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Homogenization for Poisson equations in domains with concentrated holes

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Abstract. We consider solutions u^ε of Poisson problems with the Dirichlet condition on domains Ω_ε with holes concentrated at subsets of a domain Ω non-periodically. We show u^ε converges to a solution of a Poisson problem with a simple function potential. This is a generalized result of a sample model given by Cioranescu and Murat (1997). They showed a result for case that holes are distributed at Ω periodically.

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§1. Introduction

Let $\Omega \subset \mathbb{R}^d, d \geq 2$ be open and bounded with C^2 boundary. We consider a union T_ε of holes concentrated at subsets of $\mathbb{R}^d$ as Figure 1, and domains $\Omega_\varepsilon = \Omega \setminus T_\varepsilon$. We consider Poisson problems on Ω_ε with the homogeneous Dirichlet condition with $f \in L^2(\Omega)$, that is,

$$u^\varepsilon \in H_0^1(\Omega_\varepsilon), \quad -\Delta u^\varepsilon = f. \tag{1.1}$$

We will see u^ε converge to u as $\varepsilon \to 0$ which satisfies

$$u \in H_0^1(\Omega), \quad (-\Delta + V)u = f, \tag{1.2}$$

where V is a simple function. Details of assumptions for T_ε and the main result are given in Section 2.1.

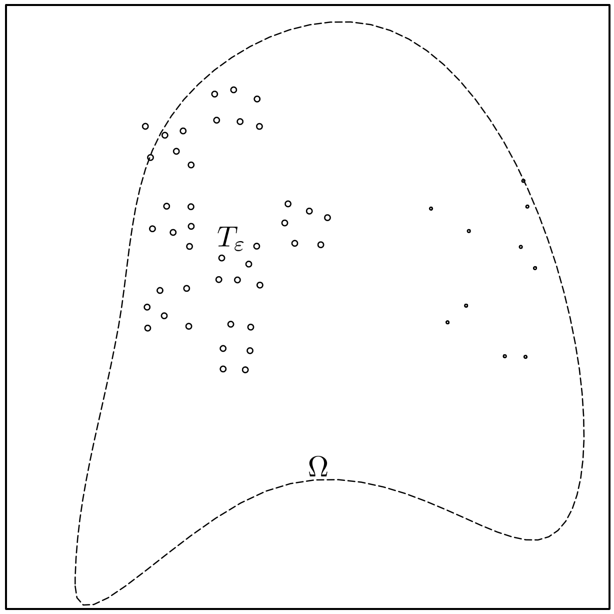

Figure 1: A domain Ω and holes T_ε.

1.1. Known results

There are many contributions to characterize the limit u of solutions u^ε on domains Ω_ε when $\Omega_\varepsilon \to \Omega$ in a proper sense. The PDE of the form (1.2) is often used to characterize the limit u. Many examples with $V = 0$ are introduced at [6], for example, $\Omega_\varepsilon \to \Omega \setminus K$ metrically with thin K.

On the other hand, there are examples for which $V \neq 0$. The case when $T_\varepsilon = \bigcup_{i \in 2\varepsilon\mathbb{Z}^d} \overline{B(i, a_\varepsilon)}$ with the critical radius a_ε is introduced at [1, Example 2.1], where a_ε satisfies the same condition for $a_{\varepsilon,k}$ of (2.3) below. In this case, V is a constant. A similar result for Robin condition is given by [5] with a different critical radius and a different constant V. These results can be regarded as a strong resolvent convergence of Laplacian, and they were improved to a norm resolvent convergence of Laplacian with Dirichlet, Robin and Neumann conditions by [2]. In these cases, V is still a constant.

Other examples for which $V \neq 0$ are also introduced at [1, Example 2.9]. If T_ε is a union of holes on a hyper plane, V is a Dirac measure supported on the hyper plane.

As for randomly perforated domains, convergence of solutions in a proper sense with holes whose centers are generated by either Poisson or stationary point process is given by [3], [4] with a constant V.

§2. Assumption and the main result

2.1. Assumption

We denote Lebesgue measure on $\mathbb{R}^d$ by $|\cdot|$. We use a class $\mathcal{J}$ of sets to determine where holes concentrate.

Definition 1. *Let*

$$\mathcal{J} = \{E \subset \mathbb{R}^d \mid |\partial E| = 0\}.$$

Remark 1. *If $E \subset \mathbb{R}^d$ and $|\overline{E}| < \infty$, $E \in \mathcal{J}$ if and only if $|\overline{E}| = |\mathring{E}|$ by $\partial E = \overline{E} \setminus |\mathring{E}|$. Elements of $\mathcal{J}$ are measurable by completeness of Lebesgue measure.*

We shall construct holes T_ε as follows (see Figure 2). Let $m \in \mathbb{N}$, $\{F_k\}_{k=1}^m \subset \mathcal{J}$ be a collection of disjoint sets and $\{N_k\}_{k=1}^m \subset \mathbb{N}$. We use $\bigsqcup$ instead of $\bigcup$ for the disjoint union of sets. Let $A \subset \mathbb{R}^d$ be measurable and bounded, and $\Lambda \subset \mathbb{R}^d$ be countable such that

$$\mathbb{R}^d = \bigsqcup_{i \in \Lambda} (A + i) \quad (A + i = \{x + i \mid x \in A\}). \tag{2.1}$$

For $x \in \mathbb{R}^d$ and $R > 0$, we denote $B(x, R) = \{y \in \mathbb{R}^d \mid |x - y| < R\}$. Choose small $C > 0$ with

$$|A| > \max_{k \le m} N_k |B(0, C)|. \tag{2.2}$$

We denote $A_i^\varepsilon = \varepsilon(A + i) = \{\varepsilon x \mid x \in A + i\}$. Remark $\mathbb{R}^d = \bigsqcup_{i \in \Lambda} A_i^\varepsilon$ follows from (2.1) for each $\varepsilon > 0$.

Definition 2. *For $E \subset \mathbb{R}^d$ and $\varepsilon > 0$, let*

$$\Lambda_\varepsilon^-(E) = \{i \in \Lambda \mid A_i^\varepsilon \subset E\}, \quad \Lambda_\varepsilon^+(E) = \{i \in \Lambda \mid A_i^\varepsilon \cap E \neq \emptyset\}.$$

For $\varepsilon > 0$ and $i \in \Lambda_\varepsilon^-(F_k)$ (such k is unique for each i), consider centers of holes $\{x_{i,j}^\varepsilon \mid j = 1, ..., N_k\} \subset \mathbb{R}^d$ with $\bigsqcup_{j=1}^{N_k} B(x_{i,j}^\varepsilon, C\varepsilon) \subset A_i^\varepsilon$ for $\varepsilon \ll 1$. We omit to write $(\varepsilon \to 0)$ for convergence of sequences indexed by $\varepsilon > 0$. Consider radii of holes $a_{\varepsilon,k}$ with the following condition for $1 \le k \le m$:

$$\varepsilon^{-d} \times \begin{cases} (-\log a_{\varepsilon,k})^{-1} & (d = 2) \\ (a_{\varepsilon,k})^{d-2} & (d \ge 3) \end{cases} \to \tilde{\mu_k} \in [0, \infty). \tag{2.3}$$

We recall that Ω is bounded, open with C^2 boundary. We denote

$$T_{\varepsilon,k} = \bigsqcup_{i \in \Lambda_\varepsilon^-(F_k), j \le N_k} \overline{B(x_{i,j}^\varepsilon, a_{\varepsilon,k})}, \quad T_\varepsilon = \bigsqcup_{k=1}^m T_{\varepsilon,k}, \quad \Omega_\varepsilon = \Omega \setminus T_\varepsilon.$$

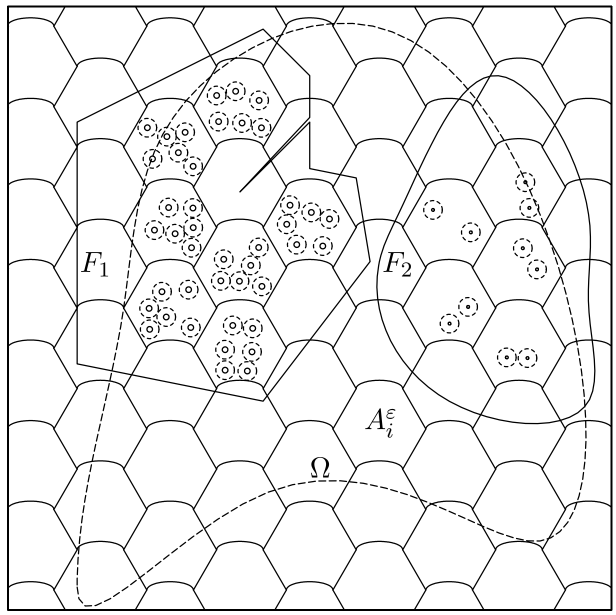

Figure 2: Construction of holes T_ε with $m = 2,\ N_1 = 6,\ N_2 = 2$.

2.2. Result

Using the surface area S_d of $\partial B(0,1)$, we write $\mu_d = \frac{S_d}{|A|} \times \begin{cases} 1 & (d=2) \\ d-2 & (d \geq 3) \end{cases}$.

For $E \subset \mathbb{R}^d$, we denote $1_E(x) = \begin{cases} 1 & (x \in E) \\ 0 & (x \notin E) \end{cases}$. Our main result is stated as follows.

Theorem 1. *Under the assumptions as in Section 2.1, u^ε in* (1.1) *converges to u weakly in $H_0^1(\Omega)$ and the limit u solves* (1.2) *with*

$$V = \mu_d \sum_{k=1}^{m} \tilde{\mu_k} N_k 1_{F_k}.$$

Remark 2. *[1, Example 2.1] is just Theorem 1 with $F_1 = \mathbb{R}^d, A = [-1,1)^d, \Lambda = 2\mathbb{Z}^d, N_1 = 1, x_{i,1}^\varepsilon = i\varepsilon$. It means holes are distributed on Ω periodically. We generalized it for the case where holes distributed concentrated at F_k non-periodically. Moreover, each F_k can have different density $\tilde{\mu_k} N_k$.*

2.3. Outline of proof

The proof of our main result is based on the theorem below.

Theorem 2 ([1, Theorem 1.2]). *Assume that $T_\varepsilon \subset \mathbb{R}^d$ is closed for each $\varepsilon > 0$. Assume there is a sequence*

$$\{w^\varepsilon\} \subset H^1(\Omega) \tag{H.1}$$

satisfying

$$w^\varepsilon = 0 \text{ on } T_\varepsilon \text{ for each } \varepsilon > 0, \tag{H.2}$$

$$w^\varepsilon \to 1 \text{ weakly in } H^1(\Omega), \tag{H.3}$$

and there is

$$V \in W^{-1,\infty}(\Omega) \tag{H.4}$$

(thus, $V \in H^{-1}(\Omega)$) such that

$$\begin{aligned} &\langle -\Delta w^\varepsilon, \varphi v^\varepsilon \rangle_{H^{-1}(\Omega)} \to \langle V, \varphi v \rangle_{H^{-1}(\Omega)} \\ &\text{if } \varphi \in C_0^\infty(\Omega),\ v^\varepsilon = 0 \text{ on } T_\varepsilon.\ v^\varepsilon \to v \text{ weakly in } H^1(\Omega). \end{aligned} \tag{H.5}$$

Then, u^ε in (1.1) *converges to $u \in H_0^1(\Omega)$ weakly in $H_0^1(\Omega)$ where u is solution to* (1.2).

We check the conditions (H.1)–(H.5) to prove Theorem 1. As mentioned in [1], it is not unusual that assuming the condition (H.5).

We first prepare some lemmas in Section 3.1, and we introduce w^ε and verify the conditions (H.1)–(H.4) in Section 3.2. Finally, we check the condition (H.5) in Section 3.3 and complete the proof of Theorem 1.

§3. Proof

3.1. Approximation of sets by tiles A_i^ε

We first state some properties for $\mathcal{J}$.

Lemma 1. *Let $E_1, E_2 \in \mathcal{J}$, then $|\overline{E_1 \cap E_2}| = |(E_1 \cap E_2)^\circ|$.*

Proof. A distributive property for sets shows $\overline{E_1 \cap E_2} \subset \overline{E_1} \cap \overline{E_2} = (\mathring{E}_1 \sqcup \partial E_1) \cap (\mathring{E}_2 \sqcup \partial E_2) = (\mathring{E}_1 \cap \mathring{E}_2) \cup E = (E_1 \cap E_2)^\circ \cup E$ with some E satisfying $|E| = 0$. □

Definition 3. *For $E \subset \mathbb{R}^d$ and $\varepsilon > 0$, let*

$$A_\varepsilon^\pm(E) = \bigsqcup_{i \in \Lambda_\varepsilon^\pm(E)} A_i^\varepsilon.$$

We remark that $A_\varepsilon^-(E) \subset E \subset A_\varepsilon^+(E)$. We will see that they are approximations of E by Lemmas 2 and 3 below.

Lemma 2. *Let $E \subset \mathbb{R}^d$ be measurable and bounded, and satisfy $|E| = |\overline{E}|$. Then $|A_\varepsilon^+(E)| \to |E|$.*

Proof. Let $d_\varepsilon = \mathrm{diam}(\varepsilon A)$. Then $d_\varepsilon \to 0$. Let $E_\varepsilon = \bigcup_{x\in E} \overline{B(x, d_\varepsilon)}$. Then $\bigcap_{\varepsilon>0} E_\varepsilon = \overline{E}$ and $|E_\varepsilon| < \infty$. Thus $|E_\varepsilon| \to |\overline{E}| = |E|$. The assertion follows from it and $E_\varepsilon \supset A_\varepsilon^+(E) \supset E$. □

Lemma 3. *Let $E \subset \mathbb{R}^d$ be a measurable set such that $|\mathring{E}| = |E|$. Then $|A_\varepsilon^-(E)| \to |E|$.*

Proof. Let $V = \mathring{E}, g(x) = \mathrm{dist}(x, \partial V),\ d_\varepsilon = \mathrm{diam}(\varepsilon A)$ and

$$V_{-\varepsilon} = V \cap g^{-1}((d_\varepsilon, \infty)).$$

Then $\bigcup_{\varepsilon>0} V_{-\varepsilon} = V$ since V is open. The assertion follows from $V_{-\varepsilon} \subset A_\varepsilon^-(V) \subset E$. We verify $V_{-\varepsilon} \subset A_\varepsilon^-(V)$. Let $x \in V_{-\varepsilon}$. There is $i \in \Lambda$ that $x \in A_i^\varepsilon$. We show $i \in \Lambda_\varepsilon^-(V)$. It is equivalence with $\mathbb{R}^d \setminus V \subset \mathbb{R}^d \setminus A_i^\varepsilon$. If $y \notin V$, we can get $p \in \partial V$ from line segment which contain $\{x, y\}$. It is $p_t = (1-t)x + ty$ with minimal $t \in [0,1]$ that $p_t \notin V$. Construction of p imply $|x-y| = |x-p|+|p-y| \geq \mathrm{dist}(x, \partial V) > d_\varepsilon$. Thus $y \notin A_i^\varepsilon$. Thus $i \in \Lambda_\varepsilon^-(V)$. □

We can count how many tiles $A_\varepsilon^\pm(E)$ has.

Lemma 4. *For $E \subset \mathbb{R}^d$ and $\varepsilon > 0$, the number of elements of $\Lambda_\varepsilon^\pm(E)$ is $\frac{|A_\varepsilon^\pm(E)|}{\varepsilon^d|A|}$.*

We say E is a cube if $E = [0, R)^d + x$ with some $x \in \mathbb{R}^d, R > 0$. We prepare lemmas related to weak star topology of $L^\infty(\mathbb{R}^d) = L^1(\mathbb{R}^d)^*$. We denote $\langle g, h\rangle_{L^1(\mathbb{R}^d)^*} = \int ghdx$ for $g \in L^\infty(\mathbb{R}^d) = L^1(\mathbb{R}^d)^*, h \in L^1(\mathbb{R}^d)$.

Lemma 5. *Let $\{g_\varepsilon\} \subset L^\infty(\mathbb{R}^d)$ be bounded and $g \in L^\infty(\mathbb{R}^d)$. If*

$$\langle g_\varepsilon, 1_E\rangle_{L^1(\mathbb{R}^d)^*} \to \langle g, 1_E\rangle_{L^1(\mathbb{R}^d)^*}$$

for any cube E, $g_\varepsilon \to g$ weakly star in $L^\infty(\mathbb{R}^d)$.

Proof. If follows from the fact that the vector space generated by $\{1_E | E :$ cube$\}$ is dense at $L^1(\mathbb{R}^d)$. And the fact follows from the facts that the set of simple functions on $\mathbb{R}^d$ is dense in $L^1(\mathbb{R}^d)$, the Lebesgue measure is outer regular and any open set can be represented as the union of disjoint countable cubes. □

Lemma 6. *If* $f_\varepsilon \to f$ *in* $L^2(\mathbb{R}^d), |f_\varepsilon| \le 1$ *for* $\varepsilon \ll 1$ *and* $g_\varepsilon \to g$ *weakly star in* $L^\infty(\mathbb{R}^d)$, *we have* $f_\varepsilon g_\varepsilon \to fg$ *weakly star in* $L^\infty(\mathbb{R}^d)$.

Proof. The existence of a subsequence of f_ε converging to f a.e. gives $|f| \le 1$ a.e. The assertion follows from $c := \sup_{\varepsilon>0} \|g_\varepsilon\|_{L^\infty(\mathbb{R}^d)} < \infty$, Lemma 5, and $|\langle f_\varepsilon g_\varepsilon - fg, 1_E\rangle| \le c\|f_\varepsilon - f\|_{L^2(\mathbb{R}^d)}\|1_E\|_{L^2(\mathbb{R}^d)} + |\langle g_\varepsilon - g, f1_E\rangle|$ for any cube E. $\square$

3.2. Error corrector w^ε

By (2.3), we have $\frac{\max_k a_{\varepsilon,k}}{\varepsilon} \to 0$. Thus $\max\limits_{k\le m} a_{\varepsilon,k} < C\varepsilon$ for $\varepsilon \ll 1$ (recall $C > 0$ is chosen to satisfy (2.2)). Let

$$w^\varepsilon_{0,k}(r) = \begin{cases} \dfrac{\log a_{\varepsilon,k} - \log r}{\log a_{\varepsilon,k} - \log C\varepsilon} & (d=2), \\ \dfrac{(a_{\varepsilon,k})^{-d+2} - r^{-d+2}}{(a_{\varepsilon,k})^{-d+2} - (C\varepsilon)^{-d+2}} & (d\ge 3), \end{cases} \quad (a_{\varepsilon,k} \le r \le C\varepsilon),$$

$$B_{\varepsilon,k} = \bigsqcup_{i\in\Lambda^-_\varepsilon(F_k), j\le N_k} B(x^\varepsilon_{i,j}, C\varepsilon), \quad B_\varepsilon = \bigsqcup_{k=1}^m B_{\varepsilon,k},$$

$$w^\varepsilon(x) = \begin{cases} 0 & (x \in T_\varepsilon), \\ w^\varepsilon_{0,k}(|x - x^\varepsilon_{i,j}|) & (x \in B(x^\varepsilon_{i,j}, C\varepsilon) \setminus B(x^\varepsilon_{i,j}, a_{\varepsilon,k})), \\ 1 & (x \notin B_\varepsilon). \end{cases}$$

Then we have

$$\Delta w^\varepsilon = 0 \text{ on } B_\varepsilon \setminus T_\varepsilon. \tag{3.1}$$

and (H.2). We need the limit of $1_{B_{\varepsilon,k}}$ to analyze w^ε.

Lemma 7. $1_{B_{\varepsilon,k}} \to \frac{N_k|B(0,C)|}{|A|} 1_{F_k} = \frac{N_k C^d S_d}{d|A|} 1_{F_k}$ *weakly star in* $L^\infty(\mathbb{R}^d)$.

Proof. Let E be a cube. By $|B_{\varepsilon,k} \cap A^\varepsilon_i| = \begin{cases} N_k|B(0,C\varepsilon)| & (i \in \Lambda^-_\varepsilon(F_k)) \\ 0 & (i \notin \Lambda^-_\varepsilon(F_k)) \end{cases}$, Lemma 4 and $B_{\varepsilon,k} \subset F_k$, we have

$$\begin{aligned} \frac{|A^-_\varepsilon(E\cap F_k)|}{\varepsilon^d|A|} N_k|B(0,C\varepsilon)| = |B_{\varepsilon,k} \cap A^-_\varepsilon(E\cap F_k)| \le \left\langle 1_{B_{\varepsilon,k}}, 1_E \right\rangle_{L^1(\mathbb{R}^d)^*} \\ \le \frac{|A^+_\varepsilon(E\cap F_k)|}{\varepsilon^d|A|} N_k|B(0,C\varepsilon)|. \end{aligned}$$

By Lemmas 1 to 3,

$$\frac{|A_\varepsilon^-(E\cap F_k)|}{\varepsilon^d|A|}N_k|B(0,C\varepsilon)| \to \frac{|E\cap F_k|N_k|B(0,C)|}{|A|} = \left\langle \frac{N_k|B(0,C)|}{|A|}1_{F_k}, 1_E \right\rangle .$$

These, Lemma 5 and $|B(0,C)| = \frac{S_d C^d}{d}$ imply the assertion. □

Lemma 8. *We have* (H.1) *and* (H.3)

Proof. For $i \in \Lambda_\varepsilon^-(F_k),\ j \le N_k,\ k \le m$, We have

$$\begin{aligned}\|\nabla w^\varepsilon\|^2_{L^2(B(x^\varepsilon_{i,j},C\varepsilon)\setminus\overline{B(x^\varepsilon_{i,j},a_{\varepsilon,k})})} &= S_d\int_{a_{\varepsilon,k}}^{C\varepsilon}|\partial_r w^\varepsilon_{0,k}(r)|^2 r^{d-1}dr\\ &= S_d\begin{cases}\dfrac{1}{\log C\varepsilon-\log a_{\varepsilon,k}} & (d=2),\\ \dfrac{d-2}{(a_{\varepsilon,k})^{-d+2}-(C\varepsilon)^{-d+2}} & (d\ge 3),\end{cases}\end{aligned}$$

which along with $|w^\varepsilon| \le 1$ implies w^ε is an extension of an $H^1_{loc}(B_\varepsilon\backslash T_\varepsilon)$ function by the boundary values on $\partial(B_\varepsilon \setminus T_\varepsilon)$. Thus, ∇w^ε in the distributional sense coincides with the pointwise, classical derivative and

$$\|\nabla w^\varepsilon\|^2_{L^2(A_i^\varepsilon)} = \begin{cases}\dfrac{N_kS_d}{\log C\varepsilon-\log a_{\varepsilon,k}} & (i\in\Lambda_\varepsilon^-(F_k),\ d=2),\\ \dfrac{N_kS_d(d-2)}{(a_{\varepsilon,k})^{-d+2}-(C\varepsilon)^{-d+2}} & (i\in\Lambda_\varepsilon^-(F_k),\ d\ge 3),\\ 0 & (i\notin\bigcup_{k\le m}\Lambda_\varepsilon^-(F_k)).\end{cases}$$

Using (2.3) for them, we have $c := \sup_{\varepsilon>0,i\in\Lambda}\varepsilon^{-d}\|\nabla w^\varepsilon\|^2_{L^2(A_i^\varepsilon)} < \infty$. Thus $\|\nabla w^\varepsilon\|^2_{L^2(A_i^\varepsilon)} \le c\varepsilon^d$. It and Lemmas 2 and 4 imply

$$\|\nabla w^\varepsilon\|^2_{L^2(\Omega)} \le \|\nabla w^\varepsilon\|^2_{L^2(A_\varepsilon^+(\Omega))} \le \frac{|A_\varepsilon^+(\Omega)|}{\varepsilon^d|A|}c\varepsilon^d \le \frac{c|\bigcup_{x\in\Omega}B(x,1)|}{|A|}\ (\varepsilon\ll 1),$$

which together with $|w^\varepsilon| \le 1$ implies (H.1), and $\{w^\varepsilon\} \subset H^1(\Omega)$ is bounded.

Consider any subsequences of $\{w^\varepsilon\}$ (we still denote w^ε) which converge weakly in $H^1(\Omega)$, and let $w = \text{w-}\lim_{\varepsilon\to 0} w^\varepsilon$. We show $w = 1$. Let $F = \sqcup_k F_k$. Rellich's theorem gives $w^\varepsilon 1_{\mathbb{R}^d\setminus F} = 1_{\mathbb{R}^d\setminus F}$ tend to $w1_{\mathbb{R}^d\setminus F} = 1_{\mathbb{R}^d\setminus F}$ in $L^2(\Omega)$. Thus, $w = 1$ a.e. on $\Omega \setminus F$. On the other hand, Lemma 7 gives $1_{F_k\setminus B_{\varepsilon,k}} = 1_{F_k}(1-1_{B_{\varepsilon,k}}) \to 1_{F_k}(1-c_k1_{F_k}) = (1-c_k)1_{F_k}$ weakly star in $L^\infty(\mathbb{R}^d)$ where $c_k = \frac{N_k|B(0,C)|}{|A|}$. Hence $w^\varepsilon 1_\Omega 1_{F_k\setminus B_{\varepsilon,k}} = 1_\Omega 1_{F_k\setminus B_{\varepsilon,k}}$ tends to $w1_\Omega(1-c_k)1_{F_k} = 1_\Omega(1-c_k)1_{F_k}$ weakly star in $L^\infty(\mathbb{R}^d)$ for each k by Lemma 6. Since $0 < c_k < 1$ by (2.2), we have $w = 1$ on $\Omega\cap F_k$. Since $\mathbb{R}^d = (\mathbb{R}^d\setminus F)\cup(\sqcup_k F_k)$, we have $w = 1$ on Ω. □

We use a special function to analyze a distribution $-\Delta w^\varepsilon$. Let

$$q_0^\varepsilon(r) = \frac{r^2 - (C\varepsilon)^2}{2} \ (0 \le r \le C\varepsilon),$$

$$q^\varepsilon(x) = \begin{cases} q_0^\varepsilon(|x - x_{i,j}^\varepsilon|) & (x \in B(x_{i,j}^\varepsilon, C\varepsilon)) \\ 0 & (x \notin B_\varepsilon) \end{cases}.$$

Then we have

$$-\Delta q^\varepsilon = -d \ (x \in B_\varepsilon), \ \partial_r q_0^\varepsilon(C\varepsilon) = C\varepsilon, \ q_0^\varepsilon(C\varepsilon) = 0. \tag{3.2}$$

Now we decompose the restricted distribution $(-\Delta w^\varepsilon)|_{H_0^1(\Omega_\varepsilon)}$ by using q^ε.

Lemma 9. *Suppose $v \in H_0^1(\Omega_\varepsilon)$. Then we have*

$$\langle -\Delta w^\varepsilon, v\rangle_{H^{-1}(\Omega)} = \sum_{k \le m} \frac{\partial_r w_{0,k}^\varepsilon(C\varepsilon)}{C\varepsilon}\left(\int_{B_{\varepsilon,k}} \nabla q^\varepsilon \cdot \nabla v dx + d\left\langle 1_{B_{\varepsilon,k}}, v\right\rangle_{H^{-1}(\Omega)}\right).$$

Proof. By (3.2) and integration by parts,

$$\int_{B_{\varepsilon,k}} \nabla q^\varepsilon \cdot \nabla v dx = C\varepsilon \int_{\partial B_{\varepsilon,k}} v d\sigma - d\left\langle 1_{B_{\varepsilon,k}}, v\right\rangle_{H^{-1}(\Omega)}$$

for $v \in H_0^1(\Omega_\varepsilon)$. By assumption, $\int_{\partial T_{\varepsilon,k}} v d\sigma = 0$. Using them and (3.1), we have

$$\begin{aligned}\langle -\Delta w^\varepsilon, v\rangle_{H^{-1}(\Omega)} &= \sum_{k \le m} \int_{B_{\varepsilon,k}\setminus T_{\varepsilon,k}} \nabla w^\varepsilon \cdot \nabla v dx = \sum_{k \le m} \partial_r w_{0,k}^\varepsilon(C\varepsilon) \int_{\partial B_{\varepsilon,k}} v d\sigma \\ &= \sum_{k \le m} \frac{\partial_r w_{0,k}^\varepsilon(C\varepsilon)}{C\varepsilon}\left(\int_{B_{\varepsilon,k}} \nabla q^\varepsilon \cdot \nabla v dx + d\left\langle 1_{B_{\varepsilon,k}}, v\right\rangle_{H^{-1}(\Omega)}\right).\end{aligned}$$

This completes the proof. □

The following lemma is very similar to (H.5).

Lemma 10. *Suppose that $v^\varepsilon \in H_0^1(\Omega_\varepsilon)$ and $v^\varepsilon \to v$ weakly in $H_0^1(\Omega)$, Then*

$$\langle -\Delta w^\varepsilon, v^\varepsilon\rangle_{H^{-1}(\Omega)} \to \left\langle \mu_d \sum_{k=1}^m \tilde{\mu_k} N_k 1_{F_k}, v\right\rangle_{H^{-1}(\Omega)}.$$

Proof. By (2.3), we have $\dfrac{\partial_r w_{0,k}^\varepsilon(C\varepsilon)}{C\varepsilon} \to \dfrac{\tilde{\mu_k}}{C^d} \times \begin{cases} 1 & (d = 2) \\ d-2 & (d \ge 3) \end{cases}$. We also have

$$\left|\int_{B_{\varepsilon,k}} \nabla q^\varepsilon \cdot \nabla v^\varepsilon dx\right| \le C\varepsilon \sup_{\delta > 0} \left\|v^\delta\right\|_{W^{1,1}(\Omega)} \to 0.$$

Rellich's theorem gives $|\langle 1_{B_{\varepsilon,k}}, v^\varepsilon - v\rangle_{H^{-1}(\Omega)}| \le \|1\|_{L^2(\Omega)}\|v^\varepsilon - v\|_{L^2(\Omega)} \to 0$. It and Lemma 7 give

$$\begin{aligned}\langle 1_{B_{\varepsilon,k}}, v^\varepsilon\rangle_{H^{-1}(\Omega)} &= \langle 1_{B_{\varepsilon,k}}, v^\varepsilon - v\rangle_{H^{-1}(\Omega)} + \langle 1_{B_{\varepsilon,k}}, 1_\Omega v\rangle_{L^1(\mathbb{R}^d)^*} \\ &\to \left\langle \frac{N_k C^d S_d}{d|A|} 1_{F_k}, v\right\rangle_{H^{-1}(\Omega)}.\end{aligned}$$

The assertion follows from these limit and Lemma 9. □

3.3. Proof of Theorem 1

Proof. Since $V = \mu_d \sum_{k=1}^m \tilde{\mu_k} N_k 1_{F_k} \in L^\infty(\Omega) = L^1(\Omega)^* \subset W^{-1,\infty}(\Omega)$, we have (H.4). We shall verify (H.5). Indeed, the multiplier of $\varphi : H^1(\Omega) \to H_0^1(\Omega)$ is a bounded operator. Thus, $\varphi v^\varepsilon \to \varphi v$ weakly in $H_0^1(\Omega)$. It and Lemma 10 imply (H.5). Since we already checked (H.1)–(H.3) in Section 3.2, Theorem 1 follows from Theorem 2. □

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