \int_0^1 G_1(t,s)[x(s) + g(x(s))]ds, (*)$$

where $g \in C(E,\ E)$, $g(D)$ is relatively compact for any bounded $D \subseteq E$ and $G_1(t,\ s) = \min\{t,\ s\}$.

It is difficult to prove that A is a strict-set-contraction operator or a condensing operator from $C([0,1], E)$ into $C([0,1], E)$. So it is necessary to establish degree theory for the operators such as A defined by (*).

Now we define a new class of operators.

Let I be a bounded, closed interval of real numbers. Assume that $C^m(I, E) = \{x,\, x$ is a mapping from I into E and $x(t)$ is m-times continuously norm differentiable$(m \geq 1)$.$\}$. Obviously $C^m(I, E)$ is a Banach space with norm $\|x\|_m = \max\{\|x\|_0, \|x'\|_0, \cdots, \|x^{(m)}\|_0\}$, here $\|x\|_0 = \max\{\|x(t)\|, t \in I\}$.

Assume that A is an operator from a bounded set $S \subseteq C^m(I, E)$ into $C^m(I, E)$, and $\alpha(S)$ denotes the Kuratowski measure of noncompactness in $C^m(I, E)$.

Now we give a new definition.

Definition 1. $A : S \to C^m(I, E)(S$:bounded) is called a semi-k-set-contraction operator if A is a bounded, continuous operator, $(AS)^{(m)}$ is equicontinuous on I, and

$$\alpha(A(D)) \leq k\alpha(D)$$

for any bounded $D \subseteq S$ with equicontinuous $D^{(m)}$, where $0 \leq k < 1$ is a constant, $(AS)^{(m)} = \{y,\, y(t) = (Ax)^{(m)}(t)$ for $t \in I,\, x \in S.\}$. And $A : C^m(I, E) \to C^m(I, E)$ is called a semi-k-set-contraction operator if the restriction $A : S \to C^m(I, E)$ is a semi-k-set-contraction operator for any bounded $S \subseteq C^m(I, E)$.

It is easy to see that this definition is different from that of the k-set-contraction operator and that of the condensing operator(see[1], [5]). For example A defined by (*), $A : C(I, E) \to C(I, E)$ and for any bounded set $S \subseteq C(I, E)$, AS is bounded and equicontinuous. Moreover, by the following lemma 1, for any equicontinuous subset $D \subseteq S$, we have

$$\begin{aligned} & \alpha(AD(t)) \\ = \;& \alpha(\{\int_0^1 G_1(t,s)[x(s) + g(x(s))]ds, x \in D\}) \\ = \;& \int_0^1 G_1(t,s)[\alpha(D(s)) + \alpha(g(D(s)))]ds \\ = \;& \int_0^1 G_1(t,s)\alpha(D(s))ds \\ \leq \;& \int_0^1 G_1(t,s)ds\alpha(D) \\ < \;& \frac{3}{4}\alpha(D). \end{aligned}$$

By lemma 2, we have

$$\alpha(AD) \leq \frac{3}{4}\alpha(D).$$

So A is a semi-$\frac{3}{4}$-set-contraction operator. In section 2, we establish the degree theory for the semi-k-set-contraction operators and prove some fixed point theorems. As their application, in section 3 we discuss the existence of the solution of two-point boundary value problems for nonlinear integrodifferential equations in Banach spaces.

The following lemmas are necessary.

Lemma 1 (see[3]). If $S \subseteq C(I, E)$ is bounded and equicontinuous, then

$$\alpha(\{\int_I x(t)dt, x \in S\}) \leq \int_I \alpha(S(t))dt. \tag{1}$$

Lemma 2 (see[2]). If $S \subseteq C^m(I, E)$ is bounded and $S^{(m)}$ is equicontinuous on I, then

$$\begin{aligned} \alpha(S) \quad = \quad & \max\{\sup\{\alpha(S(t)), t \in I\}, \sup\{\alpha(S'(t)), t \in I\}, \\ & \cdots, \quad \sup\{\alpha(S^{(m)}(t)), t \in I\}\}. \end{aligned}$$

§2. ESTABLISHMENT OF THE DEGREE THEORY

Before establishing the degree theory for the class of the semi-k-set-contraction operator A, we give some lemmas. Let $\Omega \subseteq C^m(I, E)$ be open and bounded, and $A : \Omega \to C^m(I, E)$ a semi-k-set-contraction, $f = id - A$, where id denotes the *indentity* operator. Then f is called a semi-k-set-contraction field.

Lemma 3. Assume $A : \overline{\Omega} \to C^m(I, E)$ is a semi-k-set-contraction operator, then

1) f is proper, i.e., $f^{-1}(D)$ is compact for any compact set $D \subseteq C^m(I, E)$;
2) f is a closed mapping, i.e., $f(S)$ is closed for any closed set $S \subseteq \overline{\Omega}$.

Proof. 1) Let $D_1 = f^{-1}(D)(D_1 \subseteq \overline{\Omega})$, then $D_1 \subseteq A(D_1) + D$. Since $D^{(m)}$ and $A(D_1)^{(m)}$ are equicontinuous on I, $D_1^{(m)}$ is equicontinuous on I. Consequently,

$$\alpha(D_1) \leq \alpha(A(D_1)) + \alpha(D) = \alpha(AD_1) \leq k\alpha(D_1).$$

It is easy to see that $\alpha(D_1) = 0$. So D_1 is relatively compact. Consequently, D_1 is compact.

2) Let $y_n \in f(S)$, $y_n \to y_0 \in C^m(I, E)$. We will prove $y_0 \in f(S)$. Suppose that $y_n = f(x_n)$, $x_n \in S$. Let $S_0 = \{y_0, y_1, y_2, \cdots.\}$. Obviously $S_0 \subseteq C^m(I, E)$ is compact. By the proof of 1), $f^{-1}(S_0) \subseteq C^m(I, E)$ is compact. So there exists a subsequence $\{x_{n_i}\}$, $x_{n_i} \to x_0 \in C^m(I, E)$. Since S is closed, $x_0 \in S$. By the continuity of f, $y_{n_i} = f(x_{n_i}) \to f(x_0)$. Consequently, $y_0 = f(x_0)$. So $f(S)$ is closed. The proof is complete.□

Lemma 4. If $D \subseteq C(I, E)$ is bounded and equicontinuous on I , then $\overline{co}(D)$ is bounded and equicontinuous on I.

The proof of lemma 4 is routine and may be omitted.

Lemma 5. Let $\{S_i\} \subseteq E$ be bounded, closed and $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots \supseteq S_n \supseteq \cdots$, $S_n \neq \emptyset, n = 1, 2, 3, \cdots$. If $\alpha(S_n) \to 0$, then $S = \bigcap_{i=1}^{\infty} S_i$ is a nonempty compact set.

This Lemma is the exercise 4, page 53, in [1]. In what follows, we give the definition of the degree for a semi-k-set-contraction field.

Definition 2. Let $\Omega \subseteq C^m(I, E)$ open and bounded, $A : \overline{\Omega} \to C^m(I, E)$ be a semi-k-contraction operator , $0 \leq k < 1$, $f = id - A$.

(1) Assume that $\theta \notin f(\partial\Omega)$. Let $D_1 = \overline{co}(A(\overline{\Omega}))$ and $D_n = \overline{co}(A(D_{n-1} \cap \overline{\Omega}))$, $n = 2, 3, \cdots$.

1) If there exists an n_0 such that $D_{n_0} = \emptyset$, then we define that deg$(f, \Omega, \theta) = 0$.

2) Now we suppose that $D_n \neq \emptyset$, $n = 1, 2, \cdots$. So $D_n \cap \overline{\Omega}$ is bounded and closed($n = 1, 2, \cdots$.). Let $D = \bigcap_{i=1}^{\infty} D_n$. Then D is bounded, convex, closed and nonempty as we show below. Obviously $D_1 \supseteq D_2$. If $D_{n-1} \supseteq D_n$, then $D_n = \overline{co}(A(D_{n-1} \cap \overline{\Omega})) \supseteq \overline{co}(A(D_n \cap \overline{\Omega})) = D_{n+1}$. So $D_{n-1} \supseteq D_n$, $n = 2, 3, \cdots$. By lemma 4, $(D_n)^{(m)}$ is equicontinuous on I and

$$\alpha(D_n) = \alpha(A(D_{n-1} \cap \overline{\Omega})) \leq k\alpha(D_{n-1} \cap \overline{\Omega}) \leq k\alpha(D_{n-1}).$$

So $\alpha(D_n) \leq k^{n-1}\alpha(D_1)$. By $k < 1$ and lemma 5, we know D is a nonempty compact set. Because of $D_{n-1} \cap \overline{\Omega} \supseteq D_n \cap \overline{\Omega}$, $D_n \bigcap \overline{\Omega} \neq \emptyset$ and $\alpha(D_n \cap \overline{\Omega}) \to 0$, we know $D \cap \overline{\Omega} = (\bigcap_{n=1}^{\infty} D_n) \cap \overline{\Omega}$ is nonempty and compact. On the other hand, from

$$A(D_n \cap \overline{\Omega}) \subseteq \overline{co}(A(D_{n-1} \cap \overline{\Omega})) = D_n$$

we have

$$A(D \cap \overline{\Omega}) \subseteq \bigcap_{n=1}^{\infty} A(D_n \cap \overline{\Omega}) \subseteq \bigcap_{n=1}^{\infty} D_n = D. \tag{2}$$

Since D is compact, $A : D \cap \overline{\Omega} \to D$ is completely continuous. So by the extention theorem of completely continuous operator(see[1], page 44), there exists a completely continuous operator $A_1 : \overline{\Omega} \to D$ such that $A_1x = Ax$ for every $x \in D \cap \overline{\Omega}$. Let $f_1 = id - A_1$. It is easy to see that $\theta \notin f_1(\partial\Omega)$. So the Leray-Schauder degree $\deg_{\text{LS}}(f_1, \Omega, \theta)$ can be defined. Let

$$\deg(f, \Omega, \theta) = \deg_{\text{LS}}(f_1, \Omega, \theta), \tag{3}$$

where $\deg_{\text{LS}}(f_1, \Omega, \theta)$ denotes the degree of completely continuous operator field $f_1 = id - A_1$. It is easy to find what we defined is independent of the choice of f_1. In fact, let $A_2 : \overline{\Omega} \to D$ be another extension of A, and $f_2 = id - A_2$. Let $H(t,\, x) = x - tA_1x - (1-t)A_2x$, $x \in \overline{\Omega}$, $0 \le t \le 1$. We will prove $H(t,\, x) \ne \theta$ for $t \in [0,\, 1]$ and $x \in \partial\Omega$. On the contrary, if there exist t_0, $0 \le t_0 \le 1$, and $x_0 \in \partial\Omega$ such that $H(t_0,\, x_0) = \theta$, i.e., $x_0 = t_0A_1x_0 + (1-t_0)A_2x_0$. Since $A_1x_0 \in D$, $A_2x_0 \in D$ and D is convex, we know $x_0 \in D$. So $x_0 = t_0A_1x_0 + (1-t_0)A_2x_0 = Ax_0$. This contradicts to $\theta \notin f(\partial\Omega)$. Hence

$$\deg_{\mathrm{LS}}(f_1, \Omega, \theta) = \deg_{\mathrm{LS}}(f_2, \Omega, \theta). \tag{4}$$

(2) Suppose $p \notin f(\partial\Omega)$. It is easy to see $\theta \notin (f-p)(\partial\Omega)$ and set

$$\deg(f, \Omega, p) = \deg(f-p, \Omega, \theta). \tag{5}$$

Now we have successfully defined the degree $\deg(f,\, \Omega,\, p)$ for a semi-k-set-contraction operator A.

Remark 1: If A has a fixed point $x' \in \overline{\Omega}$, we have $x' \in D_n \cap \Omega \ne \emptyset$, $n = 1,\, 2,\, \cdots$. So the fixed point set F is also non-void with $F \subseteq D \cap \overline{\Omega}$.
Remark 2: We can notice the method of establishing $\{D_n\}_n$ in definition 2 is same as that of $\{Q_n\}_n$ appearing on page 107 in [5].

Lemma 6. Assume that A is a semi-k-set-contration operator as in definition 2, $f = id - A$, $\theta \notin f(\partial\Omega)$, and 2) of Definition 2 is satisfied. If $B : \overline{\Omega} \to S$ is continuous with $Bx = Ax$ for all $x \in S \cap \overline{\Omega}$, where $S \supseteq D$(D is the same as in the definition 1) is compact and convex with $A(S \cap \overline{\Omega}) \subseteq S$. Let $g = id - B$, then

$$\deg(f, \Omega, \theta) = \deg_{\mathrm{LS}}(g, \Omega, \theta). \tag{6}$$

Proof. Assume that A_1 and f_1 are such as those of 2) in definition 2. Let

$$H(t,x) = x - tA_1x - (1-t)Bx$$

for $x \in \Omega$ and $0 \le t \le 1$. Then we have $H(t,x) \ne \theta$ for $x \in \partial\Omega$ and $0 \le t \le 1$. In fact, suppose that $H(t_0, x_0) = \theta$ for $x_0 \in \partial\Omega$, $0 \le t_0 \le 1$. Since $S \supseteq D$ is convex, $x_0 = t_0A_1x_0 + (1-t_0)Bx_0 \in S$. So $Bx_0 = Ax_0$, $x_0 = t_0A_1x_0 + (1-t_0)Ax_0$. From $Ax_0 \in D_1$ and $A_1x_0 \in D \subseteq D_1$, we have $x_0 = t_0A_1x_0 + (1-t_0)Ax_0 \in D_1$. So $Ax_0 \in D_2$, $A_1x_0 \in D \subseteq D_2$. Consequently $x_0 = t_0A_1x_0 + (1-t_0)Ax_0 \in D_2$. Proceeding as before, we have $x_0 \in D_n (n = 1,\, 2,\, 3,\, \cdots)$. Therefore $x_0 \in D$. So we have $A_1x_0 = Ax_0$, $x_0 = t_0Ax_0 + (1-t_0)Ax_0 = Ax_0$. This contradicts $\theta \notin f(\partial\Omega)$. So

$$\deg_{\mathrm{LS}}(g, \Omega, \theta) = \deg_{\mathrm{LS}}(f_1, \Omega, \theta).$$

Thus the proof is complete. □

Theorem 3. The degree of a semi-k-set-contraction field defined in Definition 2 has the following properties:

1) $\deg(id,\, \Omega,\, p) = 1$ for $p \in \Omega$;

2) $\deg(f,\, \Omega,\, p) = \deg(f,\, \Omega_1,\, p) + \deg(f,\, \Omega_2,\, p)$ whenever $\Omega_1,\, \Omega_2 \subseteq \Omega$ are open with $\Omega_1 \cap \Omega_2 = \emptyset$, $p \notin f(\overline{\Omega} \backslash (\Omega_1 \cup \Omega_2))$;

3) $\deg(id - H(t,\, \cdot),\, \Omega,\, p) = const$ for all $t \in [0,\, 1]$ whenever $H(t,\, \cdot)$ is a semi-k-set-contraction operator for all $t \in [0,\, 1]$ and as $t \to t_0$ for any t_0, $H(t,\, x)$ converges to $H(t_0,\, x)$ in $C^m(I,\, E)$ uniformly in $x \in \overline{\Omega}$, where $p \notin h_t(\partial\Omega)$, $h_t = id - H(t,\, \cdot)$;

4) if $\deg(f,\, \Omega, p) \neq 0$, then the equation $f(x) = p$ has a solution in Ω.

Moreover, set $g = id - G$, where $G : \Omega \to C^m(I,\, E)$ is a semi-k-set-contraction operator. Then

5) $\deg(f,\, \Omega,\, p) = \deg(g,\, \Omega,\, p)$ whenever $G|_{\partial\Omega} = A|_{\partial\Omega}$;

6) $\deg(f,\, \Omega,\, p) = \deg(f,\, \Omega_1,\, p)$ for every open subset Ω_1 of Ω such that $p \notin f(\overline{\Omega} - \Omega_1)$;

7) $\deg(f,\, \Omega,\, \cdot)$ is constant on every connected subset of $C^m(I, E) - f(\partial\Omega)$.

Proof. We might well suppose that $p = \theta$. Since 1) is same as the normality of strict-set-contraction field in [5], we can omit the proof. First we prove 2). We discuss three possibilities.

1' Suppose 2) of (1) in definition 2 for Ω_1 and Ω_2 is true. Obviously 2) of (1) in definition 2 for Ω is true. Now we get

$$\begin{cases} D^{(1)} \cap \overline{\Omega}_1 \neq \emptyset, \quad D^{(2)} \cap \overline{\Omega}_2 \neq \emptyset, \quad D \cap \overline{\Omega} \neq \emptyset, \quad D^{(1)} \subseteq D, \quad D^{(2)} \subseteq D, \\ A(D^{(1)} \cap \overline{\Omega}_1) \subseteq D^{(1)}, \quad A(D^{(2)} \cap \overline{\Omega}_2) \subseteq D^{(2)}, \quad A(D \cap \overline{\Omega}) \subseteq D, \end{cases}$$

where $D^{(1)}$ and $D^{(2)}$ are obtained as D in 2) of (1) in definition 2 for $A|_{\overline{\Omega}_1}$ and $A|_{\overline{\Omega}_2}$ respectivelyt.. And D is the same as in 2) of (1) in definition 2 . Let $A_1 : \overline{\Omega} \to D$ is the completely continuous operator as in 2) of (1) in definition 2, and $f_1 = id - A_1$. According to (3), we get

$$\deg(f, \Omega, \theta) = \deg_{\mathrm{LS}}(f_1, \Omega, \theta).$$

By virtue of lemma 6, we have

$$\begin{cases} \deg(f, \Omega_1, \theta) = \deg_{\mathrm{LS}}(f_1, \Omega_1, \theta), \\ \deg(f, \Omega_2, \theta) = \deg_{\mathrm{LS}}(f_1, \Omega_2, \theta). \end{cases} \tag{$**$}$$

By virtue of the degree theory of Leray-Schauder, we get

$$\deg_{\mathrm{LS}}(f_1, \Omega, \theta) = \deg_{\mathrm{LS}}(f_1, \Omega_1, \theta) + \deg_{\mathrm{LS}}(f_1, \Omega_2, \theta).$$

According to above conclusion, we get

$$\deg(f, \Omega, \theta) = \deg(f, \Omega_1, \theta) + \deg(f, \Omega_2, \theta).$$

2' Suppose that one of Ω_1 and Ω_2 satisfies 2) of (1) in definition 2(for example, Ω_1), one of Ω_1 and Ω_2 satifies 1) of (1) in definition 2(for example, Ω_2). Obviously Ω satisfies (1)2) in definition 2. Therefore

$$\deg(f, \Omega_2, \theta) = 0.$$

By virtue of lemma 6, we have

$$\deg(f, \Omega_1, \theta) = \deg_{\mathrm{LS}}(f_1, \Omega_1, \theta),$$

where f_1 is as in 1'. Now we will prove

$$\deg_{\mathrm{LS}}(f_1, \Omega_2, \theta) = 0.$$

In fact, if $\deg_{\mathrm{LS}}(f_1, \Omega_2, \theta) \neq 0$, then there exsits an $x_0 \in \Omega$ such that $f_1(x_0) = 0$, i.e. $x_0 = A_1 x_0 \in D$. So $A_1 x_0 = A x_0$, $x_0 = A x_0$. By the Remark 1, Ω_2 satifies 2) of (1) in definition 2. This is a contradiction. Now by (**), we have

$$\deg(f, \Omega, \theta) = \deg(f, \Omega_1, \theta) + \deg(f, \Omega_2, \theta).$$

3' Suppose Ω_1 and Ω_2 satisfy 1) of (1) in definition 2. Now we have

$$\deg(f, \Omega_1, \theta) = 0, \quad \deg(f, \Omega_2, \theta) = 0.$$

By the Remark 1, $\theta \notin f(\Omega_1 \cup \Omega_2)$. Hence, $\theta \notin f(\overline{\Omega})$. Then we have

$$\deg(f, \Omega, \theta) = 0.$$

So

$$\deg(f, \Omega, \theta) = \deg(f_1, \Omega_1, \theta) + \deg(f_1, \Omega_2, \theta).$$

And Since the proof of (2) includs that of (4), we can omit the proof of (4).

Next we prove 3). First we need to prove $H([0,1] \times \overline{\Omega})$ is bounded. In fact, assume that there exists a sequence $\{t_n\} \subseteq [0,1]$ and a $\{x_n\} \subseteq \overline{\Omega}$ such that

$$\|H(t_n, x_n)\|_m \to \infty, \quad n \to \infty. \tag{8}$$

We might as well suppose that $t_n \to t_0$. We have

$$\|H(t_n, x_n)\|_m \leq \|H(t_n, x_n) - H(t_0, x_n)\|_m + \|H(t_0, x_n)\|_m. \tag{9}$$

Since $H(t_0, \cdot)$ is a semi-k-set-contraction operator, $\|H(t_0, x_n)\|_m$ is bounded. And because $\|H(t_n, x) - H(t_0, x)\| \to 0 (n \to +\infty)$ uniformly in $x \in \overline{\Omega}$, $\|H(t_n, x_n) - H(t_0, x_n)\|_m$ is bounded. So $\|H(t_n, x_n)\|$ is bounded. This contradicts (9). Consequently, $H([0,1] \times \overline{\Omega})$ is bounded. Let $D_1^* = \overline{co}(H([0,1] \times \overline{\Omega}))$, and $D_n^* = \overline{co}(H([0,1] \times (\overline{\Omega} \cap D_{n-1}^*)))$, $n = 2, 3, \cdots$. Obviously $D_1^* \supseteq D_2^*$. If $D_{n-1}^* \supseteq$

D_n^*, then $D_n^* = \overline{co}(H([0,1] \times (D_{n-1}^* \cap \overline{\Omega})) \supseteq \overline{co}(H([0,1] \times (D_n^* \cap \overline{\Omega}))) = D_{n+1}^*$. So $D_{n-1}^* \supseteq D_n^*$, $n = 2, 3, \cdots$. We need to prove $D_n^{*(m)}$ is equicontinuous on I. First we will prove that $D_1^{*(m)}$ is equicontinuous. From lemma 4, we have only to prove that $H([0,1] \times \overline{\Omega})^{(m)}$ is equicontinuous. Assume that $H([0,1] \times \overline{\Omega})^{(m)}$ is not equicontinuous. Then there exists an $\varepsilon > 0$, a subsequence $\{x_n\} \subseteq H([0,1] \times \overline{\Omega})$ with $x_n = H(t_n, y_n)$, and $|t_{1,n} - t_{2,n}| < \frac{1}{n}$ such that

$$\|x_n^{(m)}(t_{1,n}) - x_n^{(m)}(t_{2,n})\| \geq \varepsilon. \tag{10}$$

We might as well suppose $t_n \to t_0$, then we have

$$\begin{aligned}
& \|H(t_n, y_n)^{(m)}(t_{1,n})) - H(t_n, y_n)^{(m)}(t_{2,n}))\| \\
\leq\ & \|H(t_n, y_n)^{(m)}(t_{1,n})) - H(t_0, y_n)^{(m)}(t_{1,n}))\| \\
& + \|H(t_0, y_n)^{(m)}(t_{2,n})) - H(t_n, y_n)^{(m)}(t_{2,n}))\| \\
& + \|H(t_0, y_n)^{(m)}(t_{1,n})) - H(t_0, y_n)^{(m)}(t_{2,n}))\| \\
=\ & I_{1,n} + I_{2,n} + I_{3,n}.
\end{aligned}$$

And because $\|H(t_n,\ x) - H(t_0,\ x)\| \to 0 (n \to +\infty)$ uniformly in $x \in \overline{\Omega}$, we have $I_{1,n} + I_{2,n} \to 0, n \to +\infty$. Since $H(t_0, \cdot)$ is a semi-k-set-contraction operator, we have $I_{3,n} \to 0, n \to +\infty$. Then $I_{1,n} + I_{2,n} + I_{3,n} \to 0, n \to +\infty$. This contradicts (10). By lemma 4, $D_1^{*(m)}$ is equicontinuous on I. By the monotonity of $\{D_n^{*(m)}\}$, $D_n^{*(m)}$ is equicontinuous.

For given $t \in [0,1]$, let $D_1(t) = \overline{co}(H(t, \overline{\Omega}))$,

$$D_n(t) = \overline{co}(H(t, D_{n-1}(t) \cap \overline{\Omega})), \quad n = 2, 3, \cdots. \tag{11}$$

Obviously $D_n(t) \subseteq D_{n-1}(t), n = 2, 3, \cdots$. If there exists an n_0 with $D_{n_0}^* \cap \overline{\Omega} = \emptyset$ for every, then $D_{n_0}(t) \cap \overline{\Omega} = \emptyset, t \in [0,1]$. Then we have

$$\deg(h_t, \Omega, \theta) \equiv 0, \quad t \in [0,1].$$

Now suppose $D_n^* \cap \overline{\Omega} \neq \emptyset (n = 1, 2, \cdots)$.

Take any $\varepsilon > 0$ and $t_0 \in [0,1]$. Then for each $n \geq 2$ there exist a finite covering $\{S_i\}_{i=1}^r$ such that $H(t_0, D_{n-1}^* \cap \overline{\Omega}) \subseteq \bigcup_{i=1}^{r} S_i$ with $d(S_i) \leq k\alpha(D_{n-1}^*) + \varepsilon$, $i = 1, 2, \cdots, r$ since $\alpha(H(t_0, D_{n-1}^* \cap \overline{\Omega})) \leq k\alpha(D_{n-1}^* \cap \overline{\Omega}) \leq k\alpha(D_{n-1}^*)$. On the other hand, from the assumption, there is a $\delta > 0$ such that $\|H(t,x) - H(t_0,x)\| < \varepsilon$ for all $x \in \overline{\Omega}$ when $|t - t_0| < \delta$. Let $S_i^\varepsilon = \{x, d(x, S_i) < \varepsilon\}, I(t_0, \delta) = (t_0 - \delta, t_0 + \delta) \cap [0,1]$. So $H(I(t_0,\delta) \times (D_{n-1}^* \cap \overline{\Omega})) \subseteq \bigcup_{i=1}^{r} S_i^\varepsilon$,

$$d(S_i^\varepsilon) \leq d(S_i) + 2\varepsilon \leq k\alpha(D_{n-1}^*) + 3\varepsilon.$$

We have $\alpha(H(I(t_0,\delta)\times(D^*_{n-1}\cap\overline{\Omega}))\leq k\alpha(D^*_{n-1})+3\varepsilon$. By the compactness of the interval $[0,1]$, there exist $t_i\in[0,1]$, $\delta_i>0$, $i=1,2,\cdots,s$ such that $[0,1]=\bigcup\limits_{i=1}^{s}I(t_i,\delta_i)$, and

$$\alpha(H(I(t_i,\delta_i)\times(D^*_{n-1}\cap\overline{\Omega})))\leq k\alpha(D^*_{n-1})+3\varepsilon,\quad i=1,2,\cdots,s.$$

So

$$\begin{aligned}\alpha(D^*_n) &= \alpha((H([0,1]\times(D^*_{n-1}\cap\overline{\Omega})))\\ &= \alpha(\bigcup_{i=1}^{s}H(I(t_i,\delta_i)\times(D^*_{n-1}\cap\overline{\Omega})))\\ &= \max\{\alpha(H(I(t_i,\delta_i)\times(D^*_{n-1}\cap\overline{\Omega}))),i=1,2,\cdots,s.\}\\ &\leq k\alpha(D^*_{n-1})+3\varepsilon.\end{aligned}$$

By the arbitrariness of ε, we have $\alpha(D^*_n)\leq k\alpha(D^*_{n-1}),n=2,3,\cdots$. Consequently, $\alpha(D^*_n)\leq k^{n-1}\alpha(D^*_1)$. This implies $\alpha(D^*_n)\to 0$. By lemma 5, $D^*=\bigcap\limits_{n=1}^{\infty}D^*_n$ is nonempty, convex and compact(recall that we are now assuming $D^*_n\cap\overline{\Omega}\neq\emptyset$ for $n=1,2,\cdots$). By the same proof, $D^*\cap\overline{\Omega}$ also is shown to be nonempty and compact. Since $H([0,1]\times(D^*_n\cap\overline{\Omega}))\subseteq\overline{co}(H([0,1]\times(D^*_n\cap\overline{\Omega})))=D^*_{n+1}\subseteq D^*_n$. So $H([0,1]\times(D^*\cap\overline{\Omega}))\subseteq\bigcap\limits_{n=1}^{\infty}H([0,1]\times(D^*_n\cap\overline{\Omega}))$ $\subseteq\bigcap\limits_{n=1}^{\infty}D^*_n=D^*$.

By the extention theorem of completely continuous function, there exists a $G:[0,1]\times\overline{\Omega}\to D^*$ such that $G(t,x)=H(t,x)$ when $(t,x)\in[0,1]\times(D^*\cap\overline{\Omega})$. Let $g_t=x-G(t,x)$. We will prove $\deg(h_t,\Omega,\theta)=\deg_{\mathrm{LS}}(g_t,\Omega,\theta)$. It is easy to see that $\theta\notin g_t(\partial\Omega)$. In fact, if there exist t_0 with $0\leq t_0\leq 1$, and $x_0\in\partial\Omega$ such that $g_{t_0}(x_0)=0$. Then $x_0=G(t_0,x_0)\in D^*$. So $G(t_0,x_0)=H(t_0,x_0)$, $x_0=H(t_0,x_0)$. This contradicts $\theta\notin h_t(\partial\Omega)$. So $\theta\notin g_t(\partial\Omega)$.

(a) If the condition 1) of definition 2 is satisfied for h_t, we have $\deg(h_t,\Omega,\theta)=0$. In this case, since $H(t,x)$ has not fixed points in $\overline{\Omega}$, $G(t,x)$ also has not fixed points in $\overline{\Omega}$. By the theory of Leray-Schauder degree, we have

$$\deg_{\mathrm{LS}}(g_t,\Omega,\theta)=0.$$

(b) If h_t satisfies the condition 2) in definition 2, by lemma 6, we have

$$\deg(h_t,\Omega,\theta)=\deg_{\mathrm{LS}}(g_t,\Omega,\theta).$$

Therefore we have

$$\deg_{\mathrm{LS}}(g_t,\Omega,\theta)=const,\quad 0\leq t\leq 1.$$

Hence

$$\deg(h_t,\Omega,\theta)=const,\quad 0\leq t\leq 1.$$

If $p \neq \theta$, let $\overline{h}_t = id - H(t,\cdot) - p$. Then by the result proved above, we have

$$\deg(\overline{h}_t, \Omega, \theta) = const.$$

Finally since the proofs of 5), 6), 7) are similar to the proofs of the relative properties of degree throry of strict-set-contraction field in [1], we omit the proofs. Thus proof is complete. □

Theorem 4. Let Ω be a bounded, convex open set in $C^m(I,E), A : \overline{\Omega} \to C^m(I,E)$ be a semi-k-set-contraction operator, $0 \leq k < 1$, $A(\partial\Omega) \subseteq \overline{\Omega}$ without fixed point in $\partial\Omega$, then $\deg(id - A, \Omega, \theta) = 1$.

Proof. Choose an $x_0 \in \Omega$ arbitrarily. Let $h_t = t(x - Ax) + (1-t)(x - x_0) = x - H(t,x)$, here $H(t,x) = tAx + (1-t)x_0$. Obviously $\|H(t_n,x) - H(t_0,x)\| \to 0 (n \to +\infty)$ uniformly in $x \in \overline{\Omega}$. And $H(t,\cdot)$ is a semi-k-set-contraction operator for all $t \in [0,1]$. In virtue of the fact: let A be a convex set in a topological vector space E with a interior point x_0, then for any $x_1 \in \overline{A}$, the open segment with end points x_0 and x_1 is contained in $\overset{\circ}{A}$(cf. N.Bourbaki, "Espace Vectoriels Topologiques", Prop.16 in Chap.2, §2, $n°6$), it is easy to see that $\theta \notin h_t(\partial\Omega)$, $0 \leq t \leq 1$. By Theorem 3, $\deg(id - A, \Omega, \theta) = \deg(id, \Omega, \theta) = 1$. The proof is complete. □

§3. EXISTENCE OF THE SOLUTION FOR TWO-POINT BOUNDARY VALUE PROBLEMS IN BANACH SPACES

Now we consider the following boundary value problem

$$\begin{cases} -x''(t) = f(t, x(t), x'(t), (Tx)(t), (Sx)(t)), 0 \leq t \leq 1; \\ ax(0) - bx'(0) = x_0, \\ cx(1) + dx'(1) = x_1, \end{cases} \tag{12}$$

where

$$(Tx)(t) = \int_0^t k(t,s)x(s)ds, \quad (Sx)(t) = \int_0^1 h(t,s)x(s)ds. \tag{13}$$

Here $k \in C(D,\ R^+)$, $D = \{(t,s) \in R^2\colon\ 0 \leq s \leq t \leq 1\}$ and $h \in C(D_0, R^+)$, $D_0 = \{(t,\ s) \in R^2\colon\ 0 \leq t,\ s \leq 1\}$. E is Banach space. And assume $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ and $J = ac + ad + bc > 0$ throughout this section.

In order to investigate BVP (12), we first consider the integral operator

$$(Ax)(t) = \int_0^1 G(t,s) f(s, x(s), x'(s), (Tx)(s), (Sx)(s))ds + y(t), \tag{14}$$

where $f \in C(I \times E \times E \times E \times E,\ P)$, $y \in C^2(I,E)$ and $y(t) \geq \theta$ for $t \in I$ and $P \subseteq E$ is a normal solid cone of E with normal constant $N \geq 1$(i.e. if we define the relation $x \leq y$ by $y - x \in P$, then '$\leq$' is an order relation in E. Moreover, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$). We denote the relation $y - x \in \overset{\circ}{P}$ by $x \ll y$).

Let

$$G(t,s) = \begin{cases} J^{-1}(at+b)(c(1-s)+d), & t \leq s; \\ J^{-1}(as+b)(c(1-t)+d), & t > s, \end{cases} \tag{15}$$

here $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ and $J = ac + ad + bc > 0$. Moreover, T and S are defined by (13). In the following, let $B_R = \{x \in E : \|x\| \leq R\}$ $(R > 0)$ and

$$k_0 = \max\{\int_0^t k(t,s)ds, t \in I\}, \quad h_0 = \max\{\int_0^1 h(t,s)ds, t \in I\}. \tag{16}$$

Furthermore, let $P(I) = \{x \in C^1(I, E) : x(t) \geq \theta \text{ for } t \in I\}$. Then $P(I)$ is a cone in $C^1(I, E)$. Usually, $P(I)$ is not normal in $C^1(I, E)$ even if P is a normal cone in E. Let

$$q_1 = \sup_{t\in[0,1]} \int_0^1 G(t,s)ds, \quad q_2 = \sup_{t\in[0,1]} \int_0^1 |G'_t(t,s)ds,$$

and

$$q = \max\{q_1, q_2\} \tag{17}$$

Then we have the following lemma 7.

Lemma 7. Let f be uniformly continuous on $I \times B_R \times B_R \times B_R \times B_R$ for any $R > 0$. Suppose that there exist constants $L_i \geq 0 (i = 1, 2, 3, 4)$ such that

$$\alpha(f(t, X, Y, Z, W)) \leq L_1\alpha(X) + L_2\alpha(Y) + L_3\alpha(Z) + L_4\alpha(W) \tag{18}$$

for any bounded $X, Y, Z, W \subseteq E$, $t \in I$ and

$$\overline{k} = q(L_1 + L_2 + k_0L_3 + h_0L_4) < 1. \tag{19}$$

Then the operator A defined by (14) is a semi-$\overline{k}$-set-contraction operator from $C^1(I, E)$ into $P(I)$.

Proof. By direct differention of (14), we have for $x \in C^1(I, E)$,

$$(Ax(t))' = \int_0^1 G'_t(t,s)f(s, x(s), x'(s), (Tx)(s), (Sx)(s))ds + y'(t), \tag{20}$$

where

$$G'_t(t,s) = \begin{cases} J^{-1}a(c(1-s)+d), & t < s; \\ J^{-1}(-c)(as+b), & t > s, \end{cases} \tag{21}$$

and

$$((Ax)(t))'' = -f(t, x(t), x'(t), (Tx)(t), (Sx)(t)) + y''(t). \tag{22}$$

It is easy to see that the uniform continuity of f on $I \times B_R \times B_R \times B_R \times B_R$ implies the boundedness of f on $I \times B_R \times B_R \times B_R \times B_R$. So A is bounded and continuous from $C^1(I, E)$ into $P(I)$. Now, let $Q \subseteq C^1(I, E)$ be bounded. By virtue of (22), $\{\|(Ax(t))''\| : x \in Q,\ t \in I\}$ is a bounded set of E. So $(A(Q))'$ is equicontinuous, and hence lemma 2 implies that

$$\alpha(A(Q)) = \max\{\sup\{\alpha(AQ(t)), t \in I\}, \sup\{\alpha((AQ)'(t)), t \in I\}\}. \tag{23}$$

On the other hand, it is easy to see that for any bounded $Q \subseteq C^1(I, E)$ with equicontinuous Q', $\{f(s,\ x(s),\ x'(s),\ (Tx)(s),\ (Sx)(s)),\ x \in Q\}$ is equicontinuous because of the uniform continuity of f. By lemma 1, lamma 2 and (18) we have

$$\begin{aligned}
& \alpha(AQ(t)) \\
= \ & \alpha\left(\{\int_0^1 G(t,s) f(s, x(s), x'(s), (Tx)(s), (Sx)(s)) ds + y(t), x \in Q\}\right) \\
\leq \ & \int_0^1 G(t,s) \alpha\left(\{f(s, x(s), x'(s), (Tx)(s), (Sx)(s)), x \in Q\}\right) ds \\
\leq \ & \int_0^1 G(t,s) \left[L_1 \alpha(Q(s)) + L_2 \alpha(Q'(s)) + L_3 \alpha((TQ)(s)) + L_4 \alpha((SQ)(s))\right] ds \\
\leq \ & \int_0^1 G(t,s) [L_1 \alpha(Q(s)) + L_2 \alpha(Q'(s)) + L_3 \int_0^s k(s,r) \alpha(Q(r)) dr \\
+ \ & L_4 \int_0^1 h(s,r) \alpha(Q(r)) dr] ds \\
\leq \ & \int_0^1 G(t,s) ds \left[L_1 + L_2 + L_3 k_0 + L_4 h_0\right] \alpha(Q) \\
\leq \ & q_1 [L_1 + L_2 + L_3 k_0 + L_4 h_0] \alpha(Q) \\
\leq \ & q [L_1 + L_2 + L_3 k_0 + L_4 h_0] \alpha(Q).
\end{aligned} \tag{24}$$

Similarly, we have

$$\begin{aligned}
& \alpha((AQ)'(t)) \\
= \ & \alpha\left(\{\int_0^1 G_t'(t,s) f(s, x(s), x'(s), (Tx)(s), (Sx)(s)) ds + y'(t), x \in Q\}\right) \\
\leq \ & \int_0^1 |G_t'(t,s)| \alpha\left(\{f(s, x(s), x'(s), (Tx)(s), (Sx)(s)), x \in Q\}\right) ds \\
\leq \ & \int_0^1 |G_t'(t,s)| \left[L_1 \alpha(Q(s)) + L_2 \alpha(Q'(s)) + L_3 k_0 \alpha(Q) + L_4 h_0 \alpha(Q)\right] ds \\
\leq \ & \int_0^1 |G_t'(t,s)| ds [L_1 + L_2 + L_3 k_0 + L_4 h_0] \alpha(Q)
\end{aligned}$$

$$\begin{aligned} &\leq q_2[L_1+L_2+L_3k_0+L_4h_0]\alpha(Q) \\ &\leq q[L_1+L_2+L_3k_0+L_4h_0]\alpha(Q). \end{aligned} \tag{25}$$

From (23), we have

$$\begin{aligned} \alpha(A(Q)) &= \max\{\sup\{\alpha(A(Q(t)), t\in I\}, \quad \sup\{\alpha((AQ)'(t)), t\in I\}\} \\ &\leq \overline{k}\alpha(Q) \end{aligned} \tag{26}$$

So A is a semi-$\overline{k}$-set-contraction operator. The proof is complete.□

Let us list some conditions for convenience:

(H_1) $x_0\geq\theta, x_1\geq\theta$, $f\in C(I\times E\times E\times E\times E, P)$ is uniformly continuous on $I\times B_R\times B_R\times B_R\times B_R$ for any $R>0$ and there exists $L_i\geq 0 (i=1,2,3,4)$ such that (18) and (19) hold;

(H_2) $\overline{\lim}_{R\to+\infty}\frac{M(R)}{R}<\frac{1}{qm}$, where $M(R)=\sup\{\|f(t,x,y,z,w)\|:(t,x,y,z,w)\in I\times B_R\times B_R\times B_R\times B_R\}, m=\max\{1,k_0,h_0\}$ and q is defined by (17);

Theorem 5. Let $(H_1),(H_2),(H_3)$ be satisfied. Then BVP (12) has at least one nonnegative solution in $C^2(I,E)$.

Proof. It is well known that the $C^2(I,E)$ solution of (12) is equivalent to $C^1(I,E)$ solution of the following integral equation

$$x(t)=\int_0^1 G(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))ds+y(t),$$

where $G(t,s)$ is the Green function given by (15) and $y(t)$ denotes the unique solution of BVP

$$\begin{cases} x''=\theta, & 0\leq t\leq 1; \\ ax(0)-bx'(0)=x_0, & cx(1)+dx'(1)=x_1, \end{cases}$$

which is given by

$$y(t)=J^{-1}\{(c(1-t)+d)x_0+(at+b)x_1\}.$$

Evidently, $y\in C^2(I,E)\cap P(I)$. Let A be defined by (14). Then condition (H_1) and lemma 7 imply that A is a semi-$\overline{k}$-set-contraction operator from $C^1(I,E)$ to $P(I)$. By (H_2), there exist $\delta>0$ and $R>2\|u_0\|$ such that for any $R'\geq R$

$$\frac{M(R')}{R'}<\frac{1}{q(m+\delta)}, \tag{27}$$

and

$$\frac{m}{m+\delta}+\frac{\|y\|_1}{R}<1 \tag{28}$$

Let $U = \{x \in C^1(I,E), \|x\|_1 < R\}$. So U is bounded convex open set. For $x \in \overline{U}$, we have $\|x\|_1 \le R$ and

$$\begin{aligned}
&\|Ax\|_0 \\
= &\max\{\|\int_0^1 G(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))ds + y(t)\|, t \in I\} \\
\le &\max\{\int_0^1 G(t,s)\|f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\|ds + \|y(t)\|, t \in I\} \\
\le &M(mR)\max\{\int_0^1 G(t,s)ds, t \in I\} + \|y\|_1 \\
\le &mR\frac{1}{q(m+\delta)}q_1 + \|y\|_1 \\
< &R(\frac{m}{m+\delta} + \frac{\|y\|_1}{R}) \\
< &R \qquad (29)
\end{aligned}$$

and

$$\begin{aligned}
&\|(Ax)'\|_0 \\
= &\max\{\|\int_0^1 G'_t(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))ds + y'(t)\|, t \in I\} \\
\le &\max\{\int_0^1 |G'_t(t,s)|\|f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\|ds + \|y'(t)\|, t \in I\} \\
\le &M(mR)\max\{\int_0^1 |G'_t(t,s)|ds, t \in I\} + \|y\|_1 \\
\le &mR\frac{1}{q(m+\delta)}q_2 + \|y\|_1 \\
< &R(\frac{m}{m+\delta} + \frac{\|y\|_1}{R}) \\
< &R \qquad (30)
\end{aligned}$$

hence $\|Ax\|_1 < R$.

In virtue of (29), (30), $A\overline{U} \subseteq U$. Then by theorem 4 we get

$$\deg(id - A, U, \theta) = 1,$$

i.e., there is a fixed point $x \in U$. The proof is complete.□

Example 1.

We consider following system of scalar valued differential equations

$$\begin{cases} -x''_n = 3(|x_n|+1)^{\frac{1}{2}} + \frac{1}{n+1}(x'^2_{n+1})^{\frac{1}{3}} + \frac{1}{2n}|\int_0^t \frac{1}{1+t+s}x_{2n}(s)ds|^{\frac{1}{3}} \\ +\frac{1}{3n}(\int_0^1 \cos(t-s)x_{3n}(s)ds)^{\frac{2}{3}} + 17, \\ x_n(0) = x_n(1) = 0, \quad n = 1, 2, \cdots. \end{cases} \qquad (31)$$

Conclusion: equation (31) has at least one positive solution.

Proof. Let $E = \{x = (x_1, x_2, \cdots, x_n, \cdots), \sup_{n\in N} |x_n| < +\infty.\}$ with norm $\|x\| = \sup_{n\in N} |x_n|$, and $P = \{x = (x_1, x_2, \cdots) \in E,\ x_n \geq 0,\ n = 1, 2, \cdots\}$. Then P is a normal solid cone of E and (31) can be regarded as a BVP of the form (12), where $a = c = 1$, $b = d = 0$, $x_0 = x_1 = \theta$, $k(t,s) = \dfrac{1}{1+t+s}$, $h(t,s) = \cos(t-s)$, $x = (x_1, x_2, \cdots)$, $y = (y_1, y_2, \cdots)$, $z = (z_1, z_2, \cdots)$, $w = (w_1, w_2, \cdots)$, and $f = g + h = (g_1, g_2, \cdots) + (h_1, h_2, \cdots)$ in which

$$g_n(t, x, y, z, w) = 3(|x_n| + 1)^{\frac{1}{2}} + 17, \tag{32}$$

and

$$h_n(y, z, w) = \frac{1}{n+1}(y_{n+1}^2)^{\frac{1}{3}} + \frac{1}{2n} z_{2n}^{\frac{1}{3}} + \frac{1}{3n} w_{3n}^{\frac{2}{3}}. \tag{33}$$

Then

$$\|f\| \leq 3(\|x\| + 1)^{\frac{1}{2}} + \frac{1}{2}(\|y\|)^{\frac{2}{3}} + \frac{1}{2}\|z\|^{\frac{1}{3}} + \frac{1}{3}\|w\|^{\frac{2}{3}} + 17. \tag{34}$$

which implies

$$M(R) \leq 3(R+1)^{\frac{1}{2}} + \frac{1}{2}R^{\frac{2}{3}} + \frac{1}{2}R^{\frac{1}{3}} + \frac{1}{3}R^{\frac{2}{3}} + 17$$

and consequently

$$\lim_{R\to+\infty} \frac{M(R)}{R} = 0.$$

This shows that condition (H_2) is satisfied.

Obviously, $f \in C(I \times E \times E \times E \times E, P)$ and f is uniformly continuous on $I \times B_R \times B_R \times B_R \times B_R$ for any $R > 0$. Now for any bounded $D \subseteq E$, it is easy to see that $\alpha(g(D)) \leq \frac{3}{2}\alpha(D)$. And for any bounded $Y \subseteq E$, $Z \subseteq P$, $W \subseteq P$, we have $\alpha(h(Y, Z, W)) = 0$. In fact, let $\{y^{(m)}\} \subseteq Y$, $\{z^{(m)}\} \subseteq Z$, $\{w^{(m)}\} \subseteq W$, and $v_n^{(m)} = h_n(y^{(m)}, z^{(m)}, w^{(m)})$. By (33), we get

$$|v_n^{(m)}| \leq \frac{1}{n+1}\|y^{(m)}\|^{\frac{2}{3}} + \frac{1}{2n}\|z^{(m)}\|^{\frac{1}{3}} + \frac{1}{3n}\|w^{(m)}\|^{\frac{1}{3}}.$$

Now by the diagonal method, we can select a subsequence $\{v^{(m_i)}\} \subseteq \{v^{(m)}\}$ such that

$$v^{(m_i)} \to v^0 \in P.$$

So $\alpha(h(Y, Z, W)) = 0$. On the other hand, it is easy to see that in this case

$$q = \frac{1}{2}, \quad m = 1.$$

So the condition (H_1) is satisfied. Consequently, our conclusion follows from theroem 5. □

The operator A defined by (31) is not a strict-set-contraction operator or a condensing operator. So the degree theory of the condensing operator or the strict-set-contraction operator is not suitable.

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